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Nonabelian Special K-Flows*

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The Kolmogorov–Sinai theory of special K-flows is enlarged to a class of nonabelian dynamical systems whose stochastic behavior is analyzed. The main result of this paper is that these dynamical systems retain the fundamental property of having homogeneous Lebesgue spectrum with countably infinite multiplicity.

1. INTRODUCTION

The motivation of this paper is to obtain a modification of the mathematical notion of classical K-flows (or "Kolmogorov dynamical system") in such a manner that it becomes general enough to encompass situations encountered in the quantum mechanical theory of dissipative phenomena. We however would like to concentrate our attention here on the mathematical aspects of the theory, and defer to a separate publication the discussion of its physical applications. In connection with the latter, let it thus suffice to say for the time being that the mathematical structure proposed in the present paper seems to occur naturally in the construction of mechanical models which accommodate quantum diffusion processes.

We briefly recall in this section the definition and spectral properties of classical K-flows, and draw the reader's attention to their algebraic structure. In the next section, this structure is used to define by analogy what we call "completely refining dynamical systems," the latter being not necessarily abelian; we show (Theor. 2.4) that these systems are ergodic, mixing and have homogeneous Lebesgue spectrum. In Section 3 the multiplicity of this spectrum is established (Theor. 3.11) to be countably infinite for the particular case of "nonabelian, special K-flows," thus generalizing to these systems

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Sinai's fundamental result [14, Theor. 3] on classical (abelian) special K-flows.

We now review briefly the fundamentals of classical K-flows; for general references, see for instance [1, 13].

DEFINITION 1.1. A classical flow is a triple (Ω, μ, T) which consists of a Lebesgue space (Ω, μ) and a mapping $T: \mathbb{R} \to \mathrm{Aut}(\Omega, \mu)$ such that: (i) $T(s)\, T(t) = T(s + t)$ for all s, t in $\mathbb{R}$; (ii) for each μ-measurable subset $X \subseteq \Omega$, $\{(x, t) \mid T(t)\, [x] \in X\}$ is measurable in $\Omega \times \mathbb{R}$.

By "Koopman formalism" we understand here the collection of wellknown (see in particular [8]) results summarized in the following proposition.

PROPOSITION 1.2. *Let $\mathscr{H}$ denote the Hilbert space $\mathscr{L}^2(\Omega, \mu)$, $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$, π: $f \in \mathscr{L}^\infty(\Omega, \mu) \mapsto \pi(f) \in \mathscr{B}(\mathscr{H})$ be defined by $[\pi(f)\Psi](\omega) = f(\omega)\, \Psi(\omega)$ for all Ψ in $\mathscr{H}$ and almost all ω in Ω. For the sake of notational simplicity we identify $\mathscr{L}^\infty(\Omega, \mu)$ and its image $\mathscr{N}$ through π. Let α: $\mathbb{R} \to \mathrm{Aut}(\mathscr{N})$ be defined by $(\alpha(t)[N])(\omega) = N(T(t)[\omega])$ for all t in $\mathbb{R}$, all N in $\mathscr{N}$ and almost all ω in Ω. Let finally $\Phi \in \mathscr{H}$ be the function $\Phi(\omega) = 1$ for all ω in Ω. Then*: (i) *the maximal abelian von Neumann algebra $\mathscr{N}$ admits Φ as cyclic and separating vector*; (ii) $\alpha(\mathbb{R})$ *is a continuous group of automorphisms of $\mathscr{N}$*; (iii) ϕ: $\mathscr{N} \to \mathbb{C}$ *defined by*

$$\langle \phi; N \rangle = (N\Phi, \Phi) = \int_\Omega N(\omega)\, \mu(d\omega),$$

is a $\alpha^*(\mathbb{R})$-invariant state on $\mathscr{N}$, where for each t in $\mathbb{R}$, $\alpha^*(t)$ denotes the dual of $\alpha(t)$; (iv) *for every t in $\mathbb{R}$, with Ψ (respectively, ω) running over $\mathscr{H}$ (respectively, Ω)*

$$[U(t)\Psi](\omega) = \Psi(T(t)[\omega]),$$

defines a unitary operator $U(t)$ on $\mathscr{H}$, in such a manner that

(a) *U: $\mathbb{R} \to \mathscr{U}(\mathscr{H})$ is continuous for the strong operator topology on $\mathscr{B}(\mathscr{H})$*;

(b) *$U(t)\, \Phi = \Phi$ for all t in $\mathbb{R}$; and*

(c) *$\alpha(t)[N] = U(t)\, NU(-t)$ for all N in $\mathscr{N}$ and all t in $\mathbb{R}$.*

DEFINITION 1.3. A classical K-flow is a classical flow for which there exists a measurable decomposition ξ of Ω satisfying the following conditions (where $\xi(t)$ denotes $T(t)[\xi]$):

(i) $\xi(-t) \leqslant \xi$ (mod 0) for all $t \geqslant 0$;

(ii) $\bigwedge_{t\in\mathbb{R}} \xi(t) = \nu$ (mod 0), where ν is the trivial decomposition consisting of Ω;

(iii) $\bigvee_{t\in\mathbb{R}} \xi(t) = \epsilon$ (mod 0), where ϵ is the decomposition consisting of the points of Ω.

DEFINITION 1.4. A strongly continuous, one-parameter group $V(\mathbb{R})$ of unitary operators acting on some Hilbert space $\mathscr{G}$ is said to have homogeneous Lebesgue spectrum of multiplicity K if there exists a decomposition $\mathscr{G} = \bigoplus_{k\in K} \mathscr{G}_k$ into mutually orthogonal subspaces invariant under $V(\mathbb{R})$ and such that for every $k \in K$ there exists a unitary operator V_k: $\mathscr{G}_k \to \mathscr{L}^2(\mathbb{R}, dx)$ with

$$V_k V_k(t)\, V_k^{-1} = U(t) \qquad \text{for all} \quad t \quad \text{in} \quad \mathbb{R},$$

where $V_k(t)$ denotes the restriction of $V(t)$ to $\mathscr{G}_k$, and

$$[U(t)\Psi](x) = \Psi(x - t) \qquad \text{for all} \quad x, t \quad \text{in} \quad \mathbb{R}.$$

PROPOSITION 1.5 [14, Theor. 3]. *Let* (Ω, μ, T, ξ) *be a classical K-flow. With the notation of Proposition* 1.2, *let*

$$\mathscr{H}_0 = \{\Psi \in \mathscr{H} \mid U(t)\Psi = \Psi\, \forall t \in \mathbb{R}\};$$

and let $\mathscr{H}_\perp$ *be the orthocomplement of* $\mathscr{H}_0$. *For each* $t \in \mathbb{R}$, *let* $U_\perp(t)$ *denote the restriction of* $U(t)$ *to* $\mathscr{H}_\perp$. *Then*: (i) $\mathscr{H}_0$ *is the one-dimensional subspace spanned by* Φ; *and* (ii) $U_\perp(\mathbb{R})$ *has homogeneous Lebesgue spectrum with countably infinite multiplicity.*

We might recall that much of the technicalities involved in the above definitions and results can be postponed if one first considers [1, 7] discrete flows, i.e., if one replaces $\mathbb{R}$ by $\mathbb{Z}$. The interest of K-systems ($\mathbb{Z}$) and K-flows ($\mathbb{R}$) is that their spectral properties ensure in particular that they are ergodic and that they satisfy strong mixing conditions; moreover they have positive entropy. This extremely stochastic behavior of K-systems is examplified by the fact that Bernouilli schemes are K-systems. Actually, the possibility of embedding a Bernouilli shift, and thus a K-system, in a mechanical system has even been taken [10] as as indication that stochastic processes are compatible with, and do occur in mechanics.

For the above reason, it becomes natural to ask whether the Kolmogorov–Sinai results persist when the classical framework reviewed above is generalized in such a manner that the maximal abelian von Neumann algebra $\mathcal{N} \approx \mathcal{L}^\infty(\Omega, \mu)$ is replaced by a nonabelian von Neumann algebra. As far as the physical potentials of such a generalization are concerned, let it suffice to say here that in analogy with the classical case where the elements of $\mathcal{L}^\infty(\Omega, \mu)$ are interpreted as the "stochastic variables" of the classical system of interest, the elements of the von Neumann algebra $\mathcal{N}$ are interpreted in quantum mechanics as the "observables" of the quantum system considered. The passage from $\mathcal{L}^\infty(\Omega, \mu)$ abelian to $\mathcal{N}$ nonabelian is indeed characteristic (see for instance [3, 6, 9, 14]) of the passage from classical to quantum mechanics.

2. Completely Self-Refining Dynamical Systems

Definition 2.1. A dynamical system is a triple $(\mathcal{N}, \Phi, \alpha)$ which consists of a von Neumann algebra $\mathcal{N}$ acting on a separable Hilbert space $\mathcal{H}$; a vector Φ in $\mathcal{H}$, normalized to 1, and cyclic in $\mathcal{H}$ with respect to $\mathcal{N}$; and a mapping α: $\mathbb{R} \to \mathrm{Aut}(\mathcal{N})$ such that: (i) $\alpha(s)\,\alpha(t) = \alpha(s + t)$ for all s, t in $\mathbb{R}$; (ii) for all t in $\mathbb{R}$ and all N in $\mathcal{N}$: $\langle \Phi; N\rangle \equiv (N\Phi, \Phi) = \langle \phi; \alpha(t)[N]\rangle$; and (iii) for all Ψ_1, Ψ_2 in $\mathcal{H}$ and all $N \in \mathcal{N}$, the mapping $t \in \mathbb{R} \mapsto (\alpha(t)[N]\,\Psi_1, \Psi_2) \in \mathbb{C}$ is continuous.

Upon reading again Proposition 1.2, one checks immediately that every classical flow defines a dynamical system. The converse however is obviously not true; in particular, it is easy to exhibit dynamical systems where $\mathcal{N}$ is not abelian, and which are thus not isomorphic to a classical flow. Hence the above definition is a genuine generalization of a classical flow. We should nevertheless notice that we kept some of the most important tools available for the study of classical flows. In particular, we have as a wellknown consequence (see for instance [3, Theor. II.2.5]) of the Gelfand–Naimark–Segal construction.

Lemma 2.2. *Let* $(\mathcal{N}, \Phi, \alpha)$ *be a dynamical system. Then there exists a strongly continuous, one parameter group* $U(\mathbb{R})$ *of unitary operators acting on* $\mathcal{H}$ *such that for all* s, t *in* $\mathbb{R}$: (i) $U(s)\,U(t) = U(s + t)$; (ii) $U(t)\Phi = \Phi$; *and* (iii) $\alpha(t)[N] = U(t)\,NU(-t)$ *for all* N *in* $\mathcal{N}$.

Definition 2.3. A completely self-refining dynamical system $(\mathcal{N}, \Phi, \alpha, \mathcal{A})$ is a dynamical system $(\mathcal{N}, \Phi, \alpha)$ for which there exists a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{N}$ satisfying the following conditions (where $\mathcal{A}(t)$ denotes $\alpha(t)[\mathcal{A}]$):

(i) $\mathscr{A}(-t) \subseteq \mathscr{A}$ for all $t \geqslant 0$,

(ii) $\bigcap_{t\in\mathbb{R}} [\mathscr{A}(t)\Phi] = \mathbb{C}\Phi$,

(iii) $\bigvee_{t\in\mathbb{R}} \mathscr{A}(t) = \mathscr{N}$.

(We used the following standard conventions: for any vector Ψ in $\mathscr{H}$ and any subset $\mathscr{M}$ of $\mathscr{B}(\mathscr{H})$, we denote by $[\mathscr{M}\Psi]$ the closed linear subspace of $\mathscr{H}$ spanned by $\{M\Psi \mid M \in \mathscr{M}\}$; for any family $\{\mathscr{A}_k \mid k \in K\}$ of von Neumann algebras acting on $\mathscr{H}$, we denote by $\bigvee_{k\in K} \mathscr{A}_k$ the smallest von Neumann algebra containing $\mathscr{A}_k$ for all k in K.)

Every classical K-flow defines canonically (via Prop. 1.2) a completely self-refining dynamical system. To see this, it suffices to use Proposition 1.2, and to identify $\mathscr{A}$ as the abelian von Neumann subalgebra of $\mathscr{L}^\infty(\Omega, \mu)$ generated by the characteristic functions χ_X of all elements X of the partition ξ; $[\mathscr{A}(t)\Phi]$ then is the closed subspace of $\mathscr{L}^2(\Omega, \mu)$ consisting of all those functions which are measurable with respect to the σ-algebra generated by the partition $\xi(t)$.

Consider now the particular case of Definition 2.1 obtained when $\mathscr{A}$ is, in addition, supposed to be abelian. Then $\{\mathscr{A}(t) \mid t \in \mathbb{R}\}$ is a totally ordered family of abelian von Neumann algebras and thus [2] the von Neumann algebra $\mathscr{N}$ generated by this family is also abelian. There exists therefore [2, Sect. I.7.3, Theor. 1] a locally compact space Ω, a positive measure μ on Ω with support Ω, and an isometric isomorphism π from $\mathscr{N}$ to $\mathscr{L}^\infty(\Omega, \mu)$; incidentally, since our definition presupposes that $\mathscr{H}$ is separable, one could impose without loss of generality that Ω is compact with countable basis. Since π is an isomorphism between von Neumann algebras, we know [2, Sect. I.4.3, Cor. 1 to Theor. 2] that π is bicontinuous for the ultraweak topology, and thus $\pi(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{L}^\infty(\Omega, \mu)$; the projectors of $\pi(\mathscr{A})$ therefore determine (mod 0) a σ-algebra $\sigma_{\mathscr{A}}$ of μ-measurable subsets of Ω, and thus [1, Appendix C or 12] a measurable partition $\xi_{\mathscr{A}}$. From thereon one is back in the realm of the classical theory [14].

Consequently a genuine generalization of the notion of classical K-flow, along the lines suggested by Definition 2.1, will require that both $\mathscr{A}$ and $\mathscr{N}$ be nonabelian von Neumann algebras. The point of the present section is that, although Definition 2.1 is more general than Definition 1.3, some of the most basic features of classical K-flows are carried over to the case of completely self-refining dynamical systems.

THEOREM 2.4. *Let* $(\mathscr{N}, \Phi, \alpha, \mathscr{A})$ *be a completely self-refining dynamical system*; U: $\mathbb{R} \to \mathscr{U}(\mathscr{H})$ *be the mapping defined by Lemma* 2.2; $\mathscr{H}_0 = \{\Psi \in \mathscr{H} \mid U(t)\Psi = \Psi\, \forall t \in \mathbb{R}\}$; $\mathscr{H}_\perp$ *be the orthogonal complement* *of $\mathscr{H}_0$. For each t in $\mathbb{R}$, let $U_\perp(t)$ denote the restriction of $U(t)$ to $\mathscr{H}_\perp$. Then:*

(i) *$\mathscr{H}_0$ is the one-dimensional subspace generated by Φ; and*

(ii) *$U_\perp(\mathbb{R})$ has homogeneous Lebesgue spectrum.*

Proof. Let $E(t)$ denote the projector $[\mathscr{A}(t)\Phi]$, and let $E^\perp(t)$ be its restriction to the orthogonal complement $\mathscr{H}^\perp$ of the one-dimensional subspace $\mathbb{C}\Phi$. Since $\alpha(s)\,\alpha(t) = \alpha(s+t)$ for all s, t in $\mathbb{R}$, and $\mathscr{A}(-t) \subseteq \mathscr{A}$ for all positive t, we have $s \leqslant t$ implies $\mathscr{A}(s) \subseteq \mathscr{A}(t)$ and thus $E(s) \subseteq E(t)$. Hence: (a) $s \leqslant t$ implies $E^\perp(s) \subseteq E^\perp(t)$. From condition (ii) in Definition 2.3, we conclude that: (b) $\bigcap_{t\in\mathbb{R}} E^\perp(t) = 0$. Let now F (respectively, $F(t)$) denote the orthocomplement of $\bigvee_{t\in\mathbb{R}} E(t)$ (respectively, $E(t)$). Clearly $\Psi \in F\mathscr{H}$ if and only if $\Psi \in F(t)\mathscr{H}$ for all $t \in \mathbb{R}$, i.e., $(\Psi, \alpha(t)[A]\Phi) = 0$ for all $t \in \mathbb{R}$ and all $A \in \mathscr{A}$. From condition (iii) in Definition 2.3, we know that for every $N \in \mathscr{N}$ there exists $\{t_n, A_n\} \subset \mathbb{R} \times \mathscr{A}$ such that $\alpha(t_n)[A_n]$ tends to N (as $n \to \infty$) in the weak-operator topology. For any $\Psi \in F\mathscr{H}$, we have thus: $(\Psi, N\Phi) = \lim_{n\to\infty} (\Psi, \alpha(t_n)[A_n]\Phi) = 0$ for every N in $\mathscr{N}$. Since on the other hand Φ is cyclic for $\mathscr{N}$, this implies that $\Psi = 0$, i.e., $F = 0$, and thus $\bigvee_{t\in\mathbb{R}} E(t) = I$. We have therefore (c) $\bigvee_{t\in\mathbb{R}} E^\perp(t)\mathscr{H} = \mathscr{H}^\perp$. Furthermore, from Lemma 2.2. (ii), we conclude that (d)

$$U^\perp(t)\, E^\perp(s)\, U^\perp(-t) = E^\perp(s+t)$$

for all $s, t \in \mathbb{R}$, where $U^\perp(t)$ denotes the restriction of $U(t)$ to $\mathscr{H}^\perp$. Since $U(t)$ is continuous in t for the strong operator topology, so is $U^\perp(t)$, and we have: (e) $w\text{-}\lim_{\epsilon\to 0} E^\perp(s+\epsilon) = E^\perp(s)$ for all $s \in \mathbb{R}$. From properties (a), (b), (c), and (e) above we conclude that $\{E^\perp(s) \mid s \in \mathbb{R}\}$ generates a projection-valued measure from $\mathbb{R}$ to $\mathscr{H}^\perp$. We next define $V^\perp\colon \mathbb{R} \to \mathscr{U}(\mathscr{H}^\perp)$ by

$$V^\perp(\tau) = \int_{\mathbb{R}} \exp(-is\tau)\, E^\perp(ds),$$

and notice that, as a result of (d):

$$U^\perp(t)\, V^\perp(\tau) = V^\perp(\tau)\, U^\perp(t) \exp(it\tau),$$

i.e., $\{U^\perp(t), V^\perp(\tau) \mid t, \tau \in \mathbb{R}\}$ forms a representation, on the separable Hilbert space $\mathscr{H}^\perp$, of the Weyl canonical commutation relations for one-degree of freedom (for a definition of these terms see for instance [3, Sects. III.1.a and III.1.b]). By von Neumann uniqueness theorem [11] (this theorem is stated (in English) and proven in [3, Theor. III.1.6]), we know that this representation is a direct sum of irreducible representations, and that each of these irreducible representations is unitary equivalent to the Schrödinger representation in $\mathscr{L}^2(\mathbb{R}, dx)$, where $[U_0(t)\Psi](x) = \Psi(x - t)$ and $[V_0(\tau)\Psi](x) = \exp(-ix\tau)\,\Psi(x)$. From this we conclude evidently that $U^{\perp}(\mathbb{R})$ has homogeneous Lebesgue spectrum in $\mathscr{H}^{\perp}$. From that fact follow in turn: $\mathscr{H}_0 = \mathbb{C}\Phi$ and $\mathscr{H}_{\perp} = \mathscr{H}^{\perp}$, which completes the proof.

Except for the use of von Neumann uniqueness theorem, the above proof is a step-by-step generalization of Sinai's [1] proof originally established for classical K-systems only. We however do not claim anything yet on the multiplicity of the spectrum of $U_{\perp}(\mathbb{R})$. Theorem 2.4 has nevertheless already some interesting consequences. We first notice that $\mathscr{H}_0$ one-dimensional implies (see for instance [3, Theor. II.2.8]) that ϕ is extremal $\alpha^*(\mathbb{R})$-invariant on $\mathscr{N}$, i.e., cannot be decomposed into a proper mixture of $\alpha^*(\mathbb{R})$-invariant states on $\mathscr{N}$. In this sense we can thus claim the following.

COROLLARY 2.5. *Every completely self-refining dynamical system is ergodic.*

We can actually claim more on the basis of the results so far established. In particular, as a consequence of Riemann–Lebesgue's lemma, Theorem 2.4 implies that for every N, M in $\mathscr{N}$:

$$\lim_{|t|\to\infty} \langle\phi; N\alpha_t[M]\rangle = \langle\phi; N\rangle\langle\phi; M\rangle,$$

a result which we can state formally as the following .

COROLLARY 2.6. *Every completely self-refining dynamical system is strongly mixing.*

We shall use in the sequel another consequence (see for instance [3, Theor. II.2.8]) of the fact that $\mathscr{H}_0$ is one-dimensional.

COROLLARY 2.7. *Let $(\mathscr{N}, \Phi, \alpha, \mathscr{A})$ be a completely self-refining dynamical system, and $U(\mathbb{R})$ be as in Lemma 2.2. Then:*

$$\mathscr{N}' \cap U(\mathbb{R})' = \mathbb{C}I.$$

3. NONABELIAN SPECIAL K-FLOWS

Throughout this section $\mathscr{N}$, Φ, α and $\mathscr{A}$ will consistently refer to the building blocks of a completely self-refining dynamical system $(\mathscr{N}, \Phi, \alpha, \mathscr{A})$ as introduced in Section 2. The present section is devoted to a sharpening of this concept by means of additional assumptions. Unless explicitly noted otherwise, each assumption, once stated, will be kept throughout.

We first notice that in the classical case Φ, being cyclic for the abelian von Neumann algebra $\mathcal{N}$, is a fortiori cyclic for $\mathcal{N}' \supseteq \mathcal{N}$ and thus separating $\mathcal{N}$. If however $\mathcal{N}$ is allowed to become nonabelian, this becomes a separate assumption which we now want to make.

Assumption 3.1. Φ is separating for $\mathcal{N}$.

Remark 3.2. (i) Clearly Φ separating for $\mathcal{N}$ and $\bigcap_{t\in\mathbb{R}} [\mathscr{A}(t)\Phi] = \mathbb{C}\Phi$ imply that $\bigcap_{t\in\mathbb{R}} \mathscr{A}(t) = \mathbb{C}I$. The symmetry in Definition 2.4 would be increased if we could replace condition (ii) by the latter condition. We however do not know when this substitution is indeed possible without altering the main content of the theory. (ii) As we shall see Assumption 3.1 is central to the theory to be developed in this section. It is therefore interesting to know that it can be formulated in several equivalent ways (see Lemma 3.3 below), and that it is often satisfied (see for instance [3]) in physical applications.

LEMMA 3.3. *The following three conditions are equivalent*: (i) Φ *is separating for* $\mathcal{N}$; (ii) *for every invariant mean* η *on* $\mathbb{R}$, *and every* N_1, N_2, N_3, N_4 in $\mathcal{N}$: $\eta\,\langle\phi; N_1(\alpha(t)[N_2]\, N_3 - N_3\alpha(t)[N_2])\, N_4\rangle = 0$; (iii) $\mathcal{N} \cap U(\mathbb{R})' = \mathbb{C}I$.

Proof. From Corollary 2.5 we know that ϕ is extremal $\alpha^*(\mathbb{R})$-invariant. Together with the cyclicity of Φ with respect to $\mathcal{N}'$, which is equivalent to (i) above, this implies (see for instance [3, Corollary on p. 187]) that $\alpha(\mathbb{R})$ acts in an η-abelian manner on $\mathcal{N}$ with respect to every mean η on $\mathbb{R}$, i.e., that (ii) above is satisfied. Hence (i) implies (ii). Condition (ii) in turn implies (see for instance [3, Cor. 2, p. 181]) that $\mathcal{N} \cap U(\mathbb{R})' = \mathcal{N}' \cap U(\mathbb{R})'$; from Corollary 2.7 above we know that the latter is $\mathbb{C}I$. Hence (ii) implies (iii). Finally, let E be the support of ϕ in $\mathcal{N}$; since $\alpha(\mathbb{R})$ is an automorphism group of $\mathcal{N}$ and ϕ is $\alpha^*(\mathbb{R})$-invariant, E belongs to $\mathcal{N} \cap U(\mathbb{R})'$. If now condition (iii) is satisfied, the state ϕ as I for support ,i.e., ϕ is faithful. Thus $N \in \mathcal{N}$, $N\Phi = 0$ implies $\langle\phi; N^*N\rangle = 0$, and then $N^*N = 0$ and $N = 0$. Hence (iii) implies (i), thus completing the proof of the lemma.

LEMMA 3.4. *For every* $\beta \in \mathbb{R}$ *with* $0 < \beta < \infty$, *there exists a unique mapping* $\alpha^\beta\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{N})$ *such that*: (i) $\alpha^\beta(s)\,\alpha^\beta(t) = \alpha^\beta(s + t)$ *for all* s, t *in* $\mathbb{R}$; *and* (ii) *for every* N, M *in* $\mathcal{N}$ *there exists a function* $\phi^\beta_{N,M}(z)$ *holomorphic in the strip* $0 \leqslant \operatorname{Im} z \leqslant \beta$, *such that for all* t *in* $\mathbb{R}$, $\phi^\beta_{NM}(t) = \langle\phi; \alpha^\beta(t)[N]M\rangle$ *and* $\phi^\beta_{NM}(t + i\beta) = \langle\phi; M\alpha^\beta(t)N\rangle$.

Proof. Φ separating for $\mathcal{N}$ clearly implies that ϕ is a faithful normal state on the von Neumann algebra $\mathcal{N}$. The latter condition was shown by Takesaki [16] to be sufficient for the conclusion of the lemma in case $\beta = 1$. The general case is immediately obtained from this by changing the time-scale.

We now start moving away from the abelian case.

Assumption 3.5. (i) For all t in $\mathbb{R}$ and all Z in $\mathcal{N} \cap \mathcal{N}'$, $\alpha(t)[Z] = Z$; (ii) ϕ is not a trace on $\mathcal{N}$.

LEMMA 3.6. *Assumption* 3.5 *is equivalent to assuming that $\mathcal{N}$ is a type III-factor.*

Proof. Assumption 3.5 (i) is trivially satisfied if $\mathcal{N}$ is a factor. On the other hand, by Lemma 2.2, Assumption 3.5 (i) implies that $\mathcal{N} \cap \mathcal{N}' \subseteq U(\mathbb{R})'$ and thus $\mathcal{N} \cap \mathcal{N}' \subseteq \mathcal{N}' \cap U(\mathbb{R})'$ which, by Corollary 2.7, is $\mathbb{C}I$. Hence $\mathcal{N}$ is a factor. Consequently Assumption 3.5 (i) is equivalent to the condition than $\mathcal{N}$ be a factor. The latter condition is well known (see for instance [3, Cor. 1, p. 204]) to imply that (and actually to be equivalent to) ϕ extremal amongst the states which satisfy the conclusion (ii) of Lemma 3.4. We have therefore as a consequence (see for instance [3, Theor. II.2.14]) of Assumption 3.5 (i) and of Lemma 3.3, that: either $\mathcal{N}$ is a type III-factor, or: (a) ϕ is a trace on $\mathcal{N}$, (b) $\mathcal{N}$ is a type II_1- or I_n-factor, and (c) $\alpha^\beta(t)[N] = N$ for all t in $\mathbb{R}$ and all N in $\mathcal{N}$. This concludes the proof of the lemma.

The above proof also provides the following result which we shall use in the sequel.

COROLLARY 3.7. *The group $\alpha^\beta(\mathbb{R})$ of automorphisms of $\mathcal{N}$, defined in Lemma* 3.4, *is not trivial, i.e., there exists some t in $\mathbb{R}$ and some N in $\mathcal{N}$ for which $\alpha^\beta(t)[N] \neq N$.*

LEMMA 3.8. (i) *There exists a mapping U^β: $\mathbb{R} \to \mathcal{U}(\mathcal{H})$, continuous for the strong-operator topology and such that for all s, t in $\mathbb{R}$*: (a) $U^\beta(s)\, U^\beta(t) = U^\beta(s + t)$, (b) $U^\beta(t)\Phi = \Phi$, *and* (c) $\alpha^\beta(t)[N] = U^\beta(t)\, N U^\beta(-t)$ *for all N in $\mathcal{N}$*; (ii) $\alpha^\beta(\mathbb{R})$ *and* $\alpha(\mathbb{R})$ *commute*; (iii) $U^\beta(\mathbb{R})'' \subseteq U(\mathbb{R})'$; (iv)

$$\mathrm{Sp}_d\, U^\beta(\mathbb{R}) = \{\lambda \in \mathbb{R} \mid \exists \Psi_\lambda \in \mathcal{H} \exists \mid U^\beta(t)\, \Psi_\lambda = \exp(-i\lambda t)\, \Psi_\lambda\ \forall t \in \mathbb{R}\}$$

is a subgroup of the additive group $\mathbb{R}$.

Proof. The conclusion of Lemma 3.4 is well known to imply (see for instance [19], or [3, Lemma on p. 196]) that ϕ is $\alpha^\beta(\mathbb{R})^*$-invariant.

The first part of the present Lemma is thus established as was Lemma 2.2. From [4, Theor. 1] we know that (ii) above is equivalent to assumption 3.5 (i). The third part of the lemma follows then from (i) and (ii) above, from Lemma 2.2, and from the cyclicity of Φ with respect to $\mathscr{N}$. Finally, the fourth part of the lemma follows from (iii) above, Theorem 2.4 (i), Assumption 3.1, Lemma 3.3 (iii), and [5, Theor. 3.2].

Assumption 3.9. $\mathscr{A}$ is stable under $\alpha^{\beta}(\mathbb{R})$.

As a consequence of a result obtained by Takesaki [18, main theorem], this assumption is strictly equivalent to the condition that there exists a σ-weakly continuous, faithful projection $\mathscr{E}_0(\cdot; \phi)$: $\mathscr{N} \to \mathscr{A}$, of norm 1, such that $\langle \phi; A^*NB \rangle = \langle \phi; A^*\mathscr{E}_0(N; \phi)B \rangle$ for all A, B in $\mathscr{A}$ and all N in $\mathscr{N}$. In particular, we have then [17]: $\langle \phi; N \rangle = \langle \phi; \mathscr{E}_0(N; \phi) \rangle$; $\mathscr{E}_0(N^*N; \phi) \geqslant \mathscr{E}_0(N; \phi)^* \, \mathscr{E}_0(N; \phi) \geqslant 0$; and

$$\mathscr{E}_0(ANB; \phi) = A\mathscr{E}_0(N; \phi)B$$

for all A, B in $\mathscr{A}$, and all N in $\mathscr{N}$. Consequently, $\mathscr{E}_0(\cdot; \phi)$ can properly be called the conditional expectation from $\mathscr{N}$ onto $\mathscr{A}$ with respect to ϕ. Moreover, since $\alpha(\mathbb{R})$ and $\alpha^{\beta}(\mathbb{R})$ commute (Lemma 3.8), each $\mathscr{A}(t)$ is stable under $\alpha^{\beta}(\mathbb{R})$. Assumption 3.9 is thus equivalent to the condition that, for each t in $\mathbb{R}$, the conditional expectation $\mathscr{E}_t(\cdot; \phi)$ from $\mathscr{N}$ to $\mathscr{A}(t)$ with respect to ϕ exists. Furthermore, the above reasoning shows similarly that Assumption 3.9 ensures the existence, for each s, t in $\mathbb{R}$ with $s \leqslant t$, of the conditional expectation $\mathscr{E}_{s,t}(\cdot; \phi)$ from $\mathscr{A}(t)$ onto $\mathscr{A}(s)$ with respect to ϕ. These remarks thus emphasize the fact that Assumption 3.9 is not merely a technical assumption, but rather is a fundamental prerequisite to any subsequent stochastic analysis of the dynamical systems studied in this paper.

DEFINITION 3.10. A nonabelian special K-flow is a completely self-refining dynamical system satisfying Assumptions 3.1, 3.5, and 3.9.

THEOREM 3.11. *Let* $(\mathscr{N}, \Phi, \alpha, \mathscr{A})$ *be a nonabelian special K-flow; let* $U(\mathbb{R})$ [*respectively* $U^{\beta}(\mathbb{R})$] *be defined as in Lemma* 2.2 [*respectively, Lemma* 3.8]; *let* $\mathscr{H}_0 \equiv \{\Psi \in \mathscr{H} \mid U(t)\Psi = \Psi \forall t \in \mathbb{R}\}$, *and* $\mathscr{H}_{\perp}$ *be its orthocomplement. Then*:

(i) $\mathscr{H}_0$ *is the one-dimensional subspace* $\mathbb{C}\Phi$;

(ii) *the restriction* $U_{\perp}(\mathbb{R})$ *of* $U(\mathbb{R})$ *to* $\mathscr{H}_{\perp}$ *has homogeneous Lebesgue spectrum with countably infinite multiplicity*;

(iii) *the spectrum of* $U^{\beta}(\mathbb{R})$ *is discrete and isomorphic to* $\mathbb{Z}$;

(iv) $\mathcal{N}$ *is a type III-factor.*

Proof. Parts (i) and (iv) of the theorem have already been established (see Theor. 2.4i and Lemma 3.6). Let now $E_{\perp}(s)$ be defined for every s in $\mathbb{R}$ as in the proof of Theorem 2.4, recalling that $\mathscr{H}_{\perp} = \mathscr{H}^{\perp}$; and let $U_{\perp}{}^{\beta}(\mathbb{R})$ be defined as the restriction of $U^{\beta}(\mathbb{R})$ to $\mathscr{H}_{\perp}$. From Assumption 3.9 and Lemma 3.8 ii, we conclude that $E_{\perp}(s)$ is an invariant subspace of $U_{\perp}{}^{\beta}(\mathbb{R})$, i.e., $E_{\perp}(s) = U_{\perp}{}^{\beta}(t)\, E_{\perp}(s)\, U_{\perp}{}^{\beta}(-t)$ for every s, t in $\mathbb{R}$. Together with Lemma 3.8 iii, this implies that $U_{\perp}{}^{\beta}(\mathbb{R}) \subseteq \{U_{\perp}(\mathbb{R}), V_{\perp}(\mathbb{R})\}'$. From Theorem 2.4 we conclude therefore that the spectrum of $U_{\perp}{}^{\beta}(\mathbb{R})$ and thus that of $U^{\beta}(\mathbb{R})$, is discrete. Lemma 3.8 iv implies consequently that the spectrum of $U^{\beta}(\mathbb{R})$ is either: (a) $\mathbb{R}$, (b) $\{0\}$, or (c) isomorphic to $\mathbb{Z}$. However (a) is excluded, since $\mathscr{H}$ is assumed to be separable. Moreover (b) would imply that $U^{\beta}(t) = I$ for all t in $\mathbb{R}$; this in turn would imply that $\alpha^{\beta}(\mathbb{R})$ is trivial, a situation which is excluded by Corollary 3.7. Hence part (iii) of the theorem is established. Finally, we already know from Theorem 2.4 that $U_{\perp}(\mathbb{R})$ has homogeneous Lebesgue spectrum. Since $\mathscr{H}$ is separable, the multiplicity of this spectrum is either finite or countably infinite. It can however not be finite without contradicting (iii) which we just established. This concludes the proof of the theorem.

Although the main point of this theorem concerns the spectrum properties of $U(\mathbb{R})$, it has an incidental consequence which might be worth pointing out here. Let G_{ϕ} be the group of automorphism of $\mathcal{N}$ leaving ϕ invariant; clearly $\alpha(\mathbb{R})$ is contained in G_{ϕ}; from Lemma 3.3 iii, we therefore conclude that the fixed points of $\mathcal{N}$ under G_{ϕ} are the scalar multiple of the identity. On the other hand, Theorem 3.11 iii implies that there exists $T > 0$ such that $\alpha^{\beta}(T)$ is the identity automorphism of $\mathcal{N}$. These two conclusions can be brought together in the following assertion.

COROLLARY 3.12. *In the sense of Takesaki* [17] ϕ *is a periodic homogeneous state on* $\mathcal{N}$.

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