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The Number of Rooted 2-Connected Triangular Maps on the Projective Plane

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In this paper we enumerate the rooted 2-connected triangular maps on the projective plane by number of vertices. A parametric expression of the generating function is obtained for such maps, from which asymptotics and a recursion are derived.

1. INTRODUCTION

A (rooted) near-triangular map on a surface is a (rooted) map on the surface [2] such that each face except possibly the root face has valency three. A (rooted) triangular map is a (rooted) near-triangular map with root face valency also being three. Although much work was done on planar maps after Tutte's breakthrough in the 1960s [9], only Brown [6] and Walsh and Lehman [10] obtained results on non-planar maps until recent work of Bender, Canfield, Robinson, and Wormald (cf. [2–4]).

Let s_{ij} and p_{ij} be the number of rooted loopless near-triangular maps on the sphere and projective plane, respectively, having i vertices and root face valency j. Define

$$S(x, y) = \sum_{i,j} s_{ij} x^{i-1} y^j = \sum_j S_j(x)\, y^j$$

and

$$P(x, y) = \sum_{i,j} p_{ij} x^{i-1} y^j = \sum_j P_j(x)\, y^j,$$

where $S_j(x)$ and $P_j(x)$ are the generating functions for all rooted loopless near-triangular maps with root face valency j on the sphere and projective plane, respectively. We prove that loopless triangular maps are 2-connected and, in Sections 2 and 3, that

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THEOREM 1.

$$P_3(x)=\frac{1}{2t}\left(1-\sqrt{\frac{1-6t}{1-2t}}\right)-\frac{1}{1-3t}, \tag{1.1}$$

where t is defined by $x=t(1-2t)^2$ and $t(0)=0$.

We then study $p_{n,3}$. Theorem 1 is used in Section 4 to obtain a recursion and a table and in Section 5 to obtain an asymptotic formula, which fits the pattern conjectured in [5]. In [7] we obtain asymptotic formulas for rooted 2-connected triangular maps on all surfaces. They also fit the conjectured pattern.

The arguments x and y are usually omitted from now on. We begin with

LEMMA 1. *Let T be a triangular map on a surface, then the following three statements are equivalent:*

(1) *T is loopless.*

(2) *T is nonsingular; i.e., the boundary of every face is a simple closed curve.*

(3) *T is 2-connected.*

Proof. (1) $\Rightarrow$ (2). Since each face of T has valency three, any closed walk along the boundary of a face can be expressed as $v_1v_2v_3v_1$. If T is loopless, then v_1, v_2, and v_3 must be distinct. Thus, the boundary of any face of T is a simple closed curve.

(2) $\Rightarrow$ (3). This is true for general maps [4].

(3) $\Rightarrow$ (1). This follows immediately from the definition of 2-connected. ∎

2. FUNCTIONAL EQUATIONS FOR S AND P

In this section, we prove:

LEMMA 2. *$S_3=S_2-x$ and $P_3=P_2$.*

THEOREM 2.

$$S=xy^2S^2+y^{-1}(S-1-y^2S_2S)+1 \tag{2.1}$$

and

$$P=2xy^2SP+y^2(yS)_y-y^2S^2+y^{-1}(P-y^2S_2P-y^2SP_2-L), \tag{2.2}$$

where

$$L = x^{-1}[S - 1 - y^2 S_2 S - y(S_2 - x)(S - 1)/S_2]. \tag{2.3}$$

Proof (*Lemma*). Let T be any rooted loopless triangular map with root edge (v, v_1) Add an edge joining v and v_1 in the root face. The resulting map is a rooted loopless near-triangular map with root face valency two and more than one edge. Conversely, given any rooted loopless near-triangular map with root face valency two and more than one edge, deleting the nonroot edge in the root face, gives a rooted loopless triangular map (see Fig. 2.1.). Note that a single nonloop edge is a map only on the sphere. ∎

Proof (*Theorem*). We reason as in [3], with which we assume familiarity. Since a single vertex is a map only on the sphere, which is counted by the last term 1 in (2.1), we will assume that our map has edges from now on. Let e be the root edge of a loopless near-triangular map T. Recall that e is called a "double edge" if the same face appears on both sides of it. As in [3], we have the following cases and subcases.

Case A. *e Is a Double Edge*

Make a cut along a simple closed curve which lies within the root face except where it intersects e and fill in any resulting holes with discs. As in [3] we have the following two subcases.

Subcase A_1. The cut creates two maps.

Reversing the process as in [3] (see Fig. 2.2.), we obtain the contributions

$$S\colon xy^2S^2 \qquad P\colon xy^2SP + xy^2PS.$$

Subcase A_2. The cut destroys a cross cap.

This only happens for the projective plane. Reversing the process as in [3] (see Fig. 2.3.), we obtain the contributions

$$S\colon \text{nothing} \qquad P\colon y^2(yS)_y,$$

but this includes the situation shown in Fig. 2.4 which introduces a loop. This arises by taking an ordered pair of maps on the sphere, merging their root vertices, and introducing e. Subtracting this, we obtain the contributions

$$S\colon \text{nothing} \qquad P\colon y^2(yS)_y - y^2S^2.$$

Case B. *e Is a Single Edge*

Note that the non-root side of e must border a triangle. After removing e, the triangle is connected to the root face. As in [3], we can reverse this

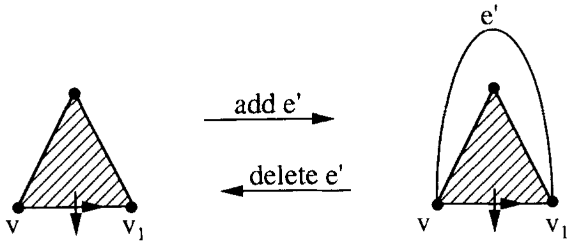


FIG. 2.1. The bijection between root face valencies 2 and 3.

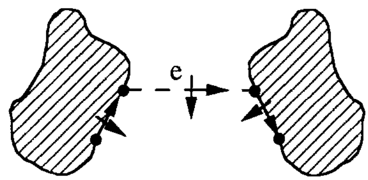


FIG. 2.2. Subcase A_1. Join two maps.

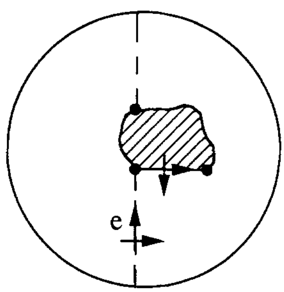


FIG. 2.3. Subcase A_2. Cross the cross-cap. (Antipodal points on the circle form the cross-cap.)

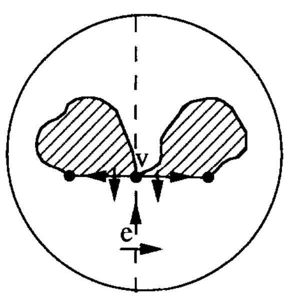


FIG. 2.4. Subcase A_2. A loop was formed.

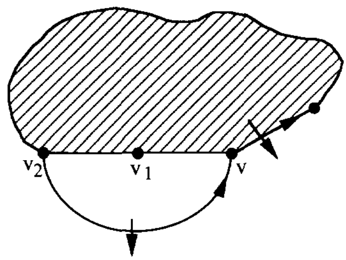


FIG. 2.5. Case B. Add a single edge.

process by taking a rooted loopless near-triangular map M and introducing a new root edge running to the root vertex v from the vertex v_2 two edges back as shown in Fig. 2.5. If $v_2 = v$, a loop will be introduced. Letting L_S, L_P be the generating function for those M with $v_2 = v$ on the sphere and the projective plane, respectively, we obtain the contributions

$$S\colon y^{-1}(S-1-L_S) \qquad P\colon y^{-1}(P-L_P).$$

Combining these cases with the following lemma proves the theorem. ∎

LEMMA 3. *$L_S = y^2 S_2 S$ and $L_P = y^2 S_2 P + y^2 S P_2 + L$, where*

$$L = x^{-1}[S-1-y^2 S_2 S - y(S_2 - x)(S-1)/S_2].$$

Proof. Make a cut along a simple closed curve that lies in the root face of M except where it cuts through v, as shown in Fig. 2.6. Let v' be the vertex split from v. After splitting v into v and v', we have a situation like Case A and its subcases.

Subcase A_1'. The cut creates two maps.
As in Subcase A_1, we obtain the contributions

$$S\colon y^2 S_2 S \qquad P\colon y^2 S_2 P + y^2 S P_2.$$

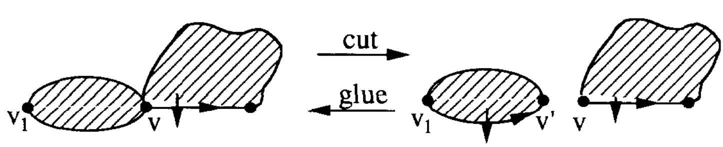


FIG. 2.6. Case A_1'. Two maps were formed.

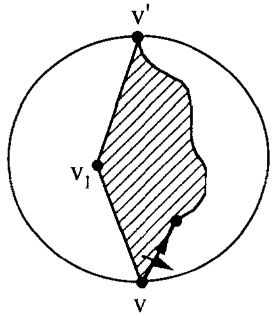


FIG. 2.7. Subcase A_2'. The cross-cap was cut along the circle.

Subcase A_2'. The cut destroys a cross cap.

The resulting map lies on the sphere and the vertex v' that occurs two edges before the root vertex v is different from v. The process is reversible as shown in Fig. 2.7, except that an edge connecting v and v' would become a loop. Therefore, the term corresponding to this case is

$$S\text{: nothing} \qquad P\text{: } L = x^{-1}(S - 1 - y^2 S_2 S - J), \tag{2.4}$$

where 1 corresponds to the single vertex map, $y^2 S_2 S$ corresponds to maps with $v = v'$ as shown in Fig. 2.6, and J corresponds to the maps with some edges joining v and v' as shown in Fig. 2.8.

Now we analyze J. Let e_1 be the leftmost edge joining v and v'. Cutting along e_1, and splitting e_1 into two edges, we get two separate pieces. The left piece is a triangular map with no edges other than e_1 joining v and v'. The right piece is a general rooted loopless near-triangular map, except that it cannot be the single vertex map. Let Δ be the generating function for maps like the left piece. Then

$$J = \Delta(S - 1)/xy^2. \tag{2.5}$$

Now we analyze Δ. Notice that the edge splitting process just described splits any map counted by S_3 into a map counted by Δ and a map counted

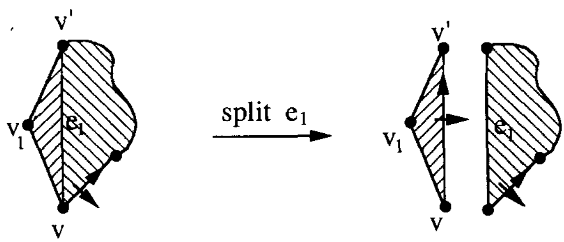


FIG. 2.8. Counting maps with edges $\{v, v'\}$. Edge e_1 is the leftmost such edge.

by S_2. Thus $y^3S_3 = \Delta y^2S_2/xy^2$. By Lemma 2, $\Delta = xy^3(S_2 - x)/S_2$. Combining this with (2.4) and (2.5) completes the proof. ▮

3. Parametric Equations

In this section we prove (1.1).

Rewriting (2.1) and (2.2) we have

$$A^2 = B \tag{3.1}$$

and

$$AP = -y^3(yS)_y + y^3S^2 + y^2SP_2 + L, \tag{3.2}$$

where

$$A = 2xy^3S + 1 - y - y^2S_2 \tag{3.3}$$

and

$$B = (y^2S_2 + y - 1)^2 - 4xy^3(y-1). \tag{3.4}$$

We use the Quadratic Method [9] to solve (3.1) and (3.2). Let $y = f(x)$ be such that $A(x, f(x)) = 0$, then

$$B(x, f) = (f^2S_2 + f - 1)^2 - 4xf^3(f-1) = 0 \tag{3.5}$$

and

$$B_y(x, f) = 2(f^2S_2 + f - 1)(2fS_2 + 1) - 4xf^2(4f-3) = 0. \tag{3.6}$$

Eliminating x from (3.5) and (3.6) and noting that $f^2S_2 + f - 1 \neq 0$ by (3.5), we obtain

$$\frac{f^2S_2 + f - 1}{f(f-1)} = \frac{2(2fS_2 + 1)}{4f - 3}.$$

Solving for S_2, substituting into (3.5), and setting $f = 1/(1-t)$, we obtain

$$x = t(1-2t)^2 \qquad \text{and} \qquad S_2 = t(1-3t). \tag{3.7}$$

Since $x = 0$ implies $S_2 = 0$, $x = 0$ implies $t = 0$. Thus $t(0) = 0$ defines the Riemann sheet for $t(x)$ near 0.

As an aside, we note that $s_{n,2}$ can be obtained from (3.7) by Lagrange inversion. Combining this with Lemmas 1 and 2 we obtain the known formula [9, p. 444] for the number of rooted 2-connected triangular maps, with n vertices, on the sphere:

$$s_{n,3}=\frac{2^{n-2}(3(n-2))!}{(n-1)!\,(2n-3)!} \qquad \text{for} \quad n\geqslant 3.$$

From (3.3) we have

$$A(x,f)=2xf^3S(x,f)+1-f-f^2S_2=0$$

and

$$A_y(x,f)=2xf^3S_y(x,f)+6xf^2S(x,f)-1-2fS_2.$$

(Hence $A_y(0,f(0))=-1$.) From (3.1),

$$2(A_y(x,f))^2=B_{yy}(x,f)=2(1-2t)(1-6t)$$

and hence

$$A_y(x,f)=-\sqrt{(1-2t)(1-6t)}.$$

(The negative square root arises because $A_y(0,f(0))=-1$.) Combining these results with (3.7), we get

$$S(x,f)=\frac{f^2S_2+f-1}{2xf^3}=\frac{1-t}{1-2t} \tag{3.8}$$

and

$$\begin{aligned} S_y(x,f)&=\frac{A_y(x,f)-6xf^2S(x,f)+1+2fS_2}{2xf^3}\\ &=\frac{(1-t)^3}{2t(1-2t)^2}\left(-\sqrt{(1-2t)(1-6t)}+\frac{(1-2t)(1-3t)}{1-t}\right). \end{aligned} \tag{3.9}$$

Setting $y=f$ in (3.2) and using Lemma 3, (3.7), (3.8), and (3.9), we obtain

$$\begin{aligned} P_2&=\frac{f^4S_y(x,f)+f^3S(x,f)-f^3S^2(x,f)-L(x,f)}{f^2S(x,f)}\\ &=\frac{1}{2t}\left(1-\sqrt{(1-6t)/(1-2t)}\right)-\frac{1}{1-3t}. \end{aligned}$$

By Lemma 2, (1.1) follows.

4. A Recursion for $p_{n,3}$

THEOREM 3. *The number of n vertex, rooted, 2-connected, triangular maps on the projective plane is*

$$p_{n,3}=q_{n,n-1}-8q_{n,n-2}+12q_{n,n-3}, \tag{4.1}$$

where $q_{n,m}$ is defined by the initial conditions

$$q_{0,0}=q_{0,1}=q_{n,-2}=q_{n,-1}=0$$

and the recursions

$$\begin{aligned}(m+1)\,q_{0,m}&=(8m-2)\,q_{0,m-1}-12(m-1)\,q_{0,m-2}\\&\quad+(m-1)\,3^{m-1}\qquad \textit{for}\quad m\geqslant 1\\ q_{n,m}&=q_{n-1,m}+4(q_{n,m-1}-q_{n,m-2})\qquad \textit{for}\quad n\geqslant 1\ \textit{and}\ m\geqslant 0.\end{aligned} \tag{4.2}$$

Proof. Let $f(z)|_{z^n}$ denote the coefficient of z^n in $f(z)$ and let $F(t)$ denote the right side of (1.1). Using (1.1) and Lagrange inversion we have

$$\begin{aligned}P_3(x)|_{x^n}&=\frac{1}{n}(1-2t)^{-2n}F'(t)\Big|_{t^{n-1}}\\&=\left[\int(1-2t)^{-2n}F'(t)\,dt\right]_{t^n}\\&=(1-2t)^{-2n}F(t)|_{t^n}-\left[\int 4n(1-2t)^{-2n-1}F(t)\,dt\right]_{t^n}\\&=(1-2t)^{-2n}F(t)|_{t^n}-4(1-2t)^{-2n-1}F(t)|_{t^{n-1}}\\&=(1-6t)(1-2t)^{-2n-1}F(t)|_{t^n}.\end{aligned}$$

With $q_{n,m}=(1-2t)^{-2n}F(t)|_{t^m}$ we have

$$\begin{aligned}P_3(x)|_{x^n}&=(1-6t)(1-2t)(1-2t)^{-2(n+1)}F(t)|_{t^n}\\&=(1-8t+12t^2)(1-2t)^{-2(n+1)}F(t)|_{t^n}\\&=q_{n+1,n}-8q_{n+1,n-1}+12q_{n+1,n-2}.\end{aligned}$$

Since $p_{n,3}=P_3(x)|_{x^{n-1}}$, we obtain (4.1). From the definition of $q_{n,m}$, we have

$$\begin{aligned}q_{n-1,m}&=(1-2t)^{-2(n-1)}F(t)|_{t^m}=(1-2t)^2(1-2t)^{-2n}F(t)|_{t^m}\\&=(1-2t)^{-2n}F(t)|_{t^m}-4t(1-2t)^{-2n}F(t)|_{t^m}\\&\quad+4t^2(1-2t)^{-2n}F(t)|_{t^m}\\&=q_{n,m}-4q_{n,m-1}+4q_{n,m-2},\end{aligned}$$

which gives the recursion for $q_{n,m}$. Note that for $m<0$, $q_{n,m}=0$.

Let $r(t) = -(1/2)\sqrt{(1-6t)/(1-2t)} = \sum r_m t^m$. Since

$$q_{0,m} = F(t)\,|_{t^m} = -\tfrac{1}{2}\sqrt{(1-6t)/(1-2t)}\,|_{t^{m+1}} - (1-3t)^{-1}\,|_{t^m},$$

we have $q_{0,m} = r_{m+1} - 3^m$. Notice that $(1-2t)(1-6t)\,r'(t) = -2r(t)$ and so

$$(m+1)\,r_{m+1} - 8mr_m + 12(m-1)\,r_{m-1} = -2r_m.$$

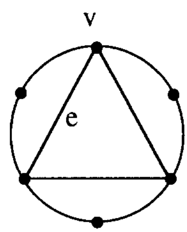
Combining these results, we obtain the recursion for $q_{0,m}$. ∎

We used the recursion to tabulate $p_{n,3}$ for $n \leqslant 20$. The 2-connected triangular maps on the projective plane for $n = 3, 4$ are shown in Fig. 4.1.

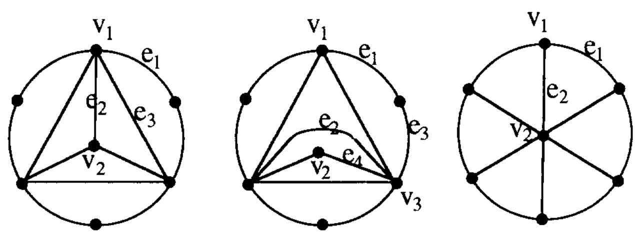
From Table I, we observe that $p_{n,3}$ is even iff n is a power of 2. Thus we have the following corollary:

COROLLARY 1. *$p_{n,3}$ is even iff n is a power of 2.*

Proof. In the following, all congruences are modulo 2. From (4.2), we have $q_{n,m} \equiv q_{n-1,m}$ for $m \geqslant 0$, $n \geqslant 1$. Hence $q_{n,m} \equiv q_{0,m}$ for $m, n \geqslant 0$. Combin-



n=3 : 1e



n=4 :

$2e_1 + 2e_2 + 2e_3$ $\qquad$ $4e_1 + 2e_2 + 1e_3 + 2e_4$ $\qquad$ $1e_1 + 2e_2$

FIG. 4.1. The 2-connected triangulations of the projective plane for $n = 3, 4$. (Antipodal points on the circle are identified to form the cross-cap.) One element of each orbit under the automorphic group is labeled and k_i in $\sum k_i e_i$ is the number of ways to root e_i.

TABLE I

The Number of n Vertex, Rooted, 2-Connected, Triangular Maps on the Projective Plane for $n \leqslant 20$

1	0	11	1,316,047,599
2	0	12	16,998,339,587
3	1	13	219,699,143,367
4	18	14	2,842,235,616,645
5	261	15	36,809,980,380,883
6	3,539	16	477,280,717,428,102
7	46,695	17	6,195,737,611,180,053
8	608,526	18	80,522,713,890,559,319
9	7,884,661	19	1,047,702,563,499,718,623
10	101,905,839	20	13,646,946,767,000,964,471

ing this with (4.1), we obtain $p_{n,3} \equiv q_{n,n-1} \equiv q_{0,n-1}$. By definition $q_{0,0} = 0$ and, for $n \geqslant 2$,

$$\begin{aligned} q_{0,n-1} &= -\frac{1}{2}\sqrt{(1-6t)/(1-2t)}\,\Big|_{t^n} - 3^{n-1} \\ &= -\frac{1-6t}{2}(1-4t(2-3t))^{-1/2}\,\Big|_{t^n} - 3^{n-1} \\ &= \left[-\frac{1-6t}{2}\sum\binom{-1/2}{k}(-4t(2-3t))^k\right]_{t^n} - 3^{n-1} \\ &= \left[-(1-6t)\sum\frac{(2k-1)!}{k!(k-1)!}(t(2-3t))^k\right]_{t^n} - 3^{n-1} \\ &\equiv \left[\sum\binom{2k-1}{k}t^{2k}\right]_{t^n} - 1. \end{aligned}$$

Therefore $p_{n,3} \equiv 1$ for n odd and $p_{n,3} \equiv \binom{2k-1}{k} - 1$ for $n = 2k$. Since $\binom{2k-1}{k}$ is odd iff k is a power of 2 (see [8, Exercise 1.2.6–11]), $p_{n,3}$ is even iff n is a power of 2. ∎

5. An Asymptotic Formula for $p_{n,3}$

We will prove

THEOREM 4. *The number of n vertex, rooted, 2-connected, triangular maps on the projective plane is asymptotic to*

$$\frac{3^{-7/4}}{2 \times \Gamma(3/4)} n^{-5/4} \left(\frac{27}{2}\right)^n. \tag{5.1}$$

Let $\sum f_n x^n \approx \sum g_n x^n$ mean that $f_n \sim g_n$, we have:

LEMMA 4. *Let $t = t(x)$ be the solution to $x = t(1-2t)^2$ with $t(0) = 0$. Then $t(x)$ is analytic at 0, and has a unique singularity at $x = 2/27$. Moreover,*

$$1 - 6t \approx \frac{2}{\sqrt{3}}\left(1 - \frac{27}{2}x\right)^{1/2}. \tag{5.2}$$

Proof. From $x = t(1-2t)^2$, it is evident that $t(x)$ is an algebraic function whose singularities are branch points. Let $F(x, t) = t(1-2t)^2 - x$. Since $F_t(x, t) = (1-2t)(1-6t)$ and $F_t(0, 0) = 1$, by the Implicit Function Theorem there exists a unique $t(x)$ which is analytic at 0 and satisfies $t(0) = 0$. To find the branch points, solve $F_t(x, t) = 0$ and $F(x, t) = 0$. We obtain $(x, t) = (0, 1/2)$ and $(2/27, 1/6)$. Since $t(0) = 0$, $(0, 1/2)$ is on the wrong Riemann sheet and so $x = 2/27$ is the only singularity of $t(x)$. As in the proof of Theorem 5 of [1], we have

$$t - 1/6 \approx -\sqrt{\frac{-2F_x(2/27, 1/6)}{F_{tt}(2/27, 1/6)}(x - 2/27)},$$

which gives (5.2). ∎

From (1.1), we have

$$P_3 \approx \frac{-1}{2 \times (1/6) \times \sqrt{1 - 1/3}} \sqrt{1 - 6t}.$$

Using Lemma 4 and the fact that $1 - 6t = 0$ at $x = 2/27$, we get

$$P_3 \approx -3^{5/4}\left(1 - \frac{27}{2}x\right)^{1/4}.$$

By Darboux's Theorem (see [1, Theorem 4]), we obtain

$$\begin{aligned} p_{n,3} = P_3(x)\big|_{x^{n-1}} &\sim -3^{5/4} \times \frac{n^{-5/4}}{\Gamma(-1/4)}\left(\frac{27}{2}\right)^{n-1} \\ &= \frac{3^{-7/4}}{2 \times \Gamma(3/4)} n^{-5/4}\left(\frac{27}{2}\right)^{n} \end{aligned}$$

and (5.1) is proved.

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