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The complexity of lattice knots

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Abstract

A family of polygonal knots K_n on the cubical lattice is constructed with the property that the quotient of length $L(K_n)$ over the crossing number $Cr(K_n)$ approaches zero as L approaches infinity. More precisely $Cr(K_n) = \mathrm{O}(L(K_n)^{4/3})$. It is shown that this construction is optimal in the sense that for any knot K on the cubical lattice with length L and Cr crossings $Cr \leqslant 3.2L^{4/3}$.

Keywords: Knots; Knotted polygons; Cubic lattice; Knot complexity

AMS classification: 57

1. Introduction

Let $\mathbb{Z}^3$ be the *cubic lattice* with each vertex a point in $\mathbb{R}^3$ with integer Cartesian coordinates augmented by all line segments joining adjacent vertices whose coordinates are different in only one coordinate. Such a line segment is called an edge or a step. A *lattice polygon* or *polygon* on the cubic lattice is a piecewise linear simple closed curve embedded in $\mathbb{Z}^3$. A lattice polygon of length n has exactly n steps in it and is denoted by P_n. A lattice polygon of length n which has the knot type $\mathcal{K}$ is denoted by $P_n(\mathcal{K})$. The set of all lattice polygons is denoted by $\mathcal{P}$. The crossing number of the knot type $\mathcal{K}$ is the same as that of $P_n(\mathcal{K})$ and is denoted by either $Cr(P_n(\mathcal{K}))$ or $Cr(\mathcal{K})$.

In [2,6] the question of what is the minimum length required for a lattice polygon to be a knot of given knot type $\mathcal{K}$ has been discussed. In [2] it is shown that the minimal length of $P_n(\mathcal{K})$ is 24 if $\mathcal{K}$ is the trefoil knot and that no shorter lattice polygons can be knotted. In [6] estimates of the minimal length of $P_n(\mathcal{K})$ are given for all knots $\mathcal{K}$ up to

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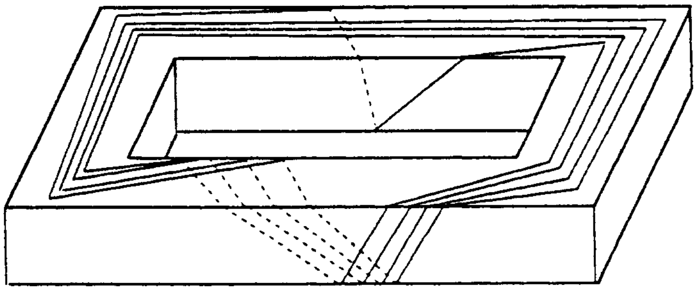

Fig. 1. A $(4,5)$-torus knot.

8 crossings and a few selected knots with 9 or 10 crossings. In all these examples the ratio of length divided by the crossing number of the knot is greater than 5. On the other hand, it has been shown in [5,7] that as n approaches infinity, most randomly chosen lattice polygons are complicated in the sense that they have many copies of connected sum components with any given knot type. Thus, the crossing number of P_n will also approach infinity with probability one. Therefore, an interesting question is: What is the asymptotic behavior of $Cr(P_n)$ as n approaches infinity? A simpler and more specific question concerns the ratio $n/Cr(P_n)$. What is the minimum value of this ratio for all $P_n \in \mathcal{P}$? Is this minimum value nonzero? In this paper we answer the above question with the following theorem.

Theorem. *Let $Cr(P_n)$ be the number of crossings of P_n. Then*

$$\inf_{P_n \in \mathcal{P}} \frac{n}{Cr(P_n)} = 0,$$

where $\mathcal{P}$ is the set of all lattice polygons.

The proof of the theorem is given in the next section.

2. The proof of the theorem

The proof of the theorem is done by constructing a (n^2, n^2+1) torus knot (as shown in Fig. 1 for $n^2 = 4$) for any large n using $O(n^3)$ steps. By [4], the crossing number of this torus knot is $n^4 - 1$ hence the theorem follows trivially.

The construction. The torus knot consists of 4 parts called bundles and denoted by B_1, B_2, B_3 and B_4. Each bundle consists of n^2 disjoint lattice paths starting and ending on two parallel planes. The intersection of the end points of the paths with such a plane are the lattice points of an $n \times n$ square as shown in Fig. 2 for $n = 4$. Any n paths of a bundle starting in a horizontal row of n points is called a horizontal layer of the bundle and any n paths of a bundle starting in a vertical row of n points is called a vertical layer of the bundle.

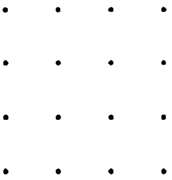

Fig. 2. The ends of the B_i's.

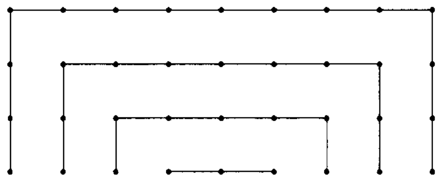

Fig. 3. A horizontal layer of B_1 and B_2.

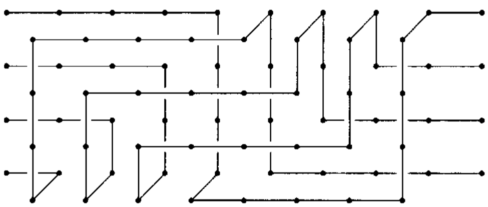

Fig. 4. The vertical layer of B_3 closest to the torus tube.

The bundles B_1 and B_2 are identical. A horizontal layer of these bundles is shown in Fig. 3 for $n = 4$.

It is easy to see that each layer takes $2n^2$ steps for B_1 and B_2. Since there are n layers each bundle, B_1 and B_2 have a total of $4n^3$ steps.

B_3 is the most complicated bundle. The n^2 paths of B_3 all twist one full rotation around a rectangular tube of length $2n + 1$ whose cross section is a $(n - 1) \times 1$ rectangle. All paths are contained in a rectangular box of dimensions $(2n + 1) \times (3n - 3) \times (2n - 1)$ with the above rectangular tube in its center. Fig. 4 shows the vertical layer of B_3 that is closest to the rectangular tube for $n = 4$.

The second closest vertical layer is obtained as follows from the first layer. First move all edges of the first layer which are parallel to the plane of page one unit in a direction perpendicular to the page. (One unit up if the edge is in front of the rectangular tube and one unit down if the edge is in the back of the rectangular tube.) Move the edges perpendicular to the page one unit down if they are on the bottom of Fig. 4 and one unit up if they are on the top of Fig. 4. In this process each path breaks into 7 disconnected pieces and adding 12 additional edges will connect the pieces together. The same procedure can be used to get from the second closest vertical layer to the third closest and so on. From Fig. 4 we can see that each path in the first vertical layer of B_3 has $4n + 1$ steps. It follows from the above that a path in the layer which is kth closest

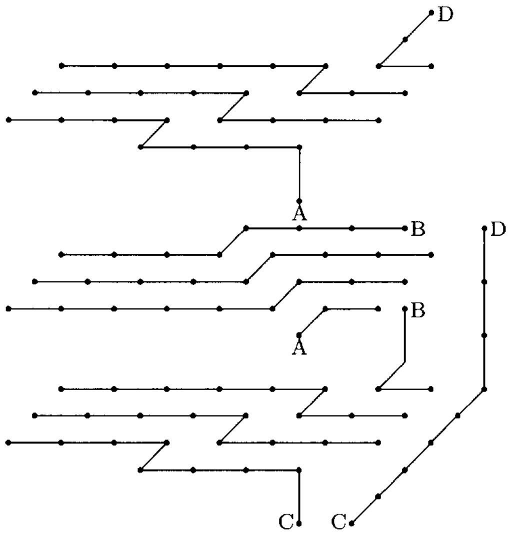

Fig. 5. The case of $n = 3$ for B_4.

to the rectangular tube has $4n + 1 + 12(k - 1)$ steps. So summing over the length of all n^2 paths we get that the total number of steps in B_3 is $10n^3 - 5n^2$. Notice also that the construction in Fig. 4 is not quite optimal, one could use a rectangular tube with a much smaller cross section. However the resulting improvement is small and one still needs $O(n^3)$ steps. The advantage in the given construction is that all paths in a given vertical layer have equal length.

The bundle B_4 is constructed in such a way to ensure that the union of the four bundles results in a knot. The construction is slightly different for even values of n and odd values of n. The cases of $n = 3$ and $n = 4$ are shown in Figs. 5 and 6. Notice that in the figures the vertices marked by the same letter are identical. We also need to point out that the figures shown are actually the mirror images (in the direction of the planes in which it starts and ends). So let us keep in mind when B_4 is put together with the other bundles, it is the mirror images shown here that will be used.

Observe that in the n paths of each horizontal layer of B_4, (a) $n - 1$ of them have $2n + 2$ steps, and (b) one of them has $2n + 4$ steps. There is only one exception to (b), which is the path that goes around the torus tube once. It has $3n + 4$ steps if n is even and $4n + 7$ steps if n is odd. So the total number of steps in B_4 is $2n^3 + 2n^2 + 3n$ if n is even and $2n^3 + 2n^2 + 4n + 3$ if n is odd. Now after putting the four bundles together (see Fig. 7) we obtain a knot with at most $16n^3 - 3n^2 + 4n + 3 \leqslant 16n^3$ steps (since $n \geqslant 2$). It is left to show that this is indeed a $(n^2, n^2 + 1)$-torus knot. This can be done by identifying a rectangular torus tube T and then observing that the constructed knot can be projected onto T and that it winds n^2 times around the longitude and $n^2 + 1$ times around the meridian of T. Fig. 7 shows the top view of this rectangular torus tube T together with the four bundles assembled. In this view the $(n - 1) \times 1$ cross section

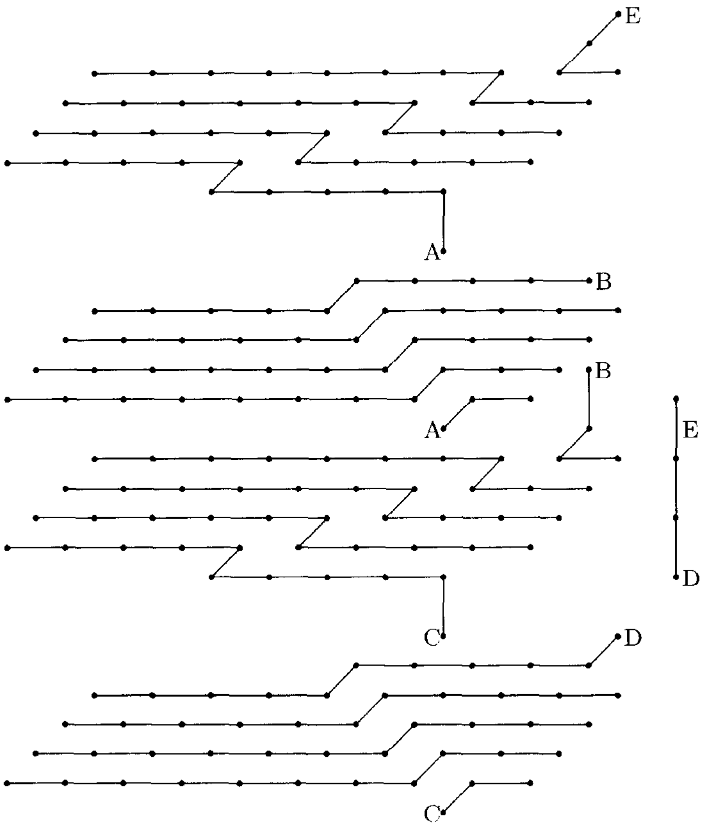

Fig. 6. The case $n = 4$ for B_4.

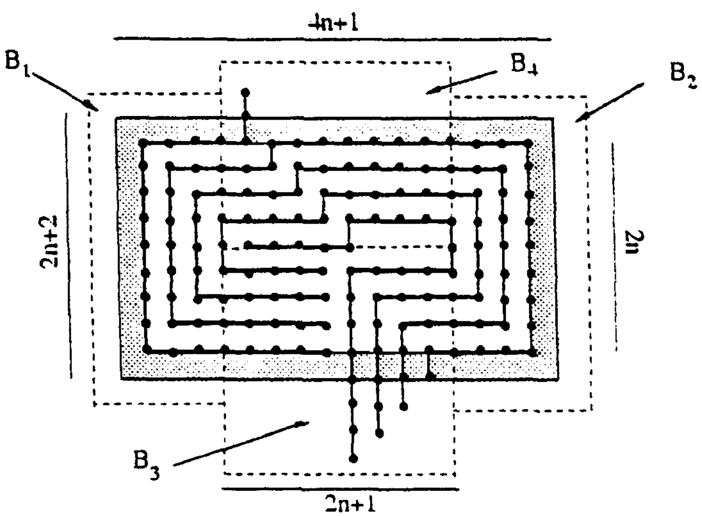

Fig. 7. The top view of T with the bundles.

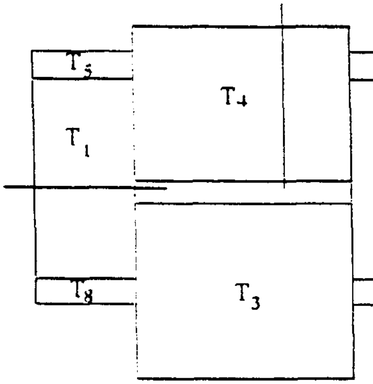

Fig. 8. The top view of the solid torus T'.

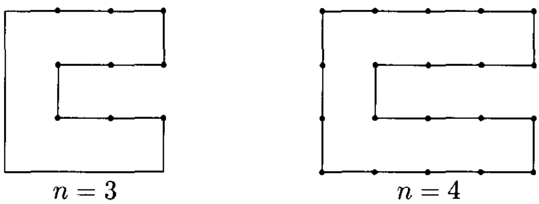

Fig. 9. The side view of T_1 and T_2 for $n = 3, 4$.

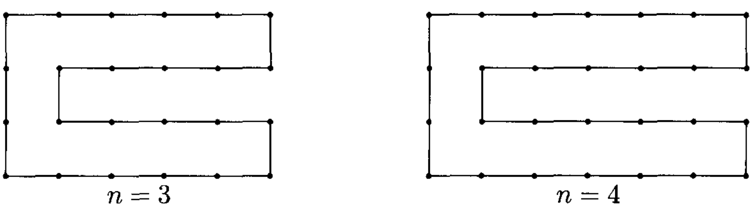

Fig. 10. The cross section of T_4 for $n = 3, 4$.

of T is only one unit thick. Notice that only the top horizontal layers of the bundles are shown in the figure and only part of the top layer is shown for B_3 for simplicity.

It is not very easy to see that the knot can indeed be projected onto T. So instead of trying to do this, we will modify the torus tube T into the surface of a new solid torus T'. T' is the union of eight parts denoted by $T_1, T_2, \ldots,$ and T_8. Fig. 8 shows the top view of these eight parts.

T_5 through T_8 are simply the solid rectangular cylinders of dimension $(n-1) \times 1 \times n$. T_1 through T_4 are topological solid cylinders with the B_i's completely lie on their side surfaces, respectively. The cross section view of T_1 and T_2 is given in Fig. 9 for $n = 3$ and $n = 4$. The cross section of T_4 is given in Fig. 10 for $n = 3$ and $n = 4$. It has been indicated in Fig. 8 where these cross sections have been taken. These look like combs

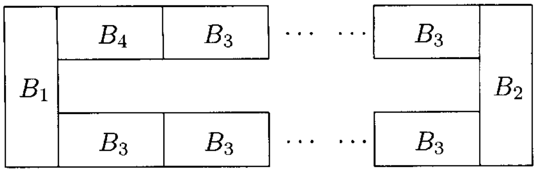

Fig. 11. A (kn^2+1, n^2)-torus knot for any odd integer k.

with $\lceil n/2 \rceil$ teeth. One can easily see that all the paths in B_1, B_2 and B_3 lie on these side surfaces.

The intersections of T_3 with the planes in which the paths of B_3 start and end are the same as that of T_1 (and T_2) with these planes. That is, they also look like combs with $\lceil n/2 \rceil$ teeth. One also has to imagine that this comb winds around the core tube (T) in the same manner the actual paths are winding around. From a topological point of view, the existence of such T_3 is much clearer. The details are left to the reader.

3. Discussions

An obvious question is how much room there is for improvement. In the above construction the ratio of the length of the knot over the number of crossing of the knot is at most $16/n$. The bulk of the length occurs in the B_3 bundle (about $10n^3$) which is also generating most of the n^4-1 crossings. If one constructs a larger (kn^2+1, n^2)-torus using k copies of the bundle B_3 as shown in Fig. 11, then this new torus knot will have $kn^4 - kn^2$ crossings. For large k one can achieve that length of the knot over the number of crossing of the knot is about $10/n$. One can also replace the bundle B_4 with another copy of the B_3 bundle and generate various torus links.

For any given knot K, let $L(K)$ be its length and $Cr(K)$ be its crossing number. If K is a smooth knot with *thickness* one (cf. [1,3]), then we have

$$Cr(K) \leqslant \frac{16(L(K))^{4/3} + 38L(K)}{4\pi} \tag{1}$$

by [1]. It is also shown in [1] that if K is a lattice knot, then

$$Cr(K) \leqslant \frac{16(2L(K))^{4/3} + 76L(K)}{4\pi}, \tag{2}$$

since every lattice knot can be modified into a smooth knot of shorter length with thickness $1/2$. If $L(K)$ is large enough, then (2) becomes

$$Cr(K) \leqslant 3.2\big(L(K)\big)^{4/3}. \tag{3}$$

On the other hand, let K_0 be the torus knot constructed in the last section, we have $L(K_0) \leqslant 16n^3$ and $Cr(K_0) = n^4 - 1 \approx n^4$ for n large. Thus $Cr(K_0) \approx 16^{-4/3}(L(K_0))^{4/3} > 0.024L^{4/3}$. This tells us that although improvement to $Cr(K_0)$ is possible, it can only be done to the coefficient of $L^{4/3}$. For example, use the knot constructed in Fig. 11 for large odd k, we can improve our result to $Cr(K_0) \approx 10^{-4/3}(L(K_0))^{4/3} > 0.046L^{4/3}$. This leads to the following results.

Corollary 1. *Let K be a smooth knot of thickness 1 and length $L(K)$. Let $Cr(K)$ be its crossing number. Then*

$$s = \sup_{K \in \mathcal{T}} \frac{Cr(K)}{\left(L(K)\right)^{4/3}}$$

exists and $0.046 \leqslant s < 2.92$, where $\mathcal{T}$ is the set of all smooth knots with thickness one.

Corollary 2. *Let P_n be a lattice polygon of n steps, then*

$$s_0 = \sup_{P_n \in \mathcal{P}} \frac{Cr(P_n)}{n^{4/3}}$$

exists and $0.046 \leqslant s_0 \leqslant 5.3$, where $\mathcal{P}$ is the set of all lattice polygons.

The construction of the knot in Fig. 11 and the fact that any lattice polygon can be modified into a smooth knot (without changing its knot type) with a shorter length and thickness $1/2$ imply the first part of the inequalities. The second part of the inequalities stems from the fact that any smooth knot of thickness one has length at least 2π and that any knotted lattice polygon has at least 24 steps.

Notice that the same question can be asked on the writhe of lattice knots (or knots with given thickness in general). It has been proved recently in [8] that

$$Wr(K) \leqslant \tfrac{1}{4}\left(L(K)\right)^{4/3}, \tag{4}$$

where $Wr(K)$ is the writhe of the knot K and K is of unit thickness. Given the close relationship between $Wr(K)$ and $Cr(K)$, (4) is not surprising. But exactly why we have $4/3$ in both cases is worth to explore. One may be able to use the method used in one case to improve the bound obtained in the other case. We are not sure at this point whether our construction provides a lower bound on $\sup_K Wr(K)/(L(K))^{3/4}$.

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