An Asymptotic Theory for Nonlinear Functional Differential Equations

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Abstract—The Hartman-Wintner theorem on asymptotic integration is established for a certain class of functional differential equations with nonlinear perturbations, by looking them as abstract ordinary differential equations. Results for delay differential equations are included in this study. Consequences and applications are also shown. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In ordinary differential equations, there is a remarkable theorem on asymptotic integration due to Hartman and Wintner (see [1,2]) for almost diagonal systems

$$y' = (\Lambda(t) + R(t))y,$$

where

$$\Lambda(t) = \text{diag}\{\lambda_1(t), \ldots, \lambda_N(t)\}.\tag{2}$$

The hypotheses on system (1) are the trichotomic condition

$$|\text{Re}(\lambda_i(t) - \lambda_k(t))| \geq \varepsilon > 0,\tag{3}$$

for all $i \neq k$ and all $t$ for some fixed integer $k$, and the integrability condition on $R$: $R = (r_{ij}) \in L^p$, for some $p : 1 \leq p \leq 2$. This theorem ensures that (1) has a vector-valued solution $y_k$ such that

$$y_k(t) = \exp \left( \int_t^\infty [\lambda_k(s) + r_{kk}(s)] \, ds \right) (e_k + o(1)),\tag{4}$$

as $t \to +\infty$, where $e_k$ is the $k^{th}$ vector of the canonical basis of $C$.

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As condition (3) shows, the Hartman-Wintner theorem deals with nonoscillatory systems and does not apply when two characteristic roots have the same real part, as shown by the classical adiabatic oscillator equation,

\[ y'' + y + t^{-1} \sin(2t)y = 0 \]

(see [3,4]).

We have extended the Hartman-Wintner theorem in [5] to a linear system

\[ y'(t) = (A(t) + R(t))y(t), \]

where \( A(t) \) and \( R(t) \) are complex matrix valued functions, locally integrable, defined on the interval \([0, +\infty)\), satisfying the following.

(A) There are complementary projections \( P_i : \mathbb{C}^n \to \mathbb{C}^n \) \( (i = 0, 1, 2) \) such that \( \text{Range } P_0 = \langle \hat{\varphi} \rangle \),

\[ A(t)P_i = P_iA(t), \quad i = 1, 2, \]

\[ A(t)\hat{\varphi} = \lambda_0(t)\hat{\varphi}, \]

and given \( \epsilon > 0 \) small enough, there is a constant \( M > 0 \) such that

\[ |\Phi(t, s)P_i| \leq M e^{-\epsilon(t-s)} \left| \exp \left( \int_s^t \lambda_0 \right) \right|, \quad t \geq s, \]

\[ |\Phi(t, s)P_2| \leq M e^{\epsilon(t-s)} \left| \exp \left( \int_s^t \lambda_0 \right) \right|, \quad t \leq s, \]

where \( t, s \in [0, +\infty) \), \( \Phi = \Phi(t, s) \) is the Cauchy matrix of the system

\[ x' = A(t)x \]

such that \( \Phi(s, s) = I \).

(B) \( R(t) \in L^p \), for some \( 1 \leq p \leq 2 \) and

\[ P_0R(t)\hat{\varphi} = r_0(t)\hat{\varphi}, \]

where \( r_0 \) is a function from \( \mathbb{R}^+ \) into \( \mathbb{C} \). Then system (5) has a solution \( y = y_k(t) \) such that

\[ y_k(t) = \exp \left( \int_0^t [\lambda_0(\tau) + r_0(\tau)] d\tau \right) (\hat{\varphi} + o(1)), \]

as \( t \to +\infty \). Note that the Hartman-Wintner trichotomy (3) is generalized by (A). In this paper, we extend these results to the nonlinear functional differential equation

\[ y' = L(y_t) + R(t, y_t) + f(t, y_t), \]

where \( y_t(s) = y(t + s) \), \( L, R(t, \cdot), f(t, \cdot) : \mathbb{C} \to \mathbb{C}^N \) are continuous mappings in each \( t \in \mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\} \) and \( C = (C[-r, 0], C^1), \| \cdot \| \) with the supremum norm \( \| \cdot \| \).

We assume that \( L, R, \) and \( f \) satisfy the following conditions.

(H1) \( L(\phi) := \int_{-r}^0 d\eta(\theta)\phi(\theta) \), where \( \eta \) is an \( N \times N \) matrix-valued bounded variation function and the characteristic equation

\[ \det \Delta(\lambda) = 0 \]

has a simple root \( \lambda = \lambda_0 \), and no other root has the real part of \( \lambda_0 \), where the matrix \( \Delta(\lambda)v = [\lambda v - L(e^{\lambda v})] \), for all \( v \in \mathbb{C}^N \).

(H2) \( R(t, \cdot) \) is linear and \( |R(t, \cdot)| \in L^p(\mathbb{R}_0^+, \mathbb{R}) \), for some \( 1 \leq p \leq 2 \).

(H3) There is a function \( \gamma : \mathbb{R}_0^+ \to \mathbb{R} \) in \( L^1 \) such that

\[ |f(t, \varphi) - f(t, \psi)| \leq \gamma(t)|\varphi - \psi|, \quad f(t, 0) = 0, \]

for all \( t \in \mathbb{R}_0^+ \) and \( \varphi, \psi \in \mathbb{C} \).
For system (10), the Hartman-Wintner trichotomy condition (3) takes the form of Condition (H1). (See Lemma 2(iv) and Lemma 6 below.)

System (10) is also nonlinear. In some sense, the proceeding result is an asymptotic result of the Levinson type [1] when system (10) is seen as a sublinear functional $L^1$ perturbation of the linear functional system,

$$x' = L(x_t) + R(t, x_t),$$

which is a Levinson type system; i.e., it satisfies a Levinson dichotomy (see [1,6]). It seems that the asymptotic behavior of systems of such generality has not been studied before.

The study of asymptotic results in delay differential equations was initiated by Bellman and Cooke [7,8]. They establish a result for the equation

$$y'(t) = (a_0 + a(t))y(t) + (b_0 + b(t))y(t - r),$$

where $a_0, b_0 \in \mathbb{R}, a, b \in L^p$, for some $1 \leq p \leq 2$, under several conditions (see [7, Theorem 9.2, p. 277]). Győri and Pituk [9] reduce such a number of conditions and require of the autonomous part that its characteristic root, with largest real part, be real and simple. These results are extended to systems by Hale [10] and Arino et al. [11]. In most of the results, the systems considered are not purely functional. More precisely, they have an ordinary perturbed equation

$$y' = A(t)y + R(t, y_t),$$

whose matrix $A(t)$ is diagonal. Thus, Ai [12] established the existence of special solutions associated with each eigenvalue of the matrix $A$ and their corresponding asymptotic behavior, for $A(t) = \Lambda + V(t)$ where $\Lambda$ and $V(t)$ are diagonal matrices such that all eigenvalues of $\Lambda$ are different, $V(t)$ is small, $|R(t, \varphi)| \leq \gamma(t)||\varphi||_\infty$, and $\gamma \in L^2$. Probably, the nearest result can be found in [13], where Cassell and Hou established similar asymptotic formulas and considered in equation (15) a diagonal matrix $A(t)$ whose eigenvalues satisfied (3) and $\gamma \in L^p$, for some $p \in [1, +\infty[.$

We present a different proof based on the reduction of system (13) and the nonlinear perturbed system (10) to an ordinary differential equation in a Banach space. This allows us to use classical tools for ordinary differential equations in FDEs. In this case, we apply a version of the proof given in [5]. We use only one operator whose fixed point is a solution of the abstract formulation and it has an explicit asymptotic formula. Our proof gives a direct method to study asymptotic formulas for other abstract cases (see [6]). A discrete version of this method is given in [14]. The spectral decomposition (trichotomy) is also new and it allows us to consider a nonlinear perturbation and several $L^{p_k}$ perturbations with different $p_k \in [1, 2]$. Asymptotic results for variable delay have been also studied in [3,15,16]. This paper is organized as follows. In Section 2, we collect some preliminary facts from Hale's theory [17] for the spectrum of the semigroup of solutions of linear functional differential equations (FDEs). In Section 3, we formulate the problem studied by Burns et al. [18] and give an abstract formulation which reduces it into a differential equation in Banach spaces, which can be seen as a perturbed ordinary differential equation. In Section 4, we give some estimates which allow us to prove that an integral operator is a contraction in Section 5, and hence, we establish the main result. Some consequences for the linear case are given in Section 6. This work is finished with some consequences and examples.

2. LINEAR AUTONOMOUS FDES

In this section, we collect some results about FDEs that are going to be needed later. See [17] for more details.

**Lemma 1.** Consider the linear autonomous FDE

$$x'(t) = L(x_t).$$

(16)
Then

(i) the solutions \( x_t \) may be expressed as \( x_t = T(t)\varphi, \) where \( \varphi \) is the initial state in \( C \) and \( \{T(t)\}_{t \geq 0} \) is a strongly continuous semigroup in \( C \);

(ii) if \( A \) is the infinitesimal generator of \( \{T(t)\}_{t \geq 0} \), then \( \sigma(A) = \{ \lambda \in C : \det \Delta(\lambda) = 0 \} \), and

(iii) \( \{e^{\lambda t} : \lambda \in \sigma(A)\} \subseteq \sigma(T(t)) \subseteq \{e^{\lambda t} : \lambda \in \sigma(A)\} \cup \{0\}, t \geq 0. \)

Hypothesis (H1) allows us to decompose the space of solutions of system (16) in the form given by the following lemma.

**Lemma 2.** Suppose that \( L \) satisfies (H1). Then we have the following.

(i) \( C = P_1 \oplus P_0 \oplus P_2, \) where

\[
\begin{align*}
P_1 &= \oplus \{M_\lambda(A) : \text{Re} \lambda < \text{Re} \lambda_0 \}, \\
P_0 &= M_{\lambda_0}(A), \quad \text{and} \\
P_2 &= \oplus \{M_\lambda(A) : \text{Re} \lambda > \text{Re} \lambda_0 \}.
\end{align*}
\]

(ii) \( \dim P_2 < \infty. \)

(iii) There is \( \varphi_0 = e^{\lambda_0}v \in C \) such that \( M_{\lambda_0}(A) = \langle \varphi_0 \rangle, \) with \( v \neq 0. \)

(iv) \( T(t) \) can be extended for all \( t \in R \) on the finite-dimensional spaces \( P_0 \) and \( P_2. \) Given \( \varepsilon > 0 \) small enough, there is a constant \( K > 0 \) such that

\[
\begin{align*}
(a) \ |T(t)/P_1| &\leq Ke^{(\text{Re} \lambda_0 - \varepsilon)t}, \quad t \geq 0, \\
(b) \ |T(t)/P_2| &\leq Ke^{(\text{Re} \lambda_0 + \varepsilon)t}, \quad t \leq 0, \quad \text{and} \\
(c) \ T(t)\varphi_0 = e^{\lambda_0 t}\varphi_0, \quad \text{for all } t \in R.
\end{align*}
\]

Note that the above result shows that the Hartman-Wintner trichotomy (H1) condition implies a condition like Condition A in [5].

In the following results, we consider the adjoint equation:

\[
x_\alpha(t) = -\int_{-r}^{0} x_\alpha(-\theta) d\eta(\theta), \quad t \leq 0, \tag{17}
\]

where \( \eta \) is defined in (H1).

**Lemma 3.** Under Hypothesis (H1) and following the notation of the above lemmas, we have that:

(i) the solutions \( x_\alpha^* \) may be expressed as \( x_\alpha^* = T^*(t)\alpha, \) where \( \alpha \) is the initial condition in \( C^* := C([0, r], C^N) \) and \( \{T^*(t)\}_{t \leq 0} \) is a strongly continuous semigroup in \( C^*; \)

(ii) if \( A^* \) is the infinitesimal generator of \( \{T^*(t)\}_{t \leq 0}, \) then \( \sigma(A) = \sigma(A^*). \)

For (17), Lemma 2 also holds; i.e., see Lemma 4.

**Lemma 4.** Suppose that \( L \) satisfies (H1); then we have the following.

(i) \( C^* = P_1^* \oplus P_0^* \oplus P_2^* \) where

\[
\begin{align*}
P_1^* &= \oplus \{M_\lambda(A^*) : \text{Re} \lambda < \text{Re} \lambda_0 \}, \\
P_0^* &= M_{\lambda_0}(A^*), \quad \text{and} \\
P_2^* &= \oplus \{M_\lambda(A^*) : \text{Re} \lambda > \text{Re} \lambda_0 \}.
\end{align*}
\]

(ii) \( \dim P_2^* < \infty. \)

(iii) \( M_{\lambda_0}(A^*) = \langle \varphi_0^* \rangle, \) where \( \varphi_0^* = c_1 e^{-\lambda_0}v^* \) with \( c_1 \neq 0. \)

(iv) \( T^*(t) \) can be extended for all \( t \in R \) on the finite-dimensional spaces \( P_0^* \) and \( P_2^*. \) Given \( \varepsilon > 0, \) there is a constant \( K > 0 \) such that

\[
\begin{align*}
(a) \ |T^*(t)/P_1^*| &\leq Ke^{(\text{Re} \lambda_0 - \varepsilon)t}, \quad t \geq 0, \\
(b) \ |T^*(t)/P_2^*| &\leq Ke^{(\text{Re} \lambda_0 + \varepsilon)t}, \quad t \leq 0, \quad \text{and} \\
(c) \ T^*(t)\varphi_0^* = e^{\lambda_0 t}\varphi_0^*, \quad \text{for all } t \in R.
\end{align*}
\]

The splittings given in Lemmas 2 and 4 are related in the following lemma.
Lemma 5. Suppose that \( L \) satisfies (H1), assume the notation of the above lemmas, and consider the bilinear form
\[
\langle \varphi^*, \varphi \rangle = \varphi^*(0)\varphi(0) - \int_{-r}^{0} \varphi^*(\xi - \theta)\, d\eta(\theta)\varphi(\xi)\, d\xi.
\]
Then we have the following.
(i) \( \varphi^* \in P_i^* \) if and only if \( \langle \varphi^*, \varphi \rangle = 0 \) for all \( \varphi \in P_j \oplus P_k, \ j, k \neq i \).
(ii) \( \varphi \in P_i \) if and only if \( \langle \varphi^*, \varphi \rangle = 0 \) for all \( \varphi^* \in P_j^* \oplus P_k^*, \ j, k \neq i \).

Throughout this paper, the notation given in the above lemmas will be assumed.

Let \( \Phi^*, \Phi \) be bases for \( P_i^* \) and \( P_i \), respectively, such that \( \langle \Phi^*, \Phi \rangle = I \), and suppose that \( \langle \varphi_0^*, \varphi_0 \rangle = 1 \); this is possible by taking
\[
c_1 = \frac{1}{(v^* \cdot v - v^* \cdot L(\varepsilon_0^* v))}
\]
which is defined in Lemma 4(iii) and it exists under Hypothesis (H1). Let
\[
X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0, \\ I, & \theta = 0, \end{cases}
\]
\[
X_0^{P_i} = [\Phi^*(0) \cdot I] \Phi, \quad X_0^{P_k} = [\varphi_0^* \cdot I] \varphi_0,
\]
and
\[
X_0^{P_i} = X_0 - X_0^{P_i} - X_0^{P_k}.
\]

Then we have the following result.

Lemma 6. Suppose that \( L \) satisfies (H1) and assume that the notation of the above lemmas is used. Then, given \( \varepsilon > 0 \), there is a constant \( K > 0 \) such that
(i) \( |T(t)X_0^{P_i}| \leq Ke^{(\text{Re}\lambda_0 - \varepsilon)t}, \ t \geq 0, \) and
(ii) \( |T(t)X_0^{P_k}| \leq Ke^{(\text{Re}\lambda_0 - \varepsilon)t}, \ t \leq 0. \)

Note again that the above lemma shows that the Hartman-Wintner’s trichotomy (H1) condition implies a condition such as Condition A in [5].

3. AN ABSTRACT FORMULATION FOR FDES

Define the Banach space \( E = C^N \times C \) with the norm \( |(\xi, \varphi)| = \max\{|\xi|, |\varphi||\}. \) Consider
\[
S(t)(v, \varphi) = (T(t)(\tilde{\varphi})(0), T(t)(\tilde{\varphi}))
\]
where
\[
\tilde{\varphi}(\theta) = \begin{cases} v, & \text{if } \theta = 0, \\ \varphi(\theta), & \text{if } -r \leq \theta < 0. \end{cases}
\]

It is easily verified that \( \{S(t)\}_{t \geq 0} \) is a semigroup of bounded operators on \( E \), which is the Cauchy operator of the abstract equation
\[
\phi' = A\phi,
\]
where \( A \) is defined on \( \text{Dom}A = \{((\xi, \varphi)) \in C^N \times W^{1,2} : \varphi(0) = \xi\} \) by
\[
A(\xi, \varphi) = (L(\varphi), \tilde{\varphi}).
\]

Then the perturbed system (10) can be seen as the linear differential equation in the Banach space \( E \),
\[
z'(t) = (A + R(t))z(t) + F(t, z(t)),
\]
where \( R(t)(\xi, \varphi) = (R(t, \varphi), 0) \) and \( F(t, z(t)) = (f(t, y_t), 0) \).
Then it can be verified that $y$ is a solution of (10) if and only if $z(t) = (y(t), Y_t)$ is a solution of (24). In addition, $\sigma(A) = \sigma(A)$ (see [18]).

Now, we will study the asymptotic integration in (24). Let $Q_i : E \to E$, $i = 0, 1, 2$, be defined by

$$Q_i(e, \varphi) = (X_0^{P_i}(0)e, X_0^{P_i}\varphi).$$

Then the $Q_i$s are complementary projections on Range$\mathcal{R}$. $S(t)$ can be extended for all $t \in \mathbb{R}$ on the finite-dimensional spaces Range$Q_0$ and Range$Q_2$. By Lemma 6, given $\varepsilon > 0$ small enough, there exists a constant $K > 0$ such that

$$\|S(t)Q_0\| = \|A Q_0\|, \quad t \geq 0, \quad (25)$$

$$|S(t)Q_1| \leq Ke^{-\varepsilon t} |e^{\lambda_0 t}|, \quad t \geq 0, \quad (26)$$

$$|S(t)Q_2| \leq Ke^{-\varepsilon t} |e^{\lambda_0 t}|, \quad t \leq 0, \quad (27)$$

$$S(t)Q_0 = e^{\lambda_0 t}Q_0, \quad t \in \mathbb{R}, \quad (28)$$

where $\| \cdot \|$ is the operator norm on $E$.

**Remark 1.** Note the following.

(i) Equations (26)–(28) are as in Hypothesis A in [5].

(ii) We have that $|\mathcal{R}(t)| \in L^p$, for some $p : 1 \leq p \leq 2$ and

$$Q_0 \mathcal{R} Q_0 = r_0(t) Q_0,$$

where $r_0(t) = c_1 v^* \cdot R(t, e^{\lambda_0 t} v)$ and $c_1$ is given by (19). This is similar to Hypothesis B in [5].

(iii) There is a function $\gamma : \mathbb{R}_+^+ \to \mathbb{R}$ in $L^1$ such that

$$|\mathcal{F}(t, \hat{e}_1) - \mathcal{F}(t, \hat{e}_2)| \leq \gamma(t) |\hat{e}_1 - \hat{e}_2|, \quad \mathcal{F}(t, 0) = 0, \quad (29)$$

for all $t \in \mathbb{R}_+^+$ and $\hat{e}_1, \hat{e}_2 \in \text{Dom } A$.

**4. SOME LEMMAS AND ESTIMATES**

**Lemma 7.** (See [2].) Let $\Psi \in L^p$, $p \geq 1$, be a nonnegative and continuous function on $\mathbb{R}_+$. For $t \geq 0$ and $\varepsilon > 0$, define

$$\varphi(t, \varepsilon) = \int_0^t \Psi(s)e^{-\varepsilon(t-s)} ds, \quad \zeta(t, \varepsilon) = \int_t^{+\infty} \Psi(s)e^{\varepsilon(t-s)} ds.$$

Then

(i) $\varphi(t, \varepsilon) \to 0$, $\zeta(t, \varepsilon) \to 0$ as $t \to +\infty$, and

(ii) $\varphi(\cdot, \varepsilon) \in L^q$, $\zeta(\cdot, \varepsilon) \in L^q$, if $q \geq p$.

**Remark 2.** Part (ii) is satisfied when $1/p + 1/q = 1$ and $p \leq 2$.

**Lemma 8.** Let $r \in L^p$, $c > 1$, and $\varepsilon > 0$. Then we have for $t > s > \sigma$,

$$\left| \exp \left( \int_s^t r \right) \right| \leq ce^{(\varepsilon/2)(t-s)},$$

for $\sigma$ sufficiently large.

Now, we consider equation (24) as a perturbation of the abstract differential equation

$$u'(t) = (A + \mathcal{R}_0)u(t), \quad (30)$$

where

$$\mathcal{R}_0 = Q_0 \mathcal{R}.$$  

Denote by $U$ its Cauchy operator. We begin with a calculation of $U$. From now on, we denote

$$\tilde{\lambda} = \lambda_0 + r_0,$$

where $\lambda_0$ is given in (H1) and $r_0$ in Remark 1(ii).
LEMMA 9. We have that

\[ U(t, s)Q_0 = \exp \left( \int_s^t \frac{d}{dt} \right) Q_0 \]  

and for \( i = 1, 2 \),

\[ U(t, s)Q_i = S(t - s)Q_i + \exp \left( \int_s^t \frac{d}{dt} \right) \left[ \int_s^t R_0(\xi) e^{-\int_s^\xi \lambda S(\xi - s)Q_1 d\xi} \right]. \]  

PROOF. By applying \( I - Q_0 \) and \( Q_0 \) to the left side of the equality

\[ \frac{d}{dt} U(t, s) = (A + R_0)U(t, s), \]

we obtain

\[ (I - Q_0)U(t, s) = S(t - s)Q_1 + S(t - s)Q_2, \quad i = 1, 2, \]

and

\[ \frac{d}{dt} Q_0 U(t, s) = \lambda(t)Q_0 U(t, s) + R_0(t)S(t - s)Q_1 + R_0(t)S(t - s)Q_2. \]

So, by the variation of constant formula, we obtain (33) and (34).

Let \( U_{01}(t, s) \) and \( U_{02}(t, s) \) be the operators defined by

\[ U_{01}(t, s)Q_1 = \exp \left( \int_s^t \frac{d}{dt} \right) \left[ \int_s^t R_0(\xi) e^{-\int_s^\xi \lambda S(\xi - s)Q_1 d\xi} \right], \]

\[ U_{02}(t, s)Q_1 = \exp \left( \int_s^t \frac{d}{dt} \right) \left[ \int_s^t R_0(\xi) e^{-\int_s^\xi \lambda S(\xi - s)Q_1 d\xi} \right]. \]

REMARK 3. We observe that

\[ Q_0 U(t, s)Q_1 = U_{01}(t, s)Q_1 + U_{02}(t, s)Q_1. \]

Thus, \( U \) has been split into four parts:

\[ U(t, s) = (I - Q_0)U(t, s)Q_1 + U_{01}(t, s)Q_1 + U_{02}(t, s)Q_1 + U(t, s)Q_2. \]

Those parts will be estimated in the following lemmas where Hypotheses (H1) and (H2) will be used.

LEMMA 10. For \( \sigma \) sufficiently large, there is a constant \( c \geq 1 \) such that

(i) \[ | \exp(-\int_s^t \lambda S(t - s)Q_1) | \leq ce^{-(\epsilon/2)(t-s)}, \quad t \geq \sigma, \quad \text{and} \]

(ii) \[ | \exp(-\int_s^t \lambda S(t - s)Q_2) | \leq ce^{(\epsilon/2)(t-s)}, \quad \sigma \leq t \leq s. \]

PROOF. Part (i) follows from

\[ \left| \exp \left( - \int_s^t \lambda \right) S(t - s)Q_1 \right| = \left| \exp \left( - \int_s^t \lambda_0 \right) S(t - s)Q_1 \right| \cdot \exp \left( \int_s^t |r_0| \right) \]

\[ \leq e^{-\epsilon(t-s)} \exp \left( \int_s^t |r_0| \right), \]

from (26) and by Lemma 8. Part (ii) follows similarly.
Lemma 11. There is a constant $K > 0$ such that

(i) $|Q_1U(t, s)Q_1| \leq K |\exp(\int_s^t \lambda)\| \cdot e^{-(s-t)}(t-s)\|$, and

(ii) $|U_0(t, s)Q_1| \leq K |\exp(\int_s^t \lambda)| \int_s^t \infty |\mathcal{R}_0(\xi)| e^{-(s-t)}(t-s) d\xi|$, for $t \geq s \geq \sigma$ and $\sigma$ sufficiently large.

Proof. From (34), $(I - Q_0)U(t, s)Q_1 = S(t - s)Q_1$, and (i) is obtained using Lemma 10(i). To prove Part (ii), by Lemma 10(i), there is a constant $K_1 > 0$ such that

$$|U_0(t, s)Q_1| \leq K_1 \left| \exp\left(\int_s^t \lambda\right) \left| \int_s^t \infty |\mathcal{R}_0(\xi)| e^{-(s-t)}(t-s) d\xi\right|, \quad t \geq s,$$

and thus, (ii) is obtained.

Lemma 12. There is a constant $K > 0$ such that

(i) $|U_0(t, s)Q_2| \leq K |\exp(\int_s^t \lambda)| \int_s^t \infty |\mathcal{R}_0(\xi)| e^{-(s-t)}(t-s) d\xi|$, and

(ii) $|U(t, s)Q_2| \leq K |\exp(\int_s^t \lambda)| \int_s^t \infty |\mathcal{R}_0(\xi)| e^{-(s-t)}(t-s) d\xi|$, for $s \geq t \geq \sigma$ and $\sigma$ sufficiently large.

Proof. By Lemma 10(i), there is a constant $K_3 > 0$ such that

$$|U_0(t, s)Q_1| \leq K_3 \left| \exp\left(\int_s^t \lambda\right) \left| \int_s^t \infty |\mathcal{R}_0(\xi)| e^{-(s-t)}(t-s) d\xi\right|,$$

and hence, (i) is obtained. To prove Part (ii), by (34),

$$U(t, s)Q_1 = \exp\left(\int_s^t \lambda\right) \left(\exp(-\int_s^t \lambda)S(t - s)Q_1 + \left(\int_s^t \mathcal{R}_0(\xi)e^{-\int_s^t \lambda}S(\xi - s)Q_1 d\xi\right)\right).$$

So, from Lemmas 8 and 10, we obtain Part (ii), for $s \geq t \geq \sigma$ and $\sigma$ sufficiently large.

Let $C_{\sigma} = C([\sigma, +\infty[, \mathbb{E})$ and define the operator $N$ defined on $C_{\sigma}$ by

$$(Nz)(t) = U(t, \sigma)w_0 + \int_{\sigma}^t (I - Q_0)U(t, s)Q_1 \mathcal{R}(s)z(s) ds - \int_t^\infty U(t, s)Q_2 \mathcal{R}(s)z(s) ds$$

$$+ \int_{\sigma}^t U_0(t, s)Q_1 \mathcal{R}(s)z(s) ds - \int_t^\infty U_0(t, s)Q_1 \mathcal{R}(s)z(s) ds$$

$$+ \int_{\sigma}^t Q_1U(t, s)\mathcal{F}(s, z(s)) ds - \int_t^\infty (Q_0 + Q_2)U(t, s)\mathcal{F}(s, z(s)),$$

where $\mathcal{R} = \mathcal{R} - \mathcal{R}_0$. 

Lemma 13. Any fixed point of $N$, defined above, is a solution of system (24).

Proof. By Remark 3 and (36),

$$Q_0U(t, s)Q_1 = U_0(t, s)Q_1 + \exp\left(\int_{t_0}^t \lambda\right) \mathcal{H}(s),$$

where $\mathcal{H}(s) = \exp(-\int_{t_0}^t \lambda)[\int_s^t \infty \mathcal{R}_0(\xi)e^{-\int_{t_0}^t \lambda}S(\xi - s)Q_1 d\xi]$. Then,

$$(Nz)(t) = \exp\left(\int_{t_0}^t \lambda\right) w_1(z) + \int_{t_0}^t U(t, s)Q_1 \mathcal{R}(s)z(s) ds - \int_t^\infty U(t, s)Q_2 \mathcal{R}(s)z(s) ds$$

$$+ \int_{t_0}^t Q_1U(t, s)\mathcal{F}(s, z(s)) ds - \int_t^\infty (Q_0 + Q_2)U(t, s)\mathcal{F}(s, z(s)),$$

where $w_1(z) = w_0 + \int_{t_0}^\infty \mathcal{H}(s)\mathcal{R}(s)z(s) ds$. By the constant variation formula, we obtain this result.
5. ASYMPTOTIC INTEGRATION

Now we can prove the following functional version of the Hartman-Wintner theorem.

**THEOREM 1.** Assume that (H1)-(H3) hold. Then (24) has a solution \( z_0 \in C(\sigma, +\infty, E) \), for \( \sigma \) large enough, satisfying

\[
\int_0^t \left[ \lambda_0 + \frac{1}{v^* \cdot v} \cdot L(e^{\lambda_0}v) \right] ds \left( w_0 + o(1) \right),
\]

as \( t \to \infty \), where \( w_0 = (\varphi_0(0), \varphi_0) \).

**PROOF.** Let \( C_\sigma = C(\sigma, +\infty, E) \) and consider the operator \( N \) defined on \( C_\sigma \) by (39) where \( \mathcal{R} = \mathcal{R} - \mathcal{R}_0 \). By Lemma 13, any fixed point of \( N \) is a solution of system (24). Hence, we will show that the operator \( N \) has a fixed point satisfying (40). Consider the norm

\[
\|z\| = \sup_{t \geq \sigma} \left| \exp \left( - \int_\sigma^t \lambda_0 \right) z(t) \right|
\]

and the Banach space \( \mathcal{C} = \{ z \in C_\sigma : \|z\| < +\infty \} \). Then we have

(i) \( N(\mathcal{C}) \subseteq \mathcal{C} \),

(ii) \( N \) has a unique fixed point \( z_0 \) in \( \mathcal{C} \), and

(iii) \( z_0 \) satisfies (40).

In fact, for a convenient constant \( K \), define

\[
H_1^\tau(t) = K \int_\sigma^t e^{-(\sigma/2)(t-s)} |\mathcal{R}(s)| ds,
\]

\[
H_2^\tau(t) = K \left[ \int_\sigma^t |\mathcal{R}(s)| e^{-(\sigma/2)(t-s)} ds \right] \left[ \int_t^{+\infty} |\mathcal{R}_0(\xi)| e^{-(\sigma/2)(\xi-t)} d\xi \right],
\]

\[
H_3^\tau(t) = K \int_t^{+\infty} |\mathcal{R}(s)| \left( \int_t^{+\infty} |\mathcal{R}_0(\xi)| e^{-(\sigma/2)(\xi-s)} d\xi \right) ds,
\]

\[
H_4^\tau(t) = K \left[ \int_t^{+\infty} e^{(\sigma/2)(t-s)} |\mathcal{R}(s)| ds + \int_t^{+\infty} \left( \int_\sigma^s |\mathcal{R}_0(\xi)| e^{-(\sigma/2)(s-\xi)} d\xi \right) |\mathcal{R}(s)| ds \right],
\]

\[
H_5^\tau(t) = K \int_\sigma^t e^{-(\sigma/2)(t-s)} |\gamma(\tau)| ds \quad \text{and} \quad H_6^\tau(t) = K \int_t^{+\infty} \gamma(s) ds.
\]

From Lemmas 10-12, we have

\[
\left| \int_\sigma^t (I - Q_0) U(t, s) Q_1 \mathcal{R}(s) z(s) ds \right| \leq \exp \left( \int_\sigma^t \lambda_0 \right) \|H_7^\tau(t)\| z(s),
\]

\[
\left| \int_\sigma^t U_{01}(t, s) Q_1 \mathcal{R}(s) z(s) ds \right| \leq \exp \left( \int_\sigma^t \lambda_0 \right) \|H_7^\tau(t)\| z(s),
\]

\[
\left| \int_t^{+\infty} U_{02}(t, s) Q_1 \mathcal{R}(s) z(s) ds \right| \leq \exp \left( \int_\sigma^t \lambda_0 \right) \|H_7^\tau(t)\| z(s),
\]

and finally,

\[
\left| \int_t^{+\infty} U(t, s) Q_2 \mathcal{R}(s) z(s) ds \right| \leq \exp \left( \int_\sigma^t \lambda_0 \right) \left[ K \int_t^{+\infty} e^{(\sigma/2)(t-s)} |\mathcal{R}(s)| ds \right.
\]

\[
+ \left. \int_t^{+\infty} \left( \int_\sigma^s |\mathcal{R}_0(\xi)| e^{-(\sigma/2)(s-\xi)} d\xi \right) |\mathcal{R}(s)| ds \right] \|z\|,
\]

\[
\leq \exp \left( \int_\sigma^t \lambda_0 \right) \|H_7^\tau(t)\| z(s),
\]
for all \( z \in \mathbb{C} \) and \( \sigma \) sufficiently large. Moreover, if \( H^\sigma = H^\sigma_1 + H^\sigma_2 + H^\sigma_3 + H^\sigma_4 + H^\sigma_5 + H^\sigma_6 \), then by (12) and (39),

\[
|Nz_1(t) - Nz_2(t)| \leq \left| \exp\left( \int_0^t \dot{\lambda} \right) \right| H^\sigma(t) |z_1 - z_2|,
\]

(41) for all \( z_1, z_2 \in \mathbb{C} \). By Lemma 7, \( H^\sigma(t) \to 0 \) as \( t \to +\infty \). From (41) with \( z_2 = 0 \), there is a constant \( c > 0 \) such that

\[
|Nz(t) - \exp\left( \int_0^t \dot{\lambda} \right) w_0| \leq c |\exp\left( \int_0^t \dot{\lambda} \right)| |z|
\]

and

\[
\lim_{t \to +\infty} |\exp\left( - \int_0^t \dot{\lambda} \right) Nz(t) - w_0| = 0.
\]

(42)

Hence, \( Nz \in \mathbb{C} \), for all \( z \in \mathbb{C} \), and (i) is proved. Now, from (41), if we take \( \sigma \) so large that \( \sup_{t \geq \sigma} H^\sigma(t) < 1 \), we see that \( N \) is a contraction and by Banach's fixed-point theorem, \( N \) has a fixed point \( z_0 \). The limit (42) proves (iii).

Taking the first coordinate in (40) and using the definition of \( \dot{\lambda} \) in (32), we have the following results.

**THEOREM 2.** Suppose that system (10) satisfies (H1)-(H3). Then, for \( \sigma \) large enough, there is a solution of (10) satisfying

\[
y(t) = \exp\left( \lambda_0 t + \frac{1}{v^* \cdot v - v^* \cdot L(e^{\lambda_0} \cdot v)} \int_0^t v^* \cdot R(s, e^{\lambda_0} \cdot v) \, ds \right) (v + o(1)),
\]

(43)
as \( t \to +\infty \).

Proceeding as in Theorem 2 of [5], we have the following theorem.

**THEOREM 3.** Suppose that in the system

\[
y'(t) = L(y_t) + \sum_{i=1}^m R_i(t, y_t) + f(t, y_t),
\]

(44)

\( L \) satisfies (H1), \( R_i(t, \cdot) \in L^{p_i} \), for some \( p_i : 1 \leq p_i \leq 2 \), \( i = 1, \ldots, m \), and there is a function \( \gamma : \mathbb{R}^*_0 \to \mathbb{R} \) in \( L^1 \) such that

\[
|f(t, \varphi) - f(t, \psi)| \leq \gamma(t)|\varphi - \psi|, \quad f(t, 0) = 0,
\]

for all \( t \in \mathbb{R}^*_0 \), \( \varphi, \psi \in \mathbb{C} \). Then, for \( \sigma \) large enough, there is a solution of (44) satisfying

\[
y(t) = \exp\left( \lambda_0 t + \frac{1}{1 - v^* \cdot L(e^{\lambda_0} \cdot v)} \int_0^t v^* \sum_{i=1}^m R_i(s, e^{\lambda_0} \cdot v) \, ds \right) (v + o(1)),
\]

(45)
as \( t \to +\infty \), where \( v \in \mathbb{C}^N \setminus \{0\} \) and \( v^* \in \mathbb{C}^{N^*} \setminus \{0\} \) are such that \( L(e^{\lambda_0} \cdot v) = \lambda_0 v \) and \( v^* \Delta(\lambda_0) = \lambda_0 v^* \).

### 6. SOME CONSEQUENCES AND APPLICATIONS

First, we present a simpler generalization of the Bellman-Cooke's result to several \( L^p \) perturbations.
COROLLARY 1. Consider the delay differential equation

\[ y'(t) = \sum_{i=1}^{n} \left( b_i + \tilde{b}_i(t) \right) y(t - r_i), \quad r_i \geq 0, \tag{46} \]

such that its characteristic equation

\[ \lambda = \sum_{i=1}^{n} b_i e^{-\lambda r_i} \quad \tag{47} \]

has a simple root \( \lambda = \lambda_0 \) with the real part different from the real part of the other roots. Suppose that \( \tilde{b}_i \) are in \( L^{p_i} \), for some \( p_i : 1 \leq p_i \leq 2 \) and \( i = 1, \ldots, n \). Then (46) has a solution of the form

\[ y(t) = \exp \left( \lambda_0 t + c_1 \int_{t_0}^{t} \left[ \sum_{i=1}^{n} \tilde{b}_i(s) e^{-\lambda_0 r_i} \right] ds \right) (1 + o(1)), \tag{48} \]
as \( t \to +\infty \) where \( c_1 = 1/(1 - \sum_{i=1}^{n} b_i r_i e^{-\lambda_0 r_i}) \).

PROOF. Take \( L(\phi) = \sum_{i=1}^{n} b_i \phi(-r_i) \) and \( R(t, \phi) = \sum_{i=1}^{n} \tilde{b}_i(t) \phi(-r_i) \), for all \( \phi \in \mathcal{C}([- \max_{i=1,\ldots,n} 0, 0], \mathbb{C}^N) \). The characteristic equation (47) satisfies (H1). Since \( \tilde{b}_i \in L^{p_i} \), for some \( p_i : 1 \leq p_i \leq 2 \), (H2) is satisfied. Therefore, from Theorem 3, (48) is obtained.

Actually, the same result is true for the following systems.

COROLLARY 2. Consider the \( N \times N \) delay differential system,

\[ y'(t) = \sum_{i=1}^{n} \left( B_i + \tilde{B}_i(t) \right) y(t - r_i), \quad r_i \geq 0, \tag{49} \]
such that the characteristic equation \( \det \Delta_0(\lambda) = 0 \), where

\[ \Delta_0(\lambda) = \lambda I - \sum_{i=1}^{n} B_i e^{-\lambda r_i} \]

has a simple root \( \lambda = \lambda_0 \) and no other root has the same real part. Suppose that \( \tilde{B}_i \in L^{p_i} \), for some \( p_i : 1 \leq p_i \leq 2 \). Then, for \( \sigma \) large, (49) has a solution \( y_0(t) \) such that

\[ y_0(t) = \exp \left( \lambda_0 t + c_1 \int_{\sigma}^{t} \left[ \sum_{i=1}^{n} e^{-\lambda_0 r_i} \tilde{B}_i(s) \right] \cdot v ds \right) (v + o(1)), \tag{50} \]

where \( c_1 = 1/(v^* \cdot v - v^* \cdot [r_i B_i e^{\lambda_0 r_i} \cdot v]) \) and \( v \in \mathbb{C}^N \) and \( v^* \in \mathbb{C}^{N*} \) are eigenvectors such that \( \Delta_0(\lambda_0) v = 0 \) and \( v^* \Delta(\lambda_0) = 0 \).

PROOF. Same proof as in the previous corollary, but taking \( L(\phi) = \sum_{i=1}^{n} B_i \phi(-r_i) \) and \( R(t, \phi) = \sum_{i=1}^{n} \tilde{B}_i(t) \phi(-r_i) \), for all \( \phi \in \mathcal{C}([- \max_{i=1,\ldots,n} 0, 0], \mathbb{C}^N) \).

COROLLARY 3. Consider the \( N \times N \) delay differential system,

\[ y'(t) = (A_0 + A(t))y(t) + \sum_{i=1}^{n} (B_i + \tilde{B}_i(t)) y(t - r_i), \tag{51} \]
such that the characteristic equation

\[ \det \Delta_0(\lambda) = 0, \]
where
\[ \Delta_0(\lambda) = \lambda I - \sum_{i=1}^{n} B_i e^{-\lambda r_i} \]
has a simple root \( \lambda = \lambda_0 \) and no other root has the same real part. Suppose that \( A_0 \in L^p_0 \) and \( B_i \in L^p \) are \( N \times N \) matrix-valued functions in \( L^p \), and \( F \) is linear in the second variable with \( |F(t, \cdot)| \in L^{p+1} \), for some \( 1 \leq p_i \leq 2 \), \( i = 0, 1, \ldots, m, m + 1 \). Then, for \( \sigma \) large enough, \( (51) \) has a solution \( y_0 \) such that
\[
y_0(t) = (v + o(1)) \exp \left( \lambda_0 t + c_1 \int_{\sigma}^{t} v^* \cdot \left[ A(s) + \sum_{i=1}^{m} e^{-\lambda_0 r_i} B_i(s) \right] \cdot v \, ds \right. \\
+ c_1 \int_{\sigma}^{t} v^* \cdot F(s, e^{\lambda_0^*} v) \, ds \right),
\]
where \( c_1 = 1/(v^* \cdot v - v^* \cdot A_0 v - v^* \cdot \sum_{i=1}^{m} r_i B_i e^{\lambda_0 r_i} v) \) and \( v \in C^N - \{0\} \), \( v^* \in C^{N*} - \{0\} \) are eigenvectors such that \( \Delta_0(\lambda_0) v = 0 \) and \( v^* \Delta_0(\lambda_0) = 0 \).

**Proof.** Again, take \( L(\phi) = A_0 \phi(0) + \sum_{i=1}^{m} r_i B_i e^{\lambda_0 r_i} \phi(0) + \int_{\sigma}^{t} B_i(t) \phi(0) \, ds + F(t, y_0) \), for all \( \phi \in C([-\max_{i=1, \ldots, m, \sigma} r_i, 0], C^N) \). The conclusion follows by using the same method given in the proof of Theorem 3.

The next corollary is a direct application of this matricial version and it was a conjecture which has been worked by several mathematicians. (See \([12,13,19,20]\).)

**Corollary 4.** (Haddock and Sacker conjecture.) Consider the \( N \times N \) delay system
\[
x'(t) = (A_0 + A(t)) x(t) + B(t) x(t - r),
\]
where
\[ A_0 = \text{diag}(\lambda_1, \ldots, \lambda_N), \quad \text{Re} \lambda_i \neq \text{Re} \lambda_j, \quad \text{if } i \neq j,
\]
and \( A, B \in L^p \), for some \( 1 \leq p \leq 2 \). Then, given \( c \in C^N \), for \( \sigma \) large enough, \( (53) \) has a solution \( y_c \) such that
\[
y_c(t) = (c + o(1)) \exp \left( \int_{\sigma}^{t} (A_0 + \text{diag} A(s) + e^{-\lambda_0 r} \text{diag} B(s)) \, ds \right),
\]
as \( t \to \infty \).

Consider the nonlinear differential equation with infinite number of delays
\[
y'(t) = \left( a + \frac{1}{t} \right) y(t - 1) + \sum_{n=1}^{\infty} \frac{1}{n!} \sin \left( \frac{2^n}{t^2} y \left( t - \frac{1}{n} \right) \right).
\]
Suppose that the characteristic equation
\[ \lambda = a e^{-\lambda} \]
has a simple root \( \lambda = \lambda_0 \) and no other root has the same real part (it is easy to check that many roots satisfying that can be found). Then \( (55) \) has a solution \( y_0(t) \) such that
\[
y_0(t) = e^{\lambda_0 t^2/(1+ae^{-\lambda})} (1 + o(1)),
\]
as \( t \to +\infty \).

In fact, we apply Theorem 2 to \( L(\varphi) = a \varphi(-1), \quad R(t, \varphi) = (1/t) \varphi(-1), \) and \( f(t, \varphi) = \sum_{n=1}^{\infty} (1/n!) \sin((2^n/t^2) \varphi(-n)), \varphi \in C([-1, 0], C^N) \).

It is easy to check that \( L \) and \( R \) satisfy (H1) and (H2), respectively. Also, \( f \) satisfies (H3) considering
\[ \gamma(t) = \left( \sum_{n=1}^{\infty} \frac{2^n}{n!} \right) \frac{1}{t^2} \in L^1. \]
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