Topology and its Applications 96 (1999) 89–92

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Representing set-inclusion by embeddability (among the subspaces of the real line)

A.E. McCluskey [a,*], T.B.M. McMaster [b], W.S. Watson [c]

[a] *Department of Mathematics, National University of Ireland, Galway, Ireland*
[b] *Department of Pure Mathematics, Queen's University of Belfast, Belfast BT7 1NN, UK*
[c] *Department of Mathematics, York University, Ontario, Canada*

Received 20 August 1997; received in revised form 5 January 1998

Abstract

We establish that the powerset $\mathcal{P}(\mathbb{R})$ of the real line $\mathbb{R}$, ordered by set-inclusion, has the same ordertype as a certain subset of $\mathcal{P}(\mathbb{R})$ ordered by homeomorphic embeddability. This is a contribution to the ongoing study of the possible ordertypes of subfamilies of $\mathcal{P}(\mathbb{R})$ under embeddability, pioneered by Banach, Kuratowski and Sierpiński. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Partial order; Ordering by homeomorphic embeddability; Transfinite induction; G_δ-sets

AMS classification: 06A06; 54H10

Introduction

The ordering by embeddability of topological spaces, although a fundamental notion in topology, is imperfectly understood at present. For instance, the question—given a topological space X, what are the possible ordertypes of families of subspaces of X under the embeddability ordering?—appears not to have been fully answered for any but the most simple instances of X. Even the 'familiar' real line $\mathbb{R}$ has yet to receive a complete analysis of the ordertypes occurring amongst its subspaces.

To facilitate the discussion we shall write $X \hookrightarrow Y$ to indicate that the space X is homeomorphically embeddable in the space Y and, adopting the terminology of [5], we shall say that a partially-ordered set (poset) P is *realized* (or *realizable*) within a family $\mathcal{F}$ of topological spaces whenever there is an injection $\theta : P \to \mathcal{F}$ for which $p \leqslant p'$ if and only if $\theta(p) \hookrightarrow \theta(p')$. The 'if' component of this condition presents the main challenge

* Corresponding author. E-mail: aisling.mccluskey@ucg.ie.

PII: S0166-8641(98)00038-8

in realizations: θ must be so designed that whenever $p \nleq p'$, no embedding of $\theta(p)$ into $\theta(p')$ is possible.

Discussion of realizability in the powerset $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ can be traced back to Banach, Kuratowski and Sierpiński [2,3,7] whose work on the extensibility of continuous maps over G_δ-subsets (in the context of Polish spaces) revealed *inter alia* that it is possible to realize, within $\mathcal{P}(\mathbb{R})$, (i) the antichain on $2^{\boldsymbol{c}}$ points [2, p. 205] and (ii) the ordinal $\boldsymbol{c}^+$ [3, p. 199]. Fresh interest in such issues was initiated in [4] where it is shown that every poset of cardinality $\boldsymbol{c}$ (or less) can be realized within $\mathcal{P}(\mathbb{R})$. The question of *precisely* which posets of cardinalities exceeding $\boldsymbol{c}$ can be so realized is as yet unresolved and, since it increasingly reveals itself to be essentially set-theoretic in nature, it appears correspondingly impervious to a purely topological attack. The authors have recently shown that no ZFC analogue of the result in [4] for cardinality $2^{\boldsymbol{c}}$ exists—that is, there is a consistent counterexample.

Accordingly, the present paper seeks to extend the current and limited fund of results in the area (see also [5,6]) by exhibiting how to realize a second natural poset of cardinality $2^{\boldsymbol{c}}$ as a family of subspaces of $\mathbb{R}$ ordered by embeddability: namely $\mathcal{P}(\mathbb{R})$ itself, ordered by set-inclusion. The demonstration develops work of Kuratowski on the realization of the antichain on $2^{\boldsymbol{c}}$ points [2].

We quote (without proof) the classical theorem [3] of M. Lavrentiev which played a key role in the earlier work of Kuratowski et al. and, consequently, in this paper.

Theorem 1. *Every homeomorphism between subsets A, B of complete metric spaces X and Y (respectively) can be extended to a homeomorphism between G_δ-subsets A^*, B^* of X and Y (respectively) such that $A \subseteq A^*$ and $B \subseteq B^*$.*

Lemma 2. *Let κ be an infinite cardinal and $\mathcal{F}$ be a family of κ-many partial injections from κ to κ. Then there is a subset A of κ such that*

(i) *$|A| = \kappa$ and*

(ii) *$B, C \in \mathcal{P}(A)$, $f \in \mathcal{F}$ and $f(B) = C$ together imply*

$$\left|(B \backslash C) \cup (C \backslash B)\right| < \kappa.$$

In particular, if $B, C \in \mathcal{P}(A)$ and $|(B \setminus C) \cup (C \setminus B)| = \kappa$ then no member of $\mathcal{F}$ maps B onto C.

Proof. Without loss of generality, $\mathcal{F}$ contains the inverse of each of its members. Indexing $\mathcal{F}$ as $\{f_\alpha\colon \alpha < \kappa\}$, it is routine to construct by transfinite induction a κ-sequence $(x_\alpha)_{\alpha<\kappa}$ so that, for each α,

$$x_\alpha \neq x_\beta \quad \text{for all } \beta < \alpha \text{ and}$$

$$x_\alpha \neq f_\gamma(x_\beta) \quad \text{for all } \beta < \alpha \text{ and } \gamma < \alpha.$$

Now let A be the set $\{x_\alpha\colon \alpha < \kappa\}$, and note that $|A| = \kappa$.

For each $\alpha < \kappa$ define a subset $\Delta(\alpha)$ of κ by the criterion

$$\delta \in \Delta(\alpha) \quad \text{if and only if} \quad f_\alpha(x_\delta) \in A \setminus \{x_\delta\}.$$

Then for a given $\delta \in \Delta(\alpha)$ we can find $\varepsilon < \kappa$ for which $f_\alpha(x_\delta) = x_\varepsilon$ but $x_\varepsilon \neq x_\delta$. Since $f_\alpha^{-1} \in \mathcal{F}$, we also have $f_\alpha^{-1} = f_\lambda$ for some $\lambda < \kappa$, and we observe that $f_\lambda(x_\varepsilon) = x_\delta$. Due to the construction of (x_α), it follows that

(a) if $\delta < \varepsilon$ then $\alpha \geqslant \varepsilon > \delta$ and

(b) if $\delta > \varepsilon$ then $\lambda \geqslant \delta$.

Hence $\delta \leqslant \max\{\alpha, \lambda\}$, and the set $\Delta(\alpha)$ is bounded in κ. Likewise $\Delta(\lambda)$ is bounded and condition (ii) follows. In fact, we have shown that each f_α when restricted and co-restricted to A acts as an identity mapping on "almost all" points. □

In the context of $\mathbb{R}$ we now specialize to the case where $\kappa = \boldsymbol{c}$ and $\mathcal{F}$ is the family of continuous real-valued injections defined on G_δ-subsets of the real line. Since, via the Lavrentiev theorem, every embedding map is a restriction of such a map, this is an appropriate family to consider.

Beginning with the poset $\mathcal{P}(\mathbb{R})$ under set-inclusion, we seek to associate with each subset H of $\mathbb{R}$ another subset $\theta(H)$ in such a way that

$$H \subseteq J \quad \text{if and only if} \quad \theta(H) \hookrightarrow \theta(J).$$

This is achieved by arranging *firstly* that the associated subsets lie within the special set A described in the above lemma and, *secondly*, that whenever $H \nsubseteq J$ we get $|\theta(H) \backslash \theta(J)| = \boldsymbol{c}$: so that embedding of $\theta(H)$ into $\theta(J)$ is rendered impossible.

Proposition 3. *The powerset of $\mathbb{R}$, ordered by set-inclusion, can be realized within the subspaces of $\mathbb{R}$.*

Proof. For each mapping $f : X \to Y$ we shall make use of the convenient notation f'' for the corresponding set-to-set mapping (see, for example, [1]) from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ specified by

$$f''(S) = f(S), \quad \text{where } S \in \mathcal{P}(X).$$

Define also $u : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^2)$ by

$$u(H) = H \times \mathbb{R}, \quad \text{where } H \in \mathcal{P}(\mathbb{R}),$$

noting that it is an order-isomorphism (with respect to set-inclusion) and that, whenever $H \neq J$ in $\mathcal{P}(\mathbb{R})$, $u(H)$ and $u(J)$ differ by $\boldsymbol{c}$-many points. Choose next a bijection $v : \mathbb{R}^2 \to \mathbb{R}$ and observe that $v'' : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R})$ is an order-isomorphism which maintains "large" set differences in the manner required. Finally, with $\mathcal{F}$ as described above and A constructed within $\mathbb{R}$ by the lemma, a bijection $w : \mathbb{R} \to A$ yields a third order-isomorphism $w'' : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(A)$.

Combining these maps, we derive

$$\theta = w''v''u$$

which is an order-embedding of $\mathcal{P}(\mathbb{R})$ into $\mathcal{P}(A)$. Now $H \subseteq J$ in $\mathcal{P}(\mathbb{R})$ implies that $\theta(H) \subseteq \theta(J)$ and, consequently, that $\theta(H) \hookrightarrow \theta(J)$; but on the other hand,

$$H \nsubseteq J \text{ in } \mathcal{P}(\mathbb{R}) \quad \text{implies } |\theta(H) \backslash \theta(J)| = \boldsymbol{c}$$

which in turn shows, using the lemma, that no member of $\mathcal{F}$ can map $\theta(H)$ into $\theta(J)$. Lastly, if $\theta(H)$ were homeomorphically embeddable into $\theta(J)$, the Lavrentiev theorem would guarantee the extension of that embedding to a member of $\mathcal{F}$: a contradiction which establishes:

$$H \subseteq J \quad \text{if and only if} \quad \theta(H) \hookrightarrow \theta(J)$$

as required. □

Note. Of course, every *subset* of the poset $(\mathcal{P}(\mathbb{R}), \subseteq)$ is similarly realizable within $(\mathcal{P}(\mathbb{R}), \hookrightarrow)$. An immediate consequence is:

Corollary 4. *Every poset E of cardinality not exceeding $\boldsymbol{c}$ can be realized within $(\mathcal{P}(\mathbb{R}), \hookrightarrow)$.*

Proof. First, augment E if necessary to have exactly $\boldsymbol{c}$ elements. Then represent E within $\mathcal{P}(E)$ in the standard way by defining, for each $x \in E$,

$$e(x) = \{y \in E\colon\ y \leqslant x\}. \qquad \square$$

References

[1] K. Kunen, Set Theory: An Introduction to Independence Proofs (North-Holland, Amsterdam, 1980).
[2] C. Kuratowski, Sur la puissance de l'ensemble des "nombres de dimension" au sens de M. Fréchet, Fund. Math. 8 (1926) 201–208.
[3] C. Kuratowski and W. Sierpiński, Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires, Fund. Math. 8 (1926) 193–200.
[4] P.T. Matthews and T.B.M. McMaster, Families of spaces having prescribed embeddability order-type, Rend. Ist. Mat. Univ. Trieste 25 (1993) 345–352.
[5] A.E. McCluskey and T.B.M. McMaster, Representing quasi-orders by embeddability ordering of families of topological spaces, Proc. Amer. Math. Soc., to appear.
[6] A.E. McCluskey and T.B.M. McMaster, Realizing quasiordered sets by subspaces of 'continuum-like' spaces, Order (submitted).
[7] W. Sierpiński, Sur un problème concernant les types de dimensions, Fund. Math. 19 (1932) 65–71.