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Countable, 1-transitive, coloured linear orderings I

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Abstract

We give a classification of all the countable, 1-transitive, coloured linear orderings for countable colour sets. This is a generalization of Morel's classification of the countable, 1-transitive linear orderings. For finite colour sets, there are $\aleph_1$ examples and for countably infinite colour sets, there are $2^{\aleph_0}$ (discussed in more detail in a subsequent paper (countable, 1-transitive, coloured linear orderings II, submitted)). We also include a classification of the countable homogeneous coloured linear orders.

Keywords: Colour; Linear order; Classification

1. Introduction

In this paper, we tackle a classification problem for coloured linear orderings, that is, linear orderings $(X, <)$ in which each point is assigned a unique member of a set C, thought of as 'colours'. If F is this 'colouring function', we write the coloured linear ordering as $(X, <, F)$. We restrict here to the countable case (so that also C may be taken to have cardinality at most $\aleph_0$), and, so that we have some chance of classification, impose a suitable homogeneity assumption. Now, if we require full homogeneity (sometimes called 'ultrahomogeneity' [4]), meaning that any isomorphism between finite substructures extends to an automorphism, then the classification consists of just the so-called 'C-coloured rationals' $\mathbb{Q}_C$ or structures built up from these and isolated points by concatenation, as we shall see in Section 4. (Gregory

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Cherlin has remarked that, for finite C, a homogeneous C-coloured linear order is also a $|C|$-tournament, in the sense defined in [2], so the results of Section 4 are also covered by his work.) Our main task however is to adapt Morel's classification [3] of the countable 1-transitive linear orders to the coloured case, where now '1-transitive' means that the automorphism group acts singly transitively on the set of points coloured by any fixed colour. In this paper, we concentrate on the case in which C is finite. In a sequel [1] we study the case of infinitely many colours.

The general definition (which applies in fact to any structure) is that a coloured linear ordering is *n-transitive* if for any two isomorphic n-element substructures, there is an automorphism taking the first to the second. It is *homogeneous* if any isomorphism between finite substructures can be extended to an automorphism. As for linear orders, an argument involving 'patching' readily shows that any 2-transitive coloured linear order is homogeneous, hence n-transitive for every n, so the only cases of interest are 1-transitivity, and homogeneity (= 2-transitivity).

Now in [6], a class of partial orders was studied, suitably related to linear orders, but with some branching allowed, called 'cycle-free' partial orders. Though some general remarks were made, the thrust of that work was the classification of countable cycle-free partial orders of a particular kind under a suitable transitivity assumption (called 'k-CS-transitivity'). It turned out that the structure of these partial orders could be 'encoded' by information about the order-type of a maximal chain in the completion, together with specifying how the elements ramified; moreover, these chains obeyed precisely the kind of 1-transitivity for linear orders we have just described, with however up to at most four orbits in all. That is, they were not actually 1-transitive, but 1-transitive provided the orbits were coloured by distinct colours. The classification of these cycle-free partial orders therefore required classification of certain (rather special) 1-transitive coloured linear orders. Arising out of this, it was natural to consider whether it would be possible to classify *all* countable 1-transitive coloured linear orders. As we shall see, this is certainly possible in the case where there are just finitely many colours; for infinitely many colours we can also give a 'classification', though it is considerably less explicit.

We may also wish to consider certain order-preserving permutations of a coloured linear order which may not preserve the colours, but permute them coherently. We say that a permutation f of a coloured set (X, F) is a *colour shuffle* if for some permutation θ of the set of colours C, $F(f(z)) = \theta F(z)$, for every $z \in X$, and for a coloured linear ordering $(X, <, F)$, the group of order-preserving permutations which are colour shuffles is called the *shuffle group*, $\mathrm{Aut}_{\mathrm{sh}}(X, <, F)$.

In the analysis of possible coloured linear orders, a key point will be to examine intervals on which only certain colours appear. In this paper, we do not need to distinguish between 'intervals' and 'convex sets'; by either we understand a subset I of X such that $x < y < z$ and $x, z \in I$ implies $y \in I$. These then have the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$, where a and b lie in the Dedekind-completion of X (or are $\pm\infty$). An interval I of a coloured linear ordering $(X, <, F)$ is *monochromatic* if there is $c \in C$ such that for all $z \in I$, $F(z) = c$.

Given any partition of the set of colours C, π, we can define an equivalence relation over X by $a \sim_\pi b$ if $a \leqslant b$ and all colours appearing in $[a, b]$ lie in the same member of π, or $b \leqslant a$ and all colours appearing in $[b, a]$ lie in the same member of π. Clearly, all $\sim_\pi$-classes are convex, and hence $<$ induces a linear ordering on the family $X/\sim_\pi$ of equivalence classes.

Lemma 1.1. *If $(X, <, F)$ is a 1-transitive, coloured linear ordering, and π is a partition of C, then all $\sim_\pi$-classes are 1-transitive, and any two $\sim_\pi$-classes that share a common colour are isomorphic.*

Proof. Let $\bar{x} = \{z \in X : z \sim_\pi x\}$, and suppose that $a \in \bar{x}$ and $b \in \bar{y}$ have the same colour. Then, since $(X, <, F)$ is 1-transitive, there is an automorphism f of X such that $f(a) = b$. We show that the restriction $f|_{\bar{x}}$ of f to $\bar{x}$ is an isomorphism from $\bar{x}$ to $\bar{y}$. Given $z \in \bar{x}$, suppose $z < a$, and let C_0 be the member of π such that $F(a) = F(b) \in C_0$. If $f(z) \notin \bar{y}$, then there is $t \in [f(z), b]$ such that $F(t) \notin C_0$. As $f(z) < t < f(a)$ and f is an automorphism, there is $w \in [z, a]$ such that $F(w) \notin C_0$, contrary to $z \sim_\pi a$. We deduce that $f(z) \in \bar{y}$. Similarly, f maps any $z > a$ in $\bar{x}$ into $\bar{y}$, so $f(\bar{x}) \subseteq \bar{y}$. Applying a similar argument to f^{-1}, we find that $f^{-1}(\bar{y}) \subseteq \bar{x}$, giving $f|_{\bar{x}}$ an isomorphism from $\bar{x}$ to $\bar{y}$. Hence, if two $\sim_\pi$-classes share a common colour, they are isomorphic.

Now in the case that $\bar{x} = \bar{y}$, it follows that $f|_{\bar{x}}$ is actually an automorphism of $\bar{x}$ taking a to b, so the argument also shows that $\bar{x}$ is a 1-transitive coloured linear order. $\square$

Note that if π is the trivial partition of C, then all $\sim_\pi$-classes are monochromatic (and convex). Therefore F induces a natural colouring function, also denoted by F, on $X/\sim_\pi$, so that $(X/\sim_\pi, <, F)$ itself becomes a coloured linear ordering. Also, $(X/\sim_\pi, <, F)$ has the property that between any two points with the same colour there is another point with a different colour, so it has no non-trivial monochromatic intervals.

A consequence of the previous lemma is that if two $\sim_\pi$-classes share a common colour, then all the colours in one appear in the other. Hence, for any partition π of C and for any $x, y \in X$, $F_\pi(x) = \{F(z) : x \sim_\pi z\}$ and $F_\pi(y) = \{F(z) : y \sim_\pi z\}$ are either identical or disjoint. Also, for all $x \in X$, $F_\pi(x)$ is contained in a member of π. This leads us to define a refinement π' of π by $\pi' = \{F_\pi(x) : x \in X\}$. Now, we intend to view the elements of π' themselves as colours. The role of π' is to colour points of $X/\sim_\pi$ differently when their colour-sets are disjoint, but, even if $\pi \neq \pi'$, it is easy to prove that $\sim_\pi$-classes and $\sim_{\pi'}$-classes are equal, as we now see.

Lemma 1.2. *For any partition π of C, π' refines π, and $\sim_\pi = \sim_{\pi'}$.*

Proof. To see that π' refines π, let c_1 and c_2 lie in the same member $F_\pi(x)$ of π'. Then $c_1 = F(y)$, $c_2 = F(z)$ with $x \sim_\pi y$, $x \sim_\pi z$. Thus, $y \sim_\pi z$, so $c_1 = F(y)$ and $c_2 = F(z)$ lie in the same member of π.

Now take any $y \leqslant z$ in X. If $y \sim_{\pi'} z$ then all members of $[y, z]$ are coloured by the same member of π', hence also by the same member of π, so $y \sim_\pi z$. Conversely, if $y \leqslant z$ and $y \sim_\pi z$, then for all $t \in [y, z]$, $y \sim_\pi t$, so $F(t) \in F_\pi(y)$. Hence all members of $[y, z]$ have colours lying in the same member of π', and so $y \sim_{\pi'} z$. $\square$

We write F' for the colouring function going from $X/\sim_\pi$ to π'.

Lemma 1.3. *Let $(X,<,F)$ be a coloured linear order. Let π be any partition of C and let π' be the refinement of π as described above. Then $(X,<,F)$ is 1-transitive if and only if $(X/\sim_\pi,<,F')$ is 1-transitive and all $\sim_\pi$-classes that share a common colour are isomorphic and 1-transitive.*

Proof. First suppose that $(X,<,F)$ is 1-transitive. By Lemma 1.1, all $\sim_\pi$-classes that share a common colour are isomorphic and 1-transitive, so it remains to show that $(X/\sim_\pi,<,F')$ is 1-transitive. Let $\bar{x},\bar{y}\in X/\sim_\pi$ be such that $F'(\bar{x})=F'(\bar{y})$, and let $x_0\in\bar{x}$ and $y_0\in\bar{y}$ be such that $F(x_0)=F(y_0)$. (It is at this point, to guarantee the existence of x_0 and y_0, that the fact that the F'-colours lie in π' is used.) Since $(X,<,F)$ is 1-transitive, $f(x_0)=y_0$ for some automorphism f of X. Define $g: X/\sim_\pi\to X/\sim_\pi$ by $g(\bar{z})=\overline{f(z)}$. Since $g(\bar{x})=g(\overline{x_0})=\overline{f(x_0)}=\overline{y_0}=\bar{y}$, G takes $\bar{x}$ to $\bar{y}$, so we just have to show that g is well defined and is an automorphism. Let $z_0,z_1\in X$ be such that $z_0\sim_\pi z_1$. Then every point between z_0 and z_1 has colour in the same member of π'. Since f preserves colour, the same applies to $f(z_0)$ and $f(z_1)$, so $f(z_0)\sim_\pi f(z_1)$. Applying the same argument to f^{-1} demonstrates that g is 1–1, and g is order-preserving since f is. To see that g preserves colour, we have

$$\begin{aligned} F'(g(\bar{z})) &= F'(\overline{f(z)}) \\ &= \{F(t): t\sim_\pi f(z)\} \\ &= \{F(f(t)): f(t)\sim_\pi f(z)\} \\ &= \{F(t): t\sim_\pi z\} \text{ since } f \text{ preserves } F, \text{ and } \sim_\pi, \\ &= F'(\bar{z}). \end{aligned}$$

For the converse, assume the given condition, and let $x,y\in X$ be such that $F(x)=F(y)$. Since $\bar{x}$ and $\bar{y}$ share a common colour, $F'(\bar{x})=F'(\bar{y})$. So there is an automorphism f of $X/\sim_\pi$ such that $f(\bar{x})=\bar{y}$ and there is an isomorphism g from $\bar{x}$ onto $\bar{y}$. Also, as $\bar{x}$ is 1-transitive and $F(x)=F(y)$, there is an automorphism h of $\bar{x}$ such that $h(x)=g^{-1}(y)\in\bar{x}$. If $g_{\bar{x}}=g\circ h$, then $g_{\bar{x}}$ is an isomorphism from $\bar{x}$ onto $\bar{y}$ such that $g_{\bar{x}}(x)=y$. Now, for every $\bar{z}\in X/\sim_\pi$, $\bar{z}\neq\bar{x}$, we choose an isomorphism $g_{\bar{z}}$ from $\bar{z}$ onto $f(\bar{z})$, and let $G(z)=g_{\bar{z}}(z)$. Since there is one isomorphism for each $\bar{z}$, G is well defined and is a bijection. Since $g_{\bar{z}}$ is an isomorphism, G preserves colours on each $\bar{z}$, and hence overall. Moreover, it also preserves the order on each $\bar{z}$. Since f preserves the ordering of the pieces, G preserves the ordering on X. $\square$

As an important special case of the above, we may take for π the trivial partition of C (into singletons). The lemma then reduces the problem of classifying a general coloured linear order to the monochromatic ones (the $\sim_\pi$-classes), known by Morel's result, and those for which all $\sim_\pi$-classes are singletons (that is, those having no non-trivial monochromatic intervals), since the general case is obtained by replacing each point by a monochromatic countable 1-transitive linear order, so that points having the same colour are replaced by the same (isomorphic) linear order, but points having different colours may be replaced by linear orders that need not be isomorphic. The main reduction, however, comes about by taking a suitable non-trivial partition π, in which case (in the finite case at any rate), the lemma effects a reduction in the total number of colours, and an inductive analysis is possible. The following lemma, which applies in both the finite and infinite cases, tells us how we can recognize when $\mathbb{Q}_C$ arises, and is a key point in the classification.

The *C-coloured version of* $\mathbb{Q}$, where C is a given (finite or countably infinite) set of colours, may be characterized as the set $\mathbb{Q}$ of rational numbers, together with a colouring $F : \mathbb{Q} \to C$ such that for every $x < y \in \mathbb{Q}$ and $c \in C$ there is $z \in \mathbb{Q}$ with $x < z < y$ and $F(z) = c$. This exists and is unique up to isomorphism (see [5, Lemma 4.1] for example) and we shall denote it by $\mathbb{Q}_C$, or by $\mathbb{Q}_n$ if $|C| = n$ (for $n = 1, 2, \ldots, \aleph_0$).

If x, y lie in the linearly ordered set $(X, <)$, and are such that $x < y$ and there is no $z \in X$ with $x < z < y$, then we say x and y are *consecutive*, and we write $x \lessdot y$.

Lemma 1.4. *Let* $(X, <, F)$ *be a countable*, 1*-transitive*, *coloured linear ordering*, *where* $|C| \geqslant 3$. *Suppose that for any partition* π (*with* $\pi \neq \{C\}$), *all* $\sim_\pi$*-classes are singletons. Then* $(X, <, F) \cong \mathbb{Q}_C$.

Proof. First, let us see that $(X, <, F)$ is dense. If not, then there are consecutive $x \lessdot y$ in X. Taking $\pi = \{\{c\} : c \in C\}$, all $\sim_\pi$-classes are singletons, so $F(x) \neq F(y)$. Since $|C| \geqslant 3$, we may alternatively let $\pi = \{\{F(x), F(y)\}\} \cup \{\{c\} : c \in C - \{F(x), F(y)\}\}$. Then $x \sim_\pi y$, contrary to all $\sim_\pi$-classes singletons. Clearly, X cannot have endpoints.

Now, given $x < y \in X$ and $c \in C$, choose u, v with $x < u < v < y$. Let $\pi = \{\{c\}, C - \{c\}\}$. By assumption, $u \nsim_\pi v$, so there is $w \in X$ such that $u \leqslant w \leqslant v$ and $F(w) = c$ (which is u or v, or if both $F(u)$ and $F(v)$ do not equal c, is given directly by the definition of $u \nsim_\pi v$). Therefore, $(X, <, F) \cong \mathbb{Q}_C$. $\square$

By this lemma, if $(X, <, F) \ncong \mathbb{Q}_C$, and $|C| \geqslant 3$ is finite, we can use Lemma 1.3 to reduce the number of colours. This will enable us, in the finite case, to classify all $(X, <, F)$ in terms of the possibilities for $(X/\sim_\pi, <, F)$.

2. Finite sets of colours

Morel [3] proved that, up to isomorphism, the countable 1-transitive linear orders are $\mathbb{Z}^\alpha$ or $\mathbb{Q} \cdot \mathbb{Z}^\alpha$ for a countable ordinal α, where $\mathbb{Z}^\alpha$ is the ordinal power. This will be employed in the sequel, as the 'basis case' (the monochromatic one). We remark that it is immediate that there must be (at least) $\aleph_1$ pairwise non-isomorphic countable, 2-coloured, 1-transitive linear orders. And this is true even if we require there to be at least two colours, since we can take a single point coloured by one colour, followed by an arbitrary monochromatic linear order from Morel's list coloured by another. We shall show that there are actually only $\aleph_1$ of them in all, and we are able to give an explicit description.

If $(Y_0, <_0, F_0), \ldots, (Y_{n-1}, <_{n-1}, F_{n-1})$, are coloured linear orderings, then we write $\mathbb{Q}_n(Y_0, \ldots, Y_{n-1})$ for the coloured linear order resulting from $\mathbb{Q}_n$ by replacing all points coloured c_i by Y_i, for $i = 0, \ldots, n-1$. By Lemma 1.3, this is 1-transitive if and only if each Y_i is 1-transitive.

The *concatenation* of $(Y_0, <_0, F_0), \ldots, (Y_{n-1}, <_{n-1}, F_{n-1})$ is the coloured linear ordering $(Y_0 \cup \cdots \cup Y_{n-1}, <, F_0 \cup \cdots \cup F_{n-1})$, where $x<y$ if $x, y \in Y_i$ and $x <_i y$ for some $i = 0, \ldots, n-1$, or $x \in Y_i$ and $y \in Y_j$ where $i<j$. We write this as $Y_0^\frown \cdots {}^\frown Y_{n-1}$.

Given linear orders Y and Z, we write $Y(Z)$ or $Y \cdot Z$ for the linear order resulting from replacing every point of Y by a copy of Z (this is the 'lexicographic product' of Y and Z). Formally, this is the cartesian product $Y \times Z$ ordered by $(y_1, z_1) < (y_2, z_2)$ if $y_1 < y_2$, or $y_1 = y_2$ and $z_1 < z_2$. In the coloured case, any colours on Y will be irrelevant, and the colours on the copies of Z will be just the same as they were in Z.

Using the results of Section 1, we can now move towards an inductive classification of all countable 1-transitive n-coloured linear orders.

Theorem 2.1. *Let $(X, <, F)$ be a countable, 1-transitive, 2-coloured linear ordering with $C = \{c_0, c_1\}$. Then $(X, <, F)$ is isomorphic to one of the following:*

(i) $\mathbb{Q}_2(Y_0, Y_1)$ *where Y_0 and Y_1 are countable, 1-transitive linear orders coloured monochromatically by c_0 and c_1, respectively;*
(ii) $Y(Y_0^\frown Y_1)$ *or* $Y(Y_1^\frown Y_0)$ *where Y is a countable, 1-transitive linear order and Y_0 and Y_1 are countable, 1-transitive linear orders coloured monochromatically by c_0 and c_1, respectively*
(*and all possibilities described in* (i) *and* (ii) *are countable* 1-*transitive* 2-*coloured linear orders*).

Proof. Let $(X, <, F)$ be a countable, 1-transitive, 2-coloured linear ordering. First, let us suppose that for the trivial partition π of C, all $\sim_\pi$-classes of $(X, <, F)$ are singletons. Then between any two points coloured the same there is a point with a different colour.

If there are consecutive $x<y$ in X, then $F(x) \neq F(y)$, by the remark just made, and as $(X, <, F)$ is 1-transitive, every $z \in X$ with $F(z) = F(x)$ must be immediately followed by some v with $F(v) = F(y)$, and every $t \in X$ with $F(t) = F(y)$ must be immediately preceded by some u with $F(u) = F(x)$. Hence, $(X, <, F)$ consists of pairs of points coloured $F(x)$ and $F(y)$, from which it follows that $(X, <, F) \cong Y(1_0^\frown 1_1)$ or $Y(1_1^\frown 1_0)$ where Y is some countable, 1-transitive linear order and 1_0 and 1_1 are singletons coloured c_0 and c_1 respectively.

Otherwise, $<$ is dense. Take any $x<y$, and $c \in C$. By density there are u, v, w with $x<u<v<w<y$. As there are only two colours, two of $F(u)$, $F(v)$, $F(w)$ are equal, $F(u) = F(v)$ for example. As all $\sim_\pi$-classes are singletons, there is a point between u and v having the other colour. Hence, both colours appear in (x, y). It follows that $(X, <, F) \cong \mathbb{Q}_2$.

Now, returning to general $(X, <, F)$, by Lemma 1.3, $(X/\sim_\pi, <, F')$ has one of the forms $\mathbb{Q}_2$, $Y(1_0^\frown 1_1)$, or $Y(1_1^\frown 1_0)$, and the result follows from the same lemma on replacing the singletons by arbitrary countable 1-transitive linear orders of the same colour. □

We remark that the two cases of Theorem 2.1(ii) are 'essentially the same', that is, up to labelling of colours, and sometimes they actually *are* isomorphic, for instance if $Y_0 \cong Y_1$ and $Y = \mathbb{Z}^\alpha$ for some $\alpha \geqslant 1$. In part (i) however, $\mathbb{Q}_2(Y_0, Y_1)$ and $\mathbb{Q}_2(Y_1, Y_0)$ are *always* isomorphic, even without relabelling the colours.

Now, using induction and Lemma 1.3, we can prove the following theorem.

Theorem 2.2. *Let* $(X, <, F)$ *be a countable*, 1-*transitive*, *coloured linear order where* $2 \leqslant |C| < \aleph_0$. *Then* $(X, <, F) \cong \mathbb{Q}_i(Y_0, \dots, Y_{i-1})$ *or* $(X, <, F) \cong Y(Y_0^\wedge \cdots {}^\wedge Y_{i-1})$, *where* $Y_0, \dots, Y_{i-1}$ *are themselves countable*, 1-*transitive linear orders for some* $i > 1$, *coloured by pairwise disjoint colour sets, and* Y *is a countable*, 1-*transitive*, *linear order*.

Proof. Using the same method as in the previous result, we may assume that between any two points in $(X, <, F)$ coloured the same there is a point with a different colour. We use induction on the number of colours. The basis case, $|C| = 2$, is the previous theorem.

Now assume the result for values less than n, and let $(X, <, F)$ be a countable, n-coloured, 1-transitive linear order. If for every proper partition π of C, all $\sim_\pi$-classes are singletons, then, by Lemma 1.4, $(X, <, F) \cong \mathbb{Q}_n$. If, however, there is a proper partition π of C such that not all $\sim_\pi$-classes are singletons, then by Lemma 1.3 $(X/\sim_\pi, <, F')$, is 1-transitive and is coloured by fewer colours than X, so by induction hypothesis has the form $\mathbb{Q}_i(Y_0, Y_1, \dots, Y_{i-1})$ or $Y(Y_0^\wedge Y_1^\wedge \cdots {}^\wedge Y_{i-1})$ where $i \geqslant 2$. The result follows on replacing each point of $X/\sim_\pi$ by the equivalence class it represents. This will involve changing the Y_j, but as their colour sets are pairwise disjoint, and $i \geqslant 2$, each of them is still coloured by fewer than n colours. □

The point of this theorem is that in the inductive representation of X, we now know that the outer 'layer' may always be taken to be of one of the two stated kinds. It provides, *in principle*, a classification of all finitely coloured countable 1-transitive linear orders. However, the induction required means that listing these gets more and more complicated as $|C|$ increases. For instance, the 4-coloured ones fall into at least 19 cases (depending on exactly how they are counted). A more systematic method of listing the possibilities is provided by the use of labelled trees, and this also provides a good way of generalizing to the case of infinitely many colours. What we do is as follows.

For every coloured countable 1-transitive linear order, we show how to associate with it a labelled tree of a certain type that 'encodes' its construction. Conversely, given such a 'coding' tree, we show how to form a coloured countable 1-transitive linear order encoded by it. This constitutes a reasonably explicit classification, if not an actual 'listing'. In this paper, finite trees suffice. Infinite trees are required corresponding to the case of C infinite (see [1]).

For us a *tree* is a finite partially ordered set, $(\tau, \prec)$, such that for every $x, y \in \tau$, there is $z \in \tau$ with $x \preccurlyeq z$ and $y \preccurlyeq z$, and such that for every $x \in \tau$, $\{y \in \tau : x \preccurlyeq y\}$ is linearly ordered. (We are thinking of our trees as growing 'downwards'.) The *root* r of a tree $(\tau, \prec)$, is its unique greatest element, and its minimal elements are called *leaves*. If $x, y \in \tau$, we say that y is a *child* of x, or x is a *parent* of y, and write $y \prec\prec x$, if $y \prec x$ and there is no $z \in \tau$ such that $y \prec z \prec x$. Distinct children of the same parent are called *siblings*.

A *labelled tree*, $(\tau, \prec, \mathscr{L})$ is a tree $(\tau, \prec)$, together with a function $\mathscr{L} : \tau \to L$, where L is a set of 'labels'. Our labels will be ordered pairs, where the first co-ordinate (except at leaves) tells us how the coloured linear ordering associated with that vertex is constructed from those associated with its children, and the second label tells us what the colour set is for that coloured linear ordering. The ordered pair at a leaf will have the form $(1, \{c\})$ for some $c \in C$ (the 1 indicating that we are at a leaf).

More precisely, a *coding tree* has the form $(\tau, \prec, \mathscr{L}, \lhd)$, where $(\tau, \prec, \mathscr{L})$ is a labelled tree in the above sense, with every label an ordered pair, and

(i) $\lhd$ is a linear ordering of the branches of τ induced by a linear ordering of the children of each vertex (meaning that one branch precedes another provided that at the first point x of difference, the child of x in the first branch precedes the child of x in the second one in the ordering of the children of x; we visualize the tree as drawn 'from left to right' on the page),

(ii) if $x \prec\prec y$ and $y \neq r$, either x has a sibling, or y has a sibling,

(iii) if r has only one child, the first entry $\mathscr{F}(r)$ of its label is $\mathbb{Z}^\alpha$ or $\mathbb{Q} \cdot \mathbb{Z}^\alpha$ where α is some countable ordinal (greater than zero in the first case),

(iv) if r has $k \geqslant 2$ children, then $\mathscr{F}(r)$ is $\mathbb{Q}_k$ or k,

(v) if $x \neq r$ and x is not a leaf, then either it has no sibling and $\mathscr{F}(x) = k$, where k is its number of children ($k > 1$ by the conditions put on the tree),
or it has a sibling, and only one child, and $\mathscr{F}(x) = \mathbb{Z}^\alpha$ or $\mathbb{Q} \cdot \mathbb{Z}^\alpha$ (α a countable ordinal, non-zero in the first case)
or x has a sibling and $k \geqslant 2$ children and the first label of its parent is $\mathbb{Q}_l$, with $2 \leqslant l < \omega$, and $\mathscr{F}(x)$ is $\mathbb{Q}_k$ or k, or the first label of its parent is m, for $2 \leqslant m < \omega$, and $\mathscr{F}(x) = \mathbb{Q}_k$,

(vi) if $x \in \tau$ is a leaf, then $\mathscr{F}(x) = 1$,

(vii) the second member $\mathscr{S}(x)$ of the label at x is a subset of the set of colours C such that $\mathscr{S}(r) = C$, if x is a leaf, then $\mathscr{S}(x)$ is a singleton, and if x is not a leaf, then $\mathscr{S}(x)$ is the disjoint union of the $\mathscr{S}(y)$ for the children y of x.

Given the above definition of 'coding tree', we have to show how any coding tree gives rise to a coloured linear order, and, conversely, given a (countable, 1-transitive) coloured linear order, how we can find a coding tree that encodes it. First the definition of what this means.

We say that a coding tree $(\tau, \prec, \mathscr{L}, \lhd)$ *encodes* the coloured linear order $(X, <, F)$ if we can assign coloured linear orders to the vertices of τ by a function θ in such a way that $\theta(r) = (X, <, F)$,

the colours occurring in $\theta(x)$ are precisely the members of $\mathscr{S}(x)$, a leaf with second co-ordinate $\{c\}$ is assigned a singleton linear order with that colour, and

if x is a non-leaf vertex, $\theta(x)$ is obtained from $\{\theta(y) : y \text{ a child of } x\}$ according to $\mathscr{F}(x)$, so if $\mathscr{F}(x)$ is $\mathbb{Q}_k$, and $y_0, \dots, y_{k-1}$ are the children, then $\theta(x) = \mathbb{Q}_k(\theta(y_0), \dots, \theta(y_{k-1}))$, if $\mathscr{F}(x)$ is k, then $\theta(x) = \theta(y_0)^\wedge \dots, ^\wedge \theta(y_{k-1})$, and if $\mathscr{F}(x)$ is Z (a 1-transitive linear order) then $\theta(x) = Z \cdot \theta(y)$ (where y is the child of x).

Theorem 2.3. (i) *Any coding tree τ encodes a uniquely determined coloured linear order, and this is countable and* 1*-transitive.*

(ii) *Any coloured countable* 1*-transitive linear order, where the set of colours is finite, is encoded by some coding tree.*

Proof. (i) This is done by induction on the number of vertices of τ. If $|\tau| = 1$ then the root is a leaf, so it encodes a singleton. Otherwise the root r has k children say, and is labelled by $(\mathbb{Q}_k, <, C)$, (k, C), or (Z, C). By induction hypothesis, the tree below each of these children encodes a unique coloured linear order, and this is countable and 1-transitive. The root must therefore encode the corresponding $\mathbb{Q}_k$-combination, concatenation, or lexicographic product of these coloured linear orders, and this is clearly unique. Since any $\mathbb{Q}_k$-combination, concatenation, or lexicographic products of countable 1-transitive coloured linear orders with pairwise disjoint colour sets is also countable and 1-transitive, the induction step follows.

We remark that this part of the proof is rather obvious in the finite colour set case, but in [1] where infinite colour sets are considered, it becomes a lot more complicated.

(ii) Let $(X, <, F)$ be the given countable 1-transitive linear order, with finite colour set C. We have to find a labelled tree encoding $(X, <, F)$, which means that we have to assign labels, and in addition say which coloured linear order is encoded by the tree below each vertex. We use induction on $|C|$.

If C has just one element, then $(X, <, F)$ is monochromatic, so is one of the linear orders Z in Morel's list. If $|X| = 1$ then it is coded by a singleton tree, with root equal to the unique leaf labelled by $(1, \{c\})$. Otherwise, it is labelled by a 2-vertex tree, the root labelled by $(Z, \{c\})$, and its unique child (leaf) labelled by $(1, \{c\})$. (In the latter case, the coloured linear order assigned to the leaf is the singleton coloured by c.)

Otherwise, C has more than one element, and we may appeal to Theorem 2.2 to write X in the form $\mathbb{Q}_k(Y_0, \dots, Y_{k-1})$ or $Y(Y_0^\wedge \cdots ^\wedge Y_{k-1})$, where $Y_0, \dots, Y_{k-1}$ are themselves countable, 1-transitive, linear orders coloured by pairwise disjoint colour sets C_i, with $|C_i| < |C|$, and Y is a countable, 1-transitive linear order. To ensure that condition (v) holds, we choose k maximal. By induction hypothesis, each Y_i is encoded by some coding tree τ_i, and so we also have coloured linear orders assigned to the vertices of each τ_i in accordance with the definition of 'encodes'. We obtain a coding tree for $(X, <, F)$ as follows.

If $X \cong \mathbb{Q}_k(Y_0, \dots, Y_{k-1})$, to form τ, we add a new root r, with label $(\mathbb{Q}_k, C)$ and with k children which are roots r_i of copies of τ_i ordered left-right $0, 1, \dots, k-1$ (for definiteness, though actually in this case the order chosen does not matter). The same coloured linear orders are assigned to the vertices of τ_i in τ as they were in τ_i.

If $X \cong Y(Y_0^\wedge \dots {}^\wedge Y_{k-1})$ where $|Y| = 1$, we add a new root r, with label (k, C) and with k children which are roots of copies of τ_i ordered left-right $0, 1, \dots, k-1$ (this time the order does matter).

If $X \cong Y(Y_0^\wedge \cdots {}^\wedge Y_{k-1})$ where $|Y| > 1$, we add a new root r with a unique child x, having labels (Y, C) and (k, C), respectively, and let $\theta(x) = Y_0^\wedge \cdots {}^\wedge Y_{k-1}$, with the children of x treated as the children of the root were in the case $|Y| = 1$.

In all these three cases we let $\theta(r) = X$, which fits the definition for each. $\square$

3. Countable homogeneous coloured linear orders

We recall that a structure is said to be *n-homogeneous* if any isomorphism between n-element substructures can be extended to an automorphism. So it is homogeneous if and only if it is n-homogeneous for every n. It is clear that for linear orders the notions of n-transitivity and n-homogeneity coincide (though for general structures they will not be the same, except for $n = 1$ when they are always equivalent). We write that $(X, <, F)$ is *$\leqslant n$-homogeneous* when $(X, <, F)$ is m-homogeneous for all $m \leqslant n$.

As for linear orders without colouring, it is easy to prove that any coloured linear order is homogeneous if and only if it is 2-homogeneous. So the only natural homogeneity assumption to make, apart from 1-transitivity, in seeking to single out families of countable, coloured linear orders, is homogeneity, which is equivalent to $\leqslant 2$-homogeneity.

Let $(X, <, F)$ be a countable, coloured, linear order. We define a relation on the colour set C by $c_0 \sim c_1$ if and only if $c_0 = c_1$ or there are $x, y, z \in X$ such that $x < z < y$, $F(x) = F(y) = c_0$, and $F(z) = c_1$.

Lemma 3.1. *Let $(X, <, F)$ be a homogeneous, coloured linear order, and $x, y, z \in X$ be such that $x < z < y$ and $F(x) = F(y)$. Then for any $u < v \in X$ with $F(u) = F(v) = F(x)$, there is $w \in X$ such that $u < w < v$ and $F(w) = F(z)$.*

Proof. By homogeneity there is an automorphism taking x to u and y to v, and we let w be the image of z under this automorphism. $\square$

Lemma 3.2. *If $(X, <, F)$ is homogeneous, then $\sim$ is an equivalence relation.*

Proof. If $c_0 \sim c_1$, there are $x, y, z \in X$ such that $x < z < y$, $F(x) = F(y) = c_0$, and $F(z) = c_1$. By homogeneity, there is an automorphism f of $(X, <, F)$ taking x to y, and hence $y < f(z)$. As f preserves colour, $F(f(z)) = c_1$, so $z < y < f(z)$ with $F(z) = F(f(z)) = c_1$, $F(y) = c_0$, giving $c_1 \sim c_0$, and establishing symmetry.

Next consider transitivity. If $c_0 \sim c_1$ and $c_1 \sim c_2$, there are $x, y, z, u, v, w \in X$ such that $F(x) = F(y) = c_0$, $F(z) = F(u) = F(v) = c_1$ and $F(w) = c_2$ with $x < z < y$ and $u < w < v$. By homogeneity, there is an automorphism f of $(X, <, F)$ taking x to y. Thus $y < f(z)$ and $F(f(z)) = c_1$. By Lemma 3.1, the existence of u, v, w gives $b \in X$ with $z < b < f(z)$ and $F(b) = c_2$. If $b < y$, then there are $x, y, b \in X$ as required to show

$c_0 \sim c_2$. If $b = y$, then $c_0 = c_2$ so also $c_0 \sim c_2$. If $y < b$, then $x < f^{-1}(b) < z < y$ and $F(f^{-1}(b)) = c_2$. Hence in each case, $c_0 \sim c_2$. $\square$

Let $\bar{c} = \{c' \in C : c' \sim c\}$. Thus the $\bar{c}$s form a partition of C. This gives a corresponding equivalence relation $\approx$ on X defined by $x \approx y$ if $F(x) \sim F(y)$. Thus if $\bar{x} = \{y \in X : x \approx y\}$, then $\bar{x} = \{y \in X : F(y) \in \overline{F(x)}\}$.

Lemma 3.3. *If $(X, <, F)$ is homogeneous, then $\bar{x}$ is convex for all $x \in X$.*

Proof. Let $x < y$ lie in $\bar{x}$. Suppose for a contradiction that there is $z \in X$ with $x < z < y$ and $z \notin \bar{x}$. Since $z \notin \bar{x}$, $F(x) \neq F(y)$. We know $x \approx y$, so $F(x) \sim F(y)$, meaning that there are $u, v, w \in X$ such that $u < w < v$, $F(u) = F(v) = F(x)$, and $F(w) = F(y)$. Since $(X, <, F)$ is homogeneous, there is an automorphism f of X taking x to u and y to w. Hence $u < f(z) < w < v$ and $F(f(z)) = F(z)$, so $F(x) = F(u) \sim F(z)$, giving $z \in \bar{x}$, contradiction. $\square$

Lemma 3.4. *If $(X, <, F)$ is a countable, coloured linear order, then $(X, <, F)$ is homogeneous if and only if $\bar{x}$ is convex and homogeneous for all $x \in X$.*

Proof. $\Rightarrow$; Suppose $(X, <, F)$ is homogeneous. By Lemma 3.3, each $\bar{x}$ is convex. Now, let $x \in X$ and φ be an isomorphism of a finite subset of $\bar{x}$ into $\bar{x}$. Since $(X, <, F)$ is homogeneous, there is an automorphism f of X extending φ. We show that $f|_{\bar{x}}$ is an automorphism of $\bar{x}$. Let $y \in \bar{x}$, so that $F(y) \in \overline{F(x)}$ and, since f preserves colour, $F(f(y)) \in \overline{F(x)}$, giving $f(y) \in \bar{x}$. This shows that $\{f(y) : y \in \bar{x}\} \subseteq \bar{x}$. Applying the same argument to f^{-1} shows that $\{f^{-1}(y) : y \in \bar{x}\} \subseteq \bar{x}$, so $\{f(y) : y \in \bar{x}\} = \bar{x}$. Thus $f|_{\bar{x}}$ is an automorphism of $\bar{x}$.

$\Leftarrow$; Suppose that for all $x \in X$, $\bar{x}$ is convex and homogeneous. Let φ be a partial isomorphism of size $\leqslant 2$ in $(X, <, F)$, so $\varphi : \{x_0, x_1\} \to \{y_0, y_1\}$ with $\varphi(x_0) = y_0$ and $\varphi(x_1) = y_1$ say. Thus $F(x_0) = F(y_0)$ and $F(x_1) = F(y_1)$. Therefore, $y_0 \in \overline{x_0}$ and $y_1 \in \overline{x_1}$.

If $F(x_0) \nsim F(x_1)$, then, by homogeneity of $\overline{x_0}$ and $\overline{x_1}$, there are automorphisms f_0 and f_1 of $\overline{x_0}$ and $\overline{x_1}$ taking x_0 to y_0 and x_1 to y_1, respectively. Let $G : X \to X$ be defined by

$$G(x) = \begin{cases} f_0(x) & \text{if } x \in \overline{x_0}, \\ f_1(x) & \text{if } x \in \overline{x_1}, \\ x & \text{if } x \notin \overline{x_0} \cup \overline{x_1}. \end{cases}$$

Since the $\approx$-classes are convex and G takes $\bar{x}$ to $\bar{x}$ for every $x \in X$, G preserves both order and colour, so G is an automorphism of $(X, <, F)$ extending φ.

If $F(x_0) \sim F(x_1)$, then by homogeneity of $\overline{x_0}$, there is an automorphism f_0 of $\overline{x_0}$ extending φ. Let $G : X \to X$ be defined by

$$G(x) = \begin{cases} f_0(x) & \text{if } x \in \overline{x_0}, \\ x & \text{if } x \notin \overline{x_0}. \end{cases}$$

Then G is an automorphism of $(X, <, F)$ extending φ. We deduce that $(X, <, F)$ is $\leqslant$2-homogeneous and hence homogeneous. $\square$

Theorem 3.5. *The countable C-coloured linear order $(X, <, F)$ is homogeneous if and only if there are a partition π of C and a linear order Y such that X is obtained from Y by replacing each point of Y either by $\mathbb{Q}_{C_i}$ or by a singleton coloured c_j where C_i or $\{c_j\}$ are elements of π, and no two points of Y are replaced by the same set.*

Proof. $\Leftarrow$; Since each $\mathbb{Q}_{C_i}$ and singleton is homogeneous, by Lemma 3.4, all coloured orders as described in the theorem are homogeneous.

$\Rightarrow$; Let $(X, <, F)$ be a countable, homogeneous, C-coloured linear order. Let $\approx$ be the equivalence relation on X defined earlier, and let $(X/\approx, <, F')$ be the quotient, where $F'(\bar{x}) = \overline{F(x)}$. Since for any distinct $\bar{x}, \bar{y} \in X/\approx$, $F(x) \nsim F(y)$, F' colours $X/\approx$ trivially. We shall show that for all $x \in X$, $\bar{x} \cong \mathbb{Q}_{F(\bar{x})}$, or $\bar{x}$ is a singleton.

Case i: $|F(\bar{x})| = 1$. By Lemma 3.4, $\bar{x}$ is homogeneous. Since it is monochromatic, it is either a singleton or isomorphic to $\mathbb{Q}$. In the latter case, $\bar{x} \cong \mathbb{Q}_{\{F(x)\}} = \mathbb{Q}_{F(\bar{x})}$.

Case ii: $|F(\bar{x})| > 1$. We show that for all $a < b \in \bar{x}$ and for all $c \in F(\bar{x})$ there is $d \in \bar{x}$ such that $a < d < b$ and $F(d) = c$.

If $F(a) = F(b) \neq c$, then, by Lemma 3.1, since $F(a) \sim c$, there is $d \in \bar{x}$ such that $a < d < b$ and $F(d) = c$.

If $F(a) = F(b) = c$, by homogeneity of $\bar{x}$ (Lemma 3.4), there are $u, v, w \in \bar{x}$ such that $u < v < w < a$ and $F(u) = F(v) = F(w) = F(a)$. Then $\varphi = \{(u, a), (w, b)\}$ is a partial isomorphism, so there is an automorphism f of $\bar{x}$ extending φ. Thus $f(v) \in \bar{x}$, $a < f(v) < b$, and $F(f(v)) = c$.

If $F(a) \neq F(b)$ and $F(a) = c$, since $F(a) \sim F(b)$, there are $u, v, w \in \bar{x}$ such that $u < w < v$, $F(u) = F(v) = F(a)$ and $F(w) = F(b)$. As $\bar{x}$ is homogeneous we may suppose that $b < u$, and there is also $z \in \bar{x}$ such that $v < z$ and $F(z) = F(b)$. Then $\varphi = \{(u, a), (z, b)\}$ is a partial isomorphism, so there is an automorphism f of $\bar{x}$ extending φ. This gives us $f(v) \in \bar{x}$ and $a < f(v) < b$, with $F(f(v)) = F(a) = c$. The case where $F(a) \neq F(b)$ and $F(b) = c$ is similar.

The final case is $F(a) \neq F(b)$, $F(a), F(b) \neq c$. Since $F(a) \sim c$, there are $u, v, w \in \bar{x}$ such that $u < w < v$, $F(u) = F(v) = F(a)$ and $F(w) = c$. By homogeneity of $\bar{x}$ we may suppose that $b < u$, and by Lemma 3.1, since $F(a) \sim F(b)$, there is $e \in \bar{x}$ such that $u < e < v$ and $F(e) = F(b)$.

If $w < e$, then $\varphi = \{(u, a), (e, b)\}$ is a partial isomorphism of $\bar{x}$, and as $\bar{x}$ is homogeneous, there is an automorphism f of $\bar{x}$ extending φ. This gives $f(w) \in \bar{x}$ with $a < f(w) < b$ and $F(f(w)) = c$.

If $e < w$, then, since $\bar{x}$ is homogeneous, there is $t \in \bar{x}$ such that $v < t$ and $F(t) = F(e) = F(b)$. Then $\varphi = \{(u, a), (t, b)\}$ is a partial isomorphism, and as $\bar{x}$ is homogeneous, there is an automorphism f of $\bar{x}$ extending φ. This gives $f(w) \in \bar{x}$ with $a < f(w) < b$ and $F(f(w)) = c$.

This shows that $\bar{x}$ is isomorphic either to a singleton coloured $F(x)$, or to $\mathbb{Q}_{F(\bar{x})}$. Since $F(\bar{x}) \cap F(\bar{y}) = \emptyset$ whenever $x \not\approx y$, each distinct equivalence class in X has a distinct colour set, disjoint from all the others. Hence, $(X, <, F)$ is isomorphic to

$(Y, <, F')$ where each point $y \in Y$ is replaced by $\bar{y}$. Thus $(X, <, F)$ is isomorphic to a countable linear order where each point is replaced by a distinct $\mathbb{Q}_{C_i}$ or singleton coloured c_j, where C_i and $\{c_j\}$ are members of a partition of C. $\square$

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