Rules admissible in transitive temporal logic $T_{S4}$, sufficient condition

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ABSTRACT

The paper develops a technique for computation inference rules admissible in temporal logic $T_{S4}$. The problem whether there exists an algorithm recognizing inference rules admissible in $T_{S4}$ is a long-standing open problem. The logic $T_{S4}$ has neither the extension property nor the co-cover property which previously were central instruments for construction decision algorithms for admissibility in modal logics (e.g. reflexive and transitive modal logic $S4$). Our paper uses a linear-compression property, a zigzag-ray property and a zigzag stretching property which hold for $T_{S4}$. The main result of the paper is a sufficient condition for admissibility inference rules in $T_{S4}$. It is shown that all rules which are valid in special finite models (with an effective upper bound on size) must be admissible in $T_{S4}$.

1. Introduction

Temporal logic has nowadays various applications in Computer Science and Artificial Intelligence. The beginning of applications to computer science is most frequently referred to [29], who employed tools of temporal logic to the specification and verification of programs (especially to concurrent programs in which the computation is performed by two or more processors working in parallel). Applications in artificial intelligence were initially concerned with finding a general framework adequate for the temporal representations required by AI programs. In modern mathematical symbolic style, temporal logic was introduced by Prior as a result of an interest in the relationship between tense and modality attributed to the Megarian philosopher Diodorus Cronus (ca. 340–280 B.C.). Applications of temporal logic (as a branch in mathematical logic) were initially focused on reasoning about knowledge, time and general properties of computation (cf. [15,48]). This approach has been noticeably influenced by well developed at the time tools of modal logics (cf. [16]). Since 1970s various forms of temporal logics more directly focused on applications have been introduced (cf. [26,29,2]). Linear temporal logic (LTL) has been quite successful in applications to systems specifications and verification (cf. [29]), to model checking (cf. [3,27,29]). Use of automatons, e.g. Büchi automaton (cf. [6,49,7,50]) for model checking in LTL and their variations, formed a robust branch in logic in computer science. Decidability problem and satisfiability problem for LTL and other temporal logics were studied from various viewpoints (cf. [1,47,22–24]). It is possible that the initial interest to temporal logic with linear time came from early papers of Bull on fmp and decidability of linear modal logics [4,5]. Since then, many interesting results concerning decidability and axiomatizability in temporal logics were obtained (cf. for instance, [9,11,10]).

Being a useful tool in computer science, temporal logics, in turn, originate computational issues concerning their own properties. Decidability and satisfiability questions, as mentioned above, were among the first problems on which the attention has been focused. These problems were profoundly investigated by usual tools of temporal logic (bounded finite
model property, filtrations, etc.) and tools of computational logic (e.g. automata, cf. references above). Another problem related to decidability, which generalizes the decidability problem, is the admissibility problem. The admissibility problem for a given logic $L$ is, for arbitrary given inference rule $r$, to compute (answer) whether $r$ is admissible in $L$. Admissible rules were (perhaps, first time explicitly) taken into account by Lorenzen [25].

Initially there were only observations concerning existence of interesting examples of admissible but not derivable rules (cf. [17,28]). Then, [8] posed the problem whether the intuitionistic logic IPC is decidable w.r.t. admissible inference rules. This problem (together with its counterpart for modal logic S4) was solved affirmatively in [31,35]. Algorithms deciding admissibility for some transitive modal logics and IPC, which are based on projective formulas and unification, were discovered by Ghilardi [12–14]. Roziere [30] found a solution of the Friedman problem for IPC by methods of proof theory. Jerábek [21] investigated complexity of algorithms recognizing admissibility in IPC and related modal logics.

If a logic $L$ itself is decidable and has a decidable basis for admissible rules, then deciding algorithms for admissibility based on parallel enumeration admissible and non-admissible follow (though, in general, we do not have then an effective bound). Therefore, the study of bases for admissible rules has been undertaken. Rybakov [32] showed that IPC and S4 do not have finite bases. Later (cf. [18,20,38]) some explicit infinite and decidable bases were constructed for some modal logics (e.g. S4, K4, Grz) and IPC. Iemhoff and Metcalfe [19] suggested a proof theory for admissible rules.

Recently, Wolter and Zakharyaschev [51] proved that modal logics situated between $K_u$ and $K4_u$ and possessing an additional universal modality $\Box_u$ are undecidable w.r.t. admissible rules. They set in [51] an open question: If the logic $S4_{u\Box}$—modal logic S4 with the universal modality $\Box_u$—is decidable? This question was answered affirmatively in [43]. In Rybakov [36,37] a refined technique for deciding admissibility, based on dropping, rarefication and the co-cover property (the extension property), has been developed. This technique allows a general and uniform way to construct algorithms recognizing admissibility for some important infinite classes of transitive modal logics. It covers all previously proved cases of decidability by admissibility (for transitive modal logics) and solves also the problems of unification, unification with parameters, problem of solution for logical equations in all these logics.

But it turns out that this technique does not work satisfactorily for intransitive and temporal logics. Therefore, the research shifted to the phase of study such particular logics. Decidability of the admissibility problem was proved for (i) the temporal logics of all integer numbers $\mathcal{L}(\mathbb{Z})$ and all natural numbers $\mathcal{L}(\mathbb{N})$ (cf. [39]), (ii) for the intransitive temporal logic of all integer intervals [40], (iii) for variations of LTL with UNTIL and SINCE on integers [41,42], (iv) for temporal logic-related LTL generated by nearly linear models—models with single branching node [44].

Recently, it was shown that the linear temporal logic LTL with UNTIL and NEXT operations (famous one from computation background) is also decidable by admissibility [46] and that all decidable temporal logics admitting universal formula are also decidable by admissibility [45].

But all these techniques do not allow us to approach standard temporal logics based on arbitrary (not linear) flows of time, such as $T_{S4}$. The reason is that such logics do not possess neither the extension property [12,13] nor the co-cover property [31,33–36] which earlier were central instruments for building algorithms deciding admissibility. Therefore, we dedicate our paper to reconstruction of a technique which would allow us to compute, to describe, inference rules admissible in temporal logics related to $T_{S4}$ (and in $T_{S4}$ itself). The main result is that we find a sufficient condition for rules to be admissible in such logics. It is shown that all rules which are valid in special finite models (with an effective upper bound on size) must be admissible in $T_{S4}$.

2. Preliminaries, definitions, notation

The paper uses standard notation and technique of relational Kripke/Hintikka semantics for temporal logics. To briefly recall basic notation and facts, the language of temporal logics consists of the language of Boolean logic extended by two new unary temporal operations: $\Box^+$ (always will be in future) and $\Box^-$ (always was in past). The operation $\Box^+$ (will be in the future) is defined as abbreviation for $\neg\Box^+\neg\Box^-$, and $\Box^-$ (was in the past) is defined for $\neg\Box^-\Box^+$. Formation rules for formulas are as usual. For a formula $\varphi$, $\Box^+\varphi$ represents the condition: there is a future state where $\varphi$ is true; $\Box^-\varphi$ means there was a state in past where $\varphi$ was true. A Kripke/Hintikka frame is a pair $\mathcal{F} := (F, R)$, where $F$ is the base of $\mathcal{F}$—a non-empty set, and $R$ is a binary (accessibility by time) relation on $F$. $|\mathcal{F}| := F, a \in F$ is a denotation for $a \in |\mathcal{F}|$. In this paper, we consider only reflexive and transitive temporal logics, so the accessibility relations $R$, in the sequel, are always reflexive and transitive. In what follows, $R^{-1}$ is the relation converse to $R$. If, for a set of propositional letters $P$, a valuation $V$ of $P$ in $|\mathcal{F}|$ is defined, i.e. $V : P \rightarrow 2^F$, in other words, $\forall p \in P(V(p) \subseteq F)$, then the tuple $\mathcal{M} := (\mathcal{F}, V)$ is called a Kripke/Hintikka model (structure). The truth values of formulas are defined at elements of $\mathcal{F}$ by the following rules:

\[
\forall p \in \text{Prop}, \forall a \in \mathcal{F}, \quad (\mathcal{F}, a) \models \neg p \iff a \notin V(p);
\]

\[
(\mathcal{F}, a) \models \neg \varphi \land \psi \iff (\mathcal{F}, a) \models \neg \varphi \quad \text{and} \quad (\mathcal{F}, a) \models \neg \psi;
\]

\[
(\mathcal{F}, a) \models \neg \varphi \lor \psi \iff (\mathcal{F}, a) \models \neg \varphi \quad \text{or} \quad (\mathcal{F}, a) \models \neg \psi;
\]

\[
(\mathcal{F}, a) \models \varphi \rightarrow \psi \iff \neg((\mathcal{F}, a) \models \neg \varphi) \quad \text{or} \quad (\mathcal{F}, a) \models \neg \psi;
\]

\[
(\mathcal{F}, a) \models \neg \neg \varphi \iff \neg((\mathcal{F}, a) \models \neg \varphi);
\]
The following holds:
\[(F, a) \models \neg \nu \rightarrow \psi \iff \exists b \in F ((aRb) \land (F, b) \models \neg \nu \psi);\]
\[(F, a) \models \nu \rightarrow \psi \iff \exists b \in F ((bRa) \land (F, b) \models \neg \nu \psi).\]

For any \(a \in F\), \(V_{\nu}(a) := \{p_1 | p_1 \in P, (F, a) \models \neg \nu p_1\}.\) For any formula \(\phi, V(\phi) := \{a | a \in F, (F, a) \models \neg \nu \phi\}.

**Definition 1.** For a Kripke–Hintikka structure \(M := (F, V)\) and a formula \(\phi, \psi\) is true in \(M\) (denotation \(\models \neg \nu \phi\)) if \(\forall a \in F \ (F, a) \models \neg \nu \psi.\) A formula \(\phi\) is true in a frame \(F\) (denotation \(\models \neg \nu \phi\)) if \(\forall \forall \forall \forall w \in F \ ((F, w) \models \neg \nu \phi).\)

**Definition 2.** For a class \(K\) of frames, the logic \(L(K)\) generated by \(K\) is the set of all formulas which are true in all models based on frames from \(K\).

The temporal logic \(T_{54}\) is the logic \(L(K_{r+tr})\), where \(K_{r+tr}\) is the class of all reflexive and transitive frames.

**Definition 3.** A logic \(L\) has the finite model property (fmp in the sequel) if \(L = L(K)\), where \(K\) is a class of finite frames.

As well known, \(T_{54} = L(K_{r+tr,f})\), where \(K_{r+tr,f}\) is the class of all finite frames from \(K_{r+tr}\), so, \(T_{54}\) has fmp. For any logic \(L(K)\), a formula \(\phi\) is a theorem of \(L(K)\) if \(\phi \in L(K)\), \(\phi\) is satisfiable in \(K\) if, for some valuation \(V\) in a frame \(F \in K\), \(\phi\) is true w.r.t. \(V\) at an world from \(F\). For reference, a Hilbert style axiomatic system of \(T_{54}\) is as follows: axioms are all classical propositional tautologies and (where \(\phi\) and \(\psi\) are arbitrary formulas)

(i) \(\square^+ (\phi \rightarrow \psi) \rightarrow (\square^+ \phi \rightarrow \square^+ \psi), \quad \square^- (\phi \rightarrow \psi) \rightarrow (\square^- \phi \rightarrow \square^- \psi),\)

(ii) \(\phi \rightarrow \square^+ \omega \phi, \quad \phi \rightarrow \square^\omega \phi,\)

(iii) \(\square^+ \phi \rightarrow \phi, \quad \square^- \phi \rightarrow \phi,\)

(iv) \(\square^+ \phi \rightarrow \square^\omega \phi, \quad \square^- \phi \rightarrow \square^- \phi;\)

inference rules are

\[
\begin{array}{c}
\phi, \psi \\
\hline
\psi, \phi
\end{array}
\quad
\begin{array}{c}
\phi \\
\hline
\square \phi
\end{array}
\quad
\begin{array}{c}
\phi \\
\hline
\square^- \phi
\end{array}
\]

A rule (inference rule) \(r\) (or, synonymously, \(r\) a consequence) \(r\) is an expression \(r := \varphi_1(x_1, \ldots, x_n) \land \ldots \land \varphi_m(x_1, \ldots, x_n) \land \psi(x_1, \ldots, x_n)\), where \(\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)\) and \(\psi(x_1, \ldots, x_n)\) are some formulas constructed out of letters \(x_1, \ldots, x_n\). Letters \(x_1, \ldots, x_n\) are called variables of \(r\). For any rule \(r\), \(Var(r) := \{x_1, \ldots, x_n\}\). The formula \(\psi(x_1, \ldots, x_n)\) is the conclusion of \(r\); formulas \(\varphi_i(x_1, \ldots, x_n)\) are premises of \(r\).

**Definition 4.** A rule \(r\) is said to be valid in a Kripke structure \((F, V)\) (we will use notation \(\models r\), or \(F \models \nu r\)) if \((F, \nu \land_{1 \leq i \leq m} \nu \varphi_i) \Rightarrow (F \models \nu \psi).\) Otherwise, we say \(r\) is refuted in \(F\), or refuted in \(F\) by \(V\), and write \(F \not\models r\). A rule \(r\) is valid in a frame \(F\) (notation \(F \models r\)) if, for all valuations \(V\) of \(Var(r)\), \(F \models r\).

This paper is devoted to study the problem of rules admissible in temporal logic \(T_{54}\). Historically, introduction into consideration of admissible inference rules may be referred to [25]. The definition of admissible rules is as follows. Let \(L\) be a logic, \(Form_L\) be the set of all formulas in the language of \(L\) and \(r := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n) \land \psi(x_1, \ldots, x_n)\) be an inference rule.

**Definition 5.** Rule \(r\) is admissible for (in) \(L\) if, \(\forall \varphi_1 \in Form_L, \ldots, \varphi_m \in Form_L,\)

\[
\left[ \land_{1 \leq i \leq m} (\varphi_i(\alpha_1, \ldots, \alpha_n) \in L) \right] \Rightarrow [\psi(\alpha_1, \ldots, \alpha_n) \in L].
\]

Thus, \(r\) is admissible if, for every substitution \(s, s(\varphi_1) \in L, \ldots, s(\varphi_m) \in L\) implies \(s(\psi) \in L\). We list below several examples of admissible rules.

**Examples 6.** The following holds:

(i) The rule \(\neg \alpha \rightarrow \beta \lor \neg (\neg \alpha \rightarrow \beta)\) [17] is admissible in the intuitionistic logic IPC but not derivable in the Heyting axiomatic system for IPC.

(ii) The rule \((\alpha \rightarrow \beta) \rightarrow (\alpha \lor (\beta \rightarrow \alpha)) \lor (\beta \rightarrow (\alpha \rightarrow \beta))\) [28] has the same properties as above.

(iii) The Lemmon–Scott rule

\[
\square (\square p \land \square \square p) \rightarrow (\square \square p \land \square \square \square p)
\]

is admissible (but non-derivable in standard Hilbert-style axiomatic systems) for modal logics S4, S4.1 and Grz (cf. [36]).
For a logic $L$, a formula $\varphi$ is said to be unifiable (in this logic $L$) if there is a substitution $\sigma$ (called unifier) such that $\sigma(\varphi) \in L$.

To give examples of rules admissible in $T_{S4}$, notice that any rule $\varphi/\psi$ with non-unifiable premise in $T_{S4}$ $\varphi$ (i.e. if no $\sigma$ such that $\sigma(\varphi) \in T_{S4}$) will be admissible. For instance, the rule $\Box^+ x \land \Box^+ \neg x/y$ has the premise which cannot be unified in $T_{S4}$, and hence is admissible.

**Example 7.** The rule
\[
\sigma(\varphi) \in L
\]
has non-unifiable premise, so is admissible, but is invalid, in $T_{S4}$.

Indeed, there is an $T_{S4}$-frame which refutes $r_1$ (e.g. the reflexive transitive model $\langle \{1, 2, 3\}, R, V \rangle$, where, $1R2, 3R2$, with the valuation $V(x) := \{1, 2\}, V(y) = \{2\}$, refutes $r_1$). Both the above-mentioned rules have non-unifiable premises in $T_{S4}$ because for any substitution $\sigma$ applied to the premises, the result cannot be true in the single-element reflexive frame. \hfill \Box

**Remark.** Notice that, in the intuitionistic propositional logic $IPC$, the situation is different if a formula $\varphi$ is not unifiable in $IPC$, $\varphi$ is inconsistent (is false in any $IPC$-model), and the rule $\varphi/\psi$ is valid in any $IPC$-model (and hence is derivable) for any formula $\psi$.

**Proposition 8.** The rule
\[
\sigma(\varphi) \notin L
\]
is not admissible in $T_{S4}$ but is admissible in the modal logic $S4$.

**Proof.** Indeed, take the substitution: $s(x) := x_1 \land \Box(\Box^+ x \land \Box^+ y \land \Box^+ z) \in T_{S4}$, which may be easily verified by computation of truth values in Kripke models. But, at the same time, $\Box^+(\Box^+ s(x) \rightarrow \Box^+ y) \land \Box^+(\Box^+ s(x) \rightarrow \Box^+ z) \in T_{S4}$. To show this, take the reflexive and transitive model $\mathcal{M} := \langle \{a, a_1, a_2, a_3\}, R, V \rangle$, where $aRa_i$ for all $i$, $V(x_1) := \{a, a_1, a_2\}$, $V(y) := \{a, a_2, a_3\}$ and $V(z) := \{a, a_1, a_2\}$. It is easy to calculate that $\mathcal{M}, a \not\vDash (\mathcal{M}, a) ; (\Box^+(\Box^+ s(x) \rightarrow \Box^+ y) \land \Box^+(\Box^+ s(x) \rightarrow \Box^+ z))$. It is well known that the rule $r_2$ is admissible in $S4$—for instance, as a derivative of the modal translation of the admissible in the intuitionistic logic Harrop’s rule mentioned above (cf. for example, [36]). \hfill \Box

So, $T_{S4}$, despite being defined by the same class of frames as the modal logic $S4$, does not always admit rules admissible in $S4$. The admissibility problem for inference rules in $L$ is to determine, given by the arbitrary inference rule $c$, if $c$ is admissible in $L$. If there is an algorithm solving this problem, we say the admissibility problem is decidable in $L$.

3. A sufficient condition for admissibility

For any frame $\mathcal{F} := \langle F, R \rangle$ and any $w \in F$, the frame $\mathcal{F}(w)^k$ generated by $w$ in $\mathcal{F}$ is the set of all $a \in \mathcal{F}$, were $a = w$ or $wQ_1w_1, w_1Q_2w_2, \ldots, w_k-1Q_kw_k, w_k = a$ for some $Q_i \in \{R, R^{-1}\}$, $w_i \in \mathcal{F}$.

**Definition 9.** A frame $\mathcal{F}$ is said to be generated (or, synonymously, rooted) if $\mathcal{F} = \mathcal{F}(w)^k$ for a world $w \in \mathcal{F}$. The world $w$ is said to be its root.

Example of a generated frame is depicted in Fig. 1.

**Remark.** It is a simple observation that the truth values of temporal formulas with temporal degree $m$ at an world $a \in \mathcal{F}$ of a frame $\mathcal{F}$ (which has an accessibility relation $R$) w.r.t. given valuation $V$ depend only on truth values of their letters at worlds situated in at most $2m$ distance from a w.r.t. swapping relations $R$ and $R^{-1}$. Because we often will use this fact in the following, we formalize and explicitly formulate this property below.
Let $\mathcal{F}$ be a finite frame, $a \in \mathcal{F}$ and $m \in N$. The set $\mathcal{F}_m^Z(a)$ of all worlds from $\mathcal{F}$ situated in $\mathcal{F}$ on at most $m$ $\epsilon$-zigzag distance from $a$ is defined as follows:

$$\mathcal{F}_m^Z(a) := \{ b \mid b \in \mathcal{F}, \exists b_1 \exists b_2 \ldots \exists b_{2m}(a R b_1 R^{-1} b_2 \ldots b_{2m-1} R^{-1} b_{2m} = b) \}. $$

The structure of the sets $\mathcal{F}_m^Z(a)$ is illustrated at Fig. 2.

For any frame (model) $\mathcal{F}$, we can consider each $\mathcal{F}_m^Z$ as a frame (model) with transferred from $\mathcal{F}$ accessibility relation (and valuation, respectively).

**Lemma 10.** For any frame $\mathcal{F}$, any $a \in \mathcal{F}$, any formula $\varphi$ of temporal degree $m$, and any valuation $V$ of letters from $\varphi$ in $\mathcal{F}$,

$$(\mathcal{F}, a) \models \varphi \Leftrightarrow (\mathcal{F}_m^Z, a) \models \varphi.$$  

**Proof.** Follows by standard easy induction on $m$. □

To study admissible rules, we need a reduction of rules to some equivalent reduced normal forms.

An inference rule $r$ is said to be in reduced normal form if $r = \varepsilon_r/x_1$, where

$$
\varepsilon_r := \bigvee_{1 \leq j \leq m} \left( \bigwedge_{1 \leq i \leq k} \left[ x_i^{(j, i, 0)} \land (\circ^+ x_i)^{(j, i, 1)} \land (\circ^- x_i)^{(j, i, 2)} \right] \right),
$$

all $x_i$ are certain variables (letters), $t(j, i, z) \in \{0, 1\}$ and, for any formula $\alpha$ above, $\alpha^0 := \alpha$, $\alpha^1 := \neg \alpha$. For a rule $r$, for the rest of the paper, $\text{Var}(r)$ is the set of all variables from $r$.

**Definition 11.** Let $r$ be an inference rule and $r_{nf}$ be a rule in the reduced normal form containing all variables of $r$. $r_{nf}$ is said to be a normal reduced form for $r$ iff the following hold: (i) $r$ and $r_{nf}$ are equivalent w.r.t. admissibility in any logic, and (ii) $r$ and $r_{nf}$ are equivalent w.r.t. validity in any frame.

Following closely proofs for Lemma 3.1.3 and Theorem 3.1.11 from [36] (or [31] where we first actually used transformations of inference rules to normal reduced forms), by similar reasoning, we can derive the following theorem.

**Theorem 12.** There exists an algorithm running in (single) exponential time, which, for any given inference rule $r$, constructs a rule $r_{nf}$ which is a normal reduced form of $r$.

**Proof.** A short draft of the proof follows. In fact, we simply shall specify the general algorithm described in Lemma 3.1.3 and Theorem 3.1.11 [36] to the language of our logic.

Assume that we are given with a rule

$$r = \psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n).$$

$$\psi(x_1, \ldots, x_n).$$

It is evident that $r$ is equivalent to the rule

$$r_0 = \psi_1(x_1, \ldots, x_n) \land \cdots \land \psi_m(x_1, \ldots, x_n) \land x_c \equiv \psi(x_1, \ldots, x_n) \land x_c$$

where $x_c$ is a new variable. Therefore, we can restrict the case to the rules in the form $r = \psi(x_1, \ldots, x_n)/x_c$.

If $\psi = \alpha \circ \beta$, where $\circ$ is a binary logical operation and both formulas $\alpha$ and $\beta$ are not simply variables or unary logical operations applied to variables (which both we call final formulas), take two new variables $x_\alpha$ and $x_\beta$ and the rule

$$r_1 := (x_\alpha \circ x_\beta) \land (x_\alpha \equiv \alpha) \land (x_\beta \equiv \beta)/x_c.$$ 

If one from formulas $\alpha$ or $\beta$ is final and another one not, we apply this transformation to the non-final formula. It is clear that $r$ and $r_1$ are equivalent w.r.t. validity in frames.

If $\psi = \star \alpha$, where $\star$ is a unary logical operation and $\alpha$ is not a variable, take a new variable $x_\alpha$ and the rule

$$r_1 := \star x_\alpha \land (x_\alpha \equiv \alpha)/x_c.$$
Again r and r₁ are equivalent. We continue this transformation over the resulting rules

\[ \bigwedge_{j \in I_1} \gamma_j \land \bigwedge_{i \in I_1} \alpha_i \equiv \alpha_i \]

until all formulas \( \alpha_i \) and \( \gamma_j \) in the premise of the resulting rules will be either atomic formulas, i.e., logical operations applied to variables, or variables. Evidently this transformation is polynomial. Further, we transform the premise of the resulting rule in the disjunctive normal form and make disjunctive normal form to be perfect (having the disjunctive members to be uniform length and containing all the components in the definition of reduced normal forms) and obtain as a result an equivalent rule r₂. This transformation, as well as all known ones for reduction of Boolean formulas to disjunctive normal forms, is exponential in time. As a result the final rule r₃ has the required form. □

For a logic \( \mathcal{L} \), a frame \( \mathcal{F} \) is \( \mathcal{L} \)-frame if \( \mathcal{L} \subseteq \mathcal{L} (\langle \mathcal{F} \rangle) \).

**Definition 13.** A logic \( \mathcal{L} \) has the linear-compression property if the following holds. For any finite model \( \mathcal{M} := \langle \mathcal{F}, V \rangle \) based on \( \mathcal{L} \)-frame \( \mathcal{F} \) (i.e. \( \mathcal{M} := \langle \mathcal{F}, V \rangle \) is a \( \mathcal{L} \)-model), where

\[ \mathcal{M} \models \neg \varphi, \quad \varphi = \bigvee_{1 \leq j \leq m} \left( \bigwedge_{1 \leq i \leq k} [x_i^{j,i,0}] \land (\neg x_i^{j,i,1}) \land (\neg x_i^{j,i,2}) \right), \]

there is a mapping \( f \) of \( \| \mathcal{F} \| \) onto the base set of a finite model \( \mathcal{M}_1 = (f (\| \mathcal{F} \|), R_1, V_1) \) with the following properties: for any disjunct \( \varphi_j = \bigwedge_{1 \leq i \leq k} [x_i^{j,i,0}] \land (\neg x_i^{j,i,1}) \land (\neg x_i^{j,i,2}) \) of \( \varphi \)

\[ \forall a \in \mathcal{F} (\langle f, a \rangle \models \neg \varphi_j \iff (f (\mathcal{F}), f (a)) \models \neg \varphi_j), \]

and the size of \( |\mathcal{M}_1| \) is linear in \( |\varphi| \).

In the following, for any frame \( \mathcal{F} := (F, R) \) and any \( a \in F \), \( C(a) \) denotes the cluster of \( \mathcal{F} \) containing \( a \), i.e., \( C(a) := \{ b \in F \mid (a, b) \in R \} \).

**Definition 14.** We say a logic \( \mathcal{L} \) has the zigzag-ray property if the following holds. For any finite \( \mathcal{L} \)-frame \( \mathcal{F} = (F, R) \), any \( a \in F \) and any finite set \( \{d_1, \ldots, d_{2n+3}\} \) disjoint with \( F \), the frame \( \mathcal{F} \cup \{d_1, \ldots, d_{2n+3}\}, R_1 \) where \( R_1 \) is the transitive close of the extension of \( R \) by \( xR_1d_1 \) for all \( x \in C(a), d_2R_1d_{2-1}, d_3R_1d_{2+1}, d_2R_1d_{2+1} \) and \( d_2R_1d_{2-1} \) is also an \( \mathcal{L} \)-frame.

We will use zigzag-ray-property for applications of Lemma 10 in order to get a sufficient condition for admissibility. For this sufficient condition, we also need the stretching property described below.

For any finite frame \( \mathcal{F} := (F, R) \) and, for any number \( k \in \mathbb{N} \), we define frames \( k \)-stretched from \( \mathcal{F} \) as follows. First, \( S_{\text{min}}(\mathcal{F}) \) is the set of all R-minimal R-clusters from \( \mathcal{F} \), and \( S_{\text{max}}(\mathcal{F}) \) is the set of all R-maximal R-clusters. Let \( S_0(\mathcal{F}) := \langle F_1, R_1 \rangle \), where \( F_1 := \{ w_1, 1 \mid w \in F \}, R_1 := \{ (w_1, 1, w_1, v_1, 1) \mid (w, v) \in R \} \):

\[
\begin{align*}
S_2(\mathcal{F}) & := \langle |S_1(\mathcal{F})| \cup S_{2,1}(\mathcal{F}) \cup S_{2,2}(\mathcal{F}), R_2 \rangle, \quad \text{where} \\
S_{2,1}(\mathcal{F}) & := \{ w_2, 1 \mid w \in F \}, \quad S_{2,2}(\mathcal{F}) := \{ w_2, 2 \mid w \in F \}, \quad \text{and} \\
R_2 & := R_1 \cup R_2, \quad \text{where} \ \ (w_2, 1, v_2, 1) \in R_2 \iff (w, v) \in R, \\
(w_1, 1, v_1, 1) & \in R_2, \iff (w \in S_{\text{min}}(\mathcal{F}) \land (w, v) \in R), \\
(w_2, 2, 1, 1) & \in R_2 \iff (v \in S_{\text{max}}(\mathcal{F}) \land (w, v) \in R).
\end{align*}
\]

If \( S_k(\mathcal{F}) \) is defined, \( S_{k+1}(\mathcal{F}) := \langle |S_k(\mathcal{F})| \cup S_{k,1,1}(\mathcal{F}) \cup S_{k,1,2}(\mathcal{F}), R_{k+1} \rangle \), where \( S_{k,1,1}(\mathcal{F}) := \{ w_{k+1,1} \mid w \in \mathcal{F} \}, S_{k,1,2}(\mathcal{F}) := \{ w_{k+1,2} \mid w \in \mathcal{F} \}, R_{k+1} := R_k \cup R_{k,1,k+1}, \( (w_{k+1,1}, v_{k+1,1}) \in R_{k+1,k+1} \iff (w, v) \in R, \\
(w_{k,2}, v_{k+1,1}) & \iff (w \in S_{\text{min}}(\mathcal{F}) \land (w, v) \in R), \\
(w_{k+1,2}, v_{k+1,1}) & \iff (v \in S_{\text{max}}(\mathcal{F}) \land (w, v) \in R).
\]

The essence of the stretching procedure is illustrated in Fig. 3.

**Definition 15.** A logic \( \mathcal{L} \) is said to have the zigzag stretching property if, for any generated finite \( \mathcal{L} \)-frame and any number \( m \in \mathbb{N} \), the frame \( S_m(\mathcal{F}) \) is again an \( \mathcal{L} \)-frame.

Recall that, for a frame \( \mathcal{F} \), \( |\mathcal{F}| \) is the base set of \( \mathcal{F} \).

**Lemma 16.** Let \( \mathcal{L} \) be a temporal logic with (1) fmp, (2) the linear-compression property, (3) the zigzag-ray property and (4) the zigzag stretching property. If a rule \( r \) in the reduced normal form is not admissible in \( \mathcal{L} \), then there are finite frames \( \mathcal{F}(b)^{\mathcal{L}} \) and \( \mathcal{F}_1 \) such that \( \mathcal{F}_1 \) is \( \mathcal{L} \)-frame, \( \mathcal{F}_1 \) refutes \( r \) by a valuation \( V \) and

(i) \( |\mathcal{F}_1| = |\mathcal{F}(b)^{\mathcal{L}}| \cup \{d_1, \ldots, d_{2m}, d_{2m+1}, d_{2m+2}, d_{2m+3}\} \), where \( d_i \not\in \mathcal{F}(b)^{\mathcal{L}} \) and the relation \( R \) on \( \mathcal{F}_1 \) is the transitive close of the extension of the relation from \( \mathcal{F}(b)^{\mathcal{L}} \) by \( xRd_1 \), for all \( x \) from the R-cluster \( C(w) \), for some single fixed \( w \in \mathcal{F}(b)^{\mathcal{L}}, d_2R_1d_{2-1}, d_3R_1d_{2+1} \) and \( d_2Rd_1; \)

(ii) \( m = |\mathcal{F}(b)^{\mathcal{L}}| + 4 \);
Indeed, if the conclusion of \( r \) is false by \( V \) at \( b \):

\[(v) \ \forall i \geq 2m + 1, \text{Val}_V(d_i) = \text{Val}_V(d_{i+1}), \text{in particular, the following holds:} \forall x_i \in \text{Var}(r) (\langle (F_1, d_i) \mid \neg \nu \diamond x_i \Leftrightarrow (F_1, d_i) \mid \neg \nu x_i \rangle \& \langle (F_1, d_i) \mid \neg \nu \diamond x_i \Leftrightarrow (F_1, d_i) \mid \neg \nu x_i \rangle).
\]

**Proof.** Indeed, if \( r := \varphi/x_1 \), where

\[
\varphi = \bigvee_{1 \leq j \leq m_1} \left( \bigwedge_{1 \leq i \leq m_1} [x_i^{(j, i, 0)} \land (\diamond x_i)^{(j, i, 1)} \land (\diamond x_i)^{(j, i, 2)}] \right),
\]

is inadmissible in \( L \), then there is a substitution \( \sigma : \text{Var}(r) \rightarrow \text{For}_L \) (let, for any \( x_i \in \text{Var}(r) \), \( \sigma (x_i) := (\alpha (p_1, \ldots, p_s)) \), which turns the premise of \( r \) to a theorem of \( L \), \( \sigma (\varphi) \in L \), but \( \sigma (x_1) \notin L \). By fmp of \( L \) we conclude that there is a finite rooted \( L \)-frame \( F (b)^\delta \) and a valuation \( \nu \) of letters \( p_1, \ldots, p_s \) in \( F (b)^\delta \) where

\[
(F (b)^\delta, b) \models \nu \sigma (x_1) \quad \text{and} \quad \forall c \in F (b)^\delta [ (F (b)^\delta, c) \models \neg \nu \sigma (\varphi) ].
\]

In order to use the zigzag-stretching property of \( L \), we set \( k := k_1 + 3 \), where \( k_1 \) is the maximum of temporal degrees of all formulas \( \alpha (p_1, \ldots, p_s) \). Now, take the model \( M \) based on the frame \( St_1(F (b)^\delta) \) (which, by the zigzag-stretching property of \( L \), is an \( L \)-frame) with the valuation \( \nu \) transferred form \( F (b)^\delta \) as follows:

\[
V(p_i) := \{ w_{j, 1} \mid w \in F (b)^\delta, (F (b)^\delta, w) \models \neg \nu p_i \}
\]

\[
\cup \{ w_{j, 2} \mid w \in F (b)^\delta, (F (b)^\delta, w) \models \neg \nu p_i \}
\]

\[
\cup \{ w_{j, 1} \mid w \in F (b)^\delta, (F (b)^\delta, w) \models \neg \nu p_i \}.
\]

We need the following auxiliary lemma. All elements of \( St_1(F (b)^\delta) \) have the form \( w_3 \), where \( w \in F (b)^\delta \) and \( \delta = (t, 1) \) or \( \delta = (t, 2) \) for some \( t \).

**Lemma 17.** For any formula \( \theta \) constructed out of letters \( p_1, \ldots, p_s \), from formulas \( \alpha (p_1, \ldots, p_s) \).

\[
\forall w_3 \in St_1(F (b)^\delta)/(St_1(F (b)^\delta), w_3) \models \neg \nu \theta \Leftrightarrow (F (b)^\delta, w) \models \neg \nu \theta .
\]

**Proof.** It is sufficient to note that the mapping \( w_3 \rightarrow \nu \) is the \( p \)-morphism w.r.t. the valuation \( \nu \), and hence this mapping preserves the truth values of formulas constructed out of letters \( p_i \). (Also the statement may be easily shown by standard induction on the temporal degree of the formula \( \theta \).) \( \square \)

To continue the proof of **Lemma 16**, by **Lemma 17**,

\[
(St_1(F (b)^\delta), b_{1, i}) \models \nu \sigma (x_1).
\]

Let \( M_{2k+3} := \langle [g_i \mid 1 \leq i \leq 2k + 3], R_g, V \rangle \), where \( g_2 R_g g_1, g_2 R_g g_3 \), and, in general, \( g_2 R_g g_{2i-1}, g_2 R_g g_{2i+1} \), and \( g_1 R_g \); let \( V(p_i) = \emptyset \) for all \( p_i \). It is clear that, for all \( i \), for all \( j \), and all \( j_1 \),

\[
(M_{2k+3}, g_i) \models \neg \nu \alpha (p_1, \ldots, p_s) \Leftrightarrow (M_{2k+3}, g_{j_1}) \models \neg \nu \alpha (p_1, \ldots, p_s).
\]

Take the model \( M := \langle \{St_1(F (b)^\delta)\} \cup M_{2k+3}, R_{M}, V \rangle \), where \( R_{M} \) is the transitive closure of the relation \( R \cup R_g \cup \{ (x, g_1) \mid x \in C(w_{k, 2}) \} \), where \( w_{k, 2} \) is a fixed world of \( S_{\text{min}}(St_1(F (b)^\delta)) \). By the zigzag-ray property of \( L \), the frame of \( M \) is an \( L \)-frame. By (3) and our choice of \( k \) as the maximum of the temporal degrees of the formulas \( \alpha (p_1, \ldots, p_s) \) plus 3, using **Lemma 10** we conclude: for all \( i \), \( \forall g_j \) with \( j \geq k \) and all \( j_1 \),

\[
(M, g_i) \models \neg \nu \alpha (p_1, \ldots, p_s) \Leftrightarrow (M, g_{j + j_1}) \models \neg \nu \alpha (p_1, \ldots, p_s).
\]
By (2), $(\sigma, (F(b)^{F_1}), b_{1,1}) \models^F \sigma(x_1)$ holds, and using that $k = k_1 + 3$, where $k_1$ is greater or equal to the temporal degree of the formula $\alpha_1(p_1, \ldots, p_n)$, by Lemma 10 we derive

$$(\mathcal{M}, b_{1,1}) \not\models^F \sigma(x_1).$$

Thus, we get that $\mathcal{M}$ refutes the formula $\sigma(x_1)$ by $V$ but, because $\sigma(\varphi) \in \mathcal{L}$ and $\mathcal{M}$ is based on an $\mathcal{L}$-frame,

$$\mathcal{M} \not\models \varphi$$

holds. Thus, if we take the model $(\mathcal{M}, V_1)$, where $V_1(x_i) := V(\sigma(x_i))$, by (5) and (6) follows

$$(\mathcal{M}, b_{1,1}) \not\models V_1 \varphi \quad \text{and} \quad (\mathcal{M}, b_{1,1}) \models V_1 x_1.$$  

Therefore by (7) and (4), the model $\mathcal{M}$ has the following properties:

$$(\mathcal{M}, b_{1,1}) \not\models V_1 \varphi; \quad \forall i \geq k + 1, \forall x_j [(\mathcal{M}, g_i) \models V_1 x_j \iff (\mathcal{M}, g_i) \not\models V_1 x_j].$$

Using the linear-compression property of $\mathcal{L}$ we take the mapping $f$ of the model $(\mathcal{M}, V_1)$ and obtain a model $\mathcal{M}_2 := (f(|M|), R_2, V_2)$ based on an $\mathcal{L}$-frame with size linear in $r$. By (8),

$$\mathcal{M}_2 \not\models V_2 \varphi \quad \text{and} \quad (\mathcal{M}_2, f(b_{1,1})) \models V_2 x_1 \quad \text{and for all } i \geq k + 1$$

$$\forall x_j [(\mathcal{M}_2, f(g_i)) \models V_2 x_j \iff (\mathcal{M}_2, f(g_i)) \not\models V_2 x_j].$$

Let $m = \|\mathcal{M}_2\| + 4$. Take the model $\mathcal{M}_3 := (|\mathcal{M}_2| \cup \{d_1, \ldots, d_{2m+3}\}, R_3, V_3)$, where $V_3$ extends $V_2$ by $\forall j, 1 \leq j \leq 2m + 3$, $\forall v_3(d_j) = V_2(f(g_{2k+3}))$ and $R_3$ is the transitive closure of the extension of $R_2$ by $C(f(g_{2k+3})) R_3 d_1, d_2 R_3 d_{2j-1}, d_2 R_3 d_{2j+1}$. By the zigzag-ray property of $\mathcal{L}$, we conclude that the model $\mathcal{M}_3$ has all properties required in Lemma 16, and the proof is complete. □

From Lemma 16, we immediately get the following theorem.

**Theorem 18.** (A sufficient condition for admissibility) Let a temporal logic $\mathcal{L}$ to have (a) fmp, (b) the linear-compression property, (c) the zigzag-ray property and (d) the zigzag stretching property. If a rule $r$ in the reduced normal form is valid at all finite models of size linear in $r$ satisfying all properties (i)-(v) from Lemma 16, then $r$ is admissible in $\mathcal{L}$.

**Corollary 19.** Let a temporal logic $\mathcal{L}$ to have (a) fmp, (b) the linear-compression property, (c) the zigzag-ray property and (d) the zigzag stretching property. If a rule $r$ is valid at all finite models of size exponential in $r$ and satisfying properties (i)-(v) from Lemma 16 (but, where in (iii) the size is exponential in $r$), then $r$ is admissible in $\mathcal{L}$.

**Proof.** Just use previous theorem and Theorem 12 and observe that in transformation of the rule $r$ to its reduced form in Theorem 12, the size of the obtained rule increases exponentially. □

We will show in the next section that Theorem 18 and this corollary are applicable to $T_{54}$. But before, we would like to comment the situation with algorithm to recognize admissibility in $T_{54}$.

**Remark.** The condition (v) from Lemma 16 was most important one for a chance to find a necessary and sufficient condition for admissibility in $T_{54}$ and to get based on this condition procedure deciding admissibility. A possible scheme to show that the conditions from Lemma 16 are necessary for admissibility was to use n-characterizing for $T_{54}$ models $\mathcal{M}_n, n \in N$. It would be sufficient, for some $\mathcal{M}_n$, to find a valuation for letters of the rule definable in the original valuation from $\mathcal{M}_n$, which would refute the rule. Actually, in this case, we need, in a sense, to extend the original valuation to inessential, superfluous worlds. And the condition (v) might allow us to stretch the valuation. But this scheme is not completed yet, and it is an open question if this can be done. Thus, it is an open question whether the sufficient conditions for admissibility from Lemma 16 are necessary. So, the problem—if the admissibility problem for $T_{54}$ is decidable—is open yet.

4. **Application to $T_{54}$**

To apply Theorem 18 and Corollary 19 to $T_{54}$ note, first, that $T_{54}$ evidently has the zigzag-ray property and the zigzag stretching property. What we need more is as follows.

**Lemma 20.** The logic $T_{54}$ has the linear-compression property.
Proof. Let $\mathcal{M} := (\mathcal{F}, V)$ be a model based on an $T_{S_4}$-frame $\mathcal{F}$, where
\[
\mathcal{M} \models \neg \nu \delta, \quad \delta = \bigvee_{1 \leq j \leq m} \varphi_j, \quad \varphi_j := \bigwedge_{1 \leq i \leq k} [\mathcal{F}(j,i,0) \land (\varphi(j,i,1) \land (\varphi(j,i,2))].
\]
For any $\varphi_j$,
\[
P(\varphi_j) := \{ x_i | t(j, i, 0) = 0 \}, \quad P(\varphi_j)^+ := \{ x_i | t(j, i, 1) = 0 \},
\]
\[
P(\varphi_j)^- := \{ x_i | t(j, i, 2) = 0 \}.
\]
Let $\mathcal{M}_1$ be the model based on the set $\mathcal{F}_1 := \{ \varphi_j | \exists a \in \mathcal{F}, (\mathcal{F}, a) \models \varphi_j \}$, with the relation $R_1$, where
\[
\varphi_j R_1 \varphi_j \Leftrightarrow [(P(\varphi_j)^+ \supseteq P(\varphi_j)^+) \& (P(\varphi_j)^- \subseteq P(\varphi_j)^-)].
\]
It is easy to see that $R_1$ is reflexive and transitive relation, and that for all $\varphi_j \in \mathcal{F}_1$, $P(\varphi_j) \subseteq P(\varphi_j)^+$, $P(\varphi_j) \subseteq P(\varphi_j)^-$ because $\mathcal{F}$ is a reflexive frame. Take the valuation $V_1$ of letters $x_i$ in $\mathcal{F}_1$ as follows:
\[
V_1(x_i) := \{ \varphi_j | x_i \in \mathcal{F}_1, x_i \in P(\varphi_j) \}.
\]
The frame $(\mathcal{F}_1, R_1)$ is an $T_{S_4}$ frame of size linear in $\delta$. Take the mapping $f$ of $\mathcal{F}$ in $\mathcal{F}_1$, where $\forall a \in \mathcal{F}, f(a) := \varphi_j a$, where $a \in V(\varphi_j)$ (i.e. $(\mathcal{F}, a) \models \varphi_j a$). This mapping is evidently a mapping onto, and by definition of $V_1$,
\[
\forall a \in \mathcal{F}, \forall x_i[(\mathcal{F}, a) \models \neg \nu x_i \Leftrightarrow (\mathcal{F}_1, f(a)) \models \neg \nu x_i].
\]
Moreover,
\[
\forall a \in \mathcal{F}, \forall x_i[(\mathcal{F}, a) \models \nu \varphi^+ \Rightarrow (\mathcal{F}_1, f(a)) \models \nu \varphi^+ x_i];
\]
\[
\forall a \in \mathcal{F}, \forall x_i[(\mathcal{F}, a) \models \nu \varphi^+ \Rightarrow (\mathcal{F}_1, f(a)) \models \nu \varphi^+ x_i].
\]
Indeed, if $(\mathcal{F}, a) \models \nu \varphi^+ x_i$, there is $b \in \mathcal{F}$ such that $aRb$ and $(\mathcal{F}, b) \models \nu x_i$. Then $(\mathcal{F}, a) \models \nu \varphi_j a$ and $x_i \in P(\varphi_j)^+$. $(\mathcal{F}, b) \models \nu \varphi_j a$ and $x_i \in P(\varphi_j)^+$. By definition of $R_1$ we infer $\varphi_j a R_1 \varphi_j b$, and $(\mathcal{F}_1, \varphi_j b) \models \nu x_i$. Hence $(\mathcal{F}_1, \varphi_j b) \models \nu \varphi^+ x_i$. Vise versa, assume $(\mathcal{F}_1, f(a)) \models \nu \varphi^+ x_i$. Then, first, $(\mathcal{F}, a) \models \nu \varphi_j a$, where $f(a) = \varphi_j a$. And, $\varphi_j a R_1 \varphi_j b$, for some $\varphi_j b$, where $(\mathcal{F}_1, \varphi_j b) \models \nu x_i$. Then, $x_i \in P(\varphi_j)^+ \subseteq P(\varphi_j)^+$. By $\varphi_j a R_1 \varphi_j b$ we conclude $x_i \in P(\varphi_j)^+$. Therefore, from $(\mathcal{F}, a) \models \nu \varphi_j a$ follows $(\mathcal{F}, a) \models \nu \varphi^+ x_i$. The case $\nu \neg x_i$ may be considered similarly. Thus, we proved:
\[
\forall a \in \mathcal{F} \forall \varphi_j[(\mathcal{F}, a) \models \nu \varphi_j \Leftrightarrow (f(\mathcal{F}), f(a)) \models \nu \varphi_j],
\]
which completes the proof of our lemma. \qed

From this lemma, we conclude that Theorem 18 and Corollary 19 are applicable to $T_{S_4}$ and give sufficient conditions for admissibility in $T_{S_4}$.

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References


