Representing set-inclusion by embeddability 
(among the subspaces of the real line)

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Abstract

We establish that the powerset \( P(\mathbb{R}) \) of the real line \( \mathbb{R} \), ordered by set-inclusion, has the same ordertype as a certain subset of \( P(\mathbb{R}) \) ordered by homeomorphic embeddability. This is a contribution to the ongoing study of the possible ordertypes of subfamilies of \( P(\mathbb{R}) \) under embeddability, pioneered by Banach, Kuratowski and Sierpiński. © 1999 Elsevier Science B.V. All rights reserved.

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Introduction

The ordering by embeddability of topological spaces, although a fundamental notion in topology, is imperfectly understood at present. For instance, the question—given a topological space \( X \), what are the possible ordertypes of families of subspaces of \( X \) under the embeddability ordering?—appears not to have been fully answered for any but the most simple instances of \( X \). Even the ‘familiar’ real line \( \mathbb{R} \) has yet to receive a complete analysis of the ordertypes occurring amongst its subspaces.

To facilitate the discussion we shall write \( X \leftrightarrow Y \) to indicate that the space \( X \) is homeomorphically embeddable in the space \( Y \) and, adopting the terminology of [5], we shall say that a partially-ordered set (poset) \( P \) is realized (or realizable) within a family \( \mathcal{F} \) of topological spaces whenever there is an injection \( \theta : P \rightarrow \mathcal{F} \) for which \( p \leq p' \) if and only if \( \theta(p) \leftrightarrow \theta(p') \). The ‘if’ component of this condition presents the main challenge

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in realizations: \( \Theta \) must be so designed that whenever \( p \neq p' \), no embedding of \( \Theta(p) \) into \( \Theta(p') \) is possible.

Discussion of realizability in the powerset \( \mathcal{P}(\mathbb{R}) \) of \( \mathbb{R} \) can be traced back to Banach, Kuratowski and Sierpiński [2,3,7] whose work on the extensibility of continuous maps over \( G_\delta \)-subsets (in the context of Polish spaces) revealed inter alia that it is possible to realize, within \( \mathcal{P}(\mathbb{R}) \), (i) the antichain on \( 2^c \) points [2, p. 205] and (ii) the ordinal \( c^+ \) [3, p. 199].

Fresh interest in such issues was initiated in [4] where it is shown that every poset of cardinality \( c \) (or less) can be realized within \( \mathcal{P}(\mathbb{R}) \). The question of precisely which posets of cardinalities exceeding \( c \) can be so realized is as yet unresolved and, since it increasingly reveals itself to be essentially set-theoretic in nature, it appears correspondingly impervious to a purely topological attack. The authors have recently shown that no ZFC analogue of the result in [4] for cardinality \( 2^c \) exists—that is, there is a consistent counterexample.

Accordingly, the present paper seeks to extend the current and limited fund of results in the area (see also [5,6]) by exhibiting how to realize a second natural poset of cardinality \( 2^c \) as a family of subspaces of \( \mathbb{R} \) ordered by embeddability: namely \( \mathcal{P}(\mathbb{R}) \) itself, ordered by set-inclusion. The demonstration develops work of Kuratowski on the realization of the antichain on \( 2^c \) points [2].

We quote (without proof) the classical theorem [3] of M. Lavrentiev which played a key role in the earlier work of Kuratowski et al. and, consequently, in this paper.

**Theorem 1.** Every homeomorphism between subsets \( A, B \) of complete metric spaces \( X \) and \( Y \) (respectively) can be extended to a homeomorphism between \( G_\delta \)-subsets \( A^*, B^* \) of \( X \) and \( Y \) (respectively) such that \( A \subseteq A^* \) and \( B \subseteq B^* \).

**Lemma 2.** Let \( \kappa \) be an infinite cardinal and \( \mathcal{F} \) be a family of \( \kappa \)-many partial injections from \( \kappa \) to \( \kappa \). Then there is a subset \( A \) of \( \kappa \) such that

(i) \( |A| = \kappa \) and

(ii) \( B, C \in \mathcal{P}(A), f \in \mathcal{F} \) and \( f(B) = C \) together imply

\[
|\{(B \setminus C) \cup (C \setminus B)\}| < \kappa.
\]

In particular, if \( B, C \in \mathcal{P}(A) \) and \( |\{(B \setminus C) \cup (C \setminus B)\}| = \kappa \) then no member of \( \mathcal{F} \) maps \( B \) onto \( C \).

**Proof.** Without loss of generality, \( \mathcal{F} \) contains the inverse of each of its members. Indexing \( \mathcal{F} \) as \( \{f_\alpha : \alpha < \kappa\} \), it is routine to construct by transfinite induction a \( \kappa \)-sequence \( (x_\alpha)_{\alpha < \kappa} \) so that, for each \( \alpha \),

\[
x_\alpha \neq x_\beta \quad \text{for all } \beta < \alpha \quad \text{and} \quad \overline{x_\alpha} \\
x_\alpha \neq f_\gamma(x_\beta) \quad \text{for all } \beta < \alpha \text{ and } \gamma < \alpha.
\]

Now let \( A \) be the set \( \{x_\alpha : \alpha < \kappa\} \), and note that \( |A| = \kappa \).

For each \( \alpha < \kappa \) define a subset \( \Delta(\alpha) \) of \( \kappa \) by the criterion

\[
\delta \in \Delta(\alpha) \quad \text{if and only if} \quad f_\alpha(x_\delta) \in A \setminus \{x_\delta\}.
\]
Then for a given \( \delta \in \Delta(\alpha) \) we can find \( \varepsilon < \kappa \) for which \( f_\alpha(x_\delta) = x_\varepsilon \) but \( x_\varepsilon \neq x_\delta \). Since \( f_\alpha^{-1} \in \mathcal{F} \), we also have \( f_\alpha^{-1} = f_\lambda \) for some \( \lambda < \kappa \), and we observe that \( f_\lambda(x_\varepsilon) = x_\delta \). Due to the construction of \((x_\alpha)\), it follows that

(a) if \( \delta < \varepsilon \) then \( \alpha \geq \varepsilon > \delta \)

(b) if \( \delta > \varepsilon \) then \( \lambda \geq \delta \).

Hence \( \delta \leq \max\{\alpha, \lambda\} \), and the set \( \Delta(\alpha) \) is bounded in \( \kappa \). Likewise \( \Delta(\lambda) \) is bounded and condition (ii) follows. In fact, we have shown that each \( f_\alpha \) when restricted and co-restricted to \( A \) acts as an identity mapping on “almost all” points.

In the context of \( \mathbb{R} \) we now specialize to the case where \( \kappa = c \) and \( \mathcal{F} \) is the family of continuous real-valued injections defined on \( G_\delta \)-subsets of the real line. Since, via the Lavrentiev theorem, every embedding map is a restriction of such a map, this is an appropriate family to consider.

Beginning with the poset \( \mathcal{P}(\mathbb{R}) \) under set-inclusion, we seek to associate with each subset \( H \) of \( \mathbb{R} \) another subset \( \theta(H) \) in such a way that

\[ H \subseteq J \text{ if and only if } \theta(H) \leftrightarrow \theta(J). \]

This is achieved by arranging firstly that the associated subsets lie within the special set \( A \) described in the above lemma and, secondly, that whenever \( H \nsubseteq J \) we get \( |\theta(H) \setminus \theta(J)| = c \); so that embedding of \( \theta(H) \) into \( \theta(J) \) is rendered impossible.

**Proposition 3.** The powerset of \( \mathbb{R} \), ordered by set-inclusion, can be realized within the subspaces of \( \mathbb{R} \).

**Proof.** For each mapping \( f : X \to Y \) we shall make use of the convenient notation \( f'' \) for the corresponding set-to-set mapping (see, for example, [1]) from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \) specified by

\[ f''(S) = f(S), \text{ where } S \in \mathcal{P}(X). \]

Define also \( u : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^2) \) by

\[ u(H) = H \times \mathbb{R}, \text{ where } H \in \mathcal{P}(\mathbb{R}), \]

noting that it is an order-isomorphism (with respect to set-inclusion) and that, whenever \( H \neq J \) in \( \mathcal{P}(\mathbb{R}) \), \( u(H) \) and \( u(J) \) differ by \( c \)-many points. Choose next a bijection \( v : \mathbb{R}^2 \to \mathbb{R} \) and observe that \( v'' : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}) \) is an order-isomorphism which maintains “large” set differences in the manner required. Finally, with \( \mathcal{F} \) as described above and \( A \) constructed within \( \mathbb{R} \) by the lemma, a bijection \( u : \mathbb{R} \to A \) yields a third order-isomorphism \( u'' : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(A) \).

Combining these maps, we derive

\[ \theta = u''v''u \]

which is an order-embedding of \( \mathcal{P}(\mathbb{R}) \) into \( \mathcal{P}(A) \). Now \( H \subseteq J \) in \( \mathcal{P}(\mathbb{R}) \) implies that \( \theta(H) \subseteq \theta(J) \) and, consequently, that \( \theta(H) \leftrightarrow \theta(J) \); but on the other hand,

\[ H \nsubseteq J \text{ in } \mathcal{P}(\mathbb{R}) \text{ implies } |\theta(H) \setminus \theta(J)| = c \]
which in turn shows, using the lemma, that no member of $\mathcal{F}$ can map $\theta(H)$ into $\theta(J)$. Lastly, if $\theta(H)$ were homeomorphically embeddable into $\theta(J)$, the Lavrentiev theorem would guarantee the extension of that embedding to a member of $\mathcal{F}$: a contradiction which establishes:

$$H \subseteq J \quad \text{if and only if} \quad \theta(H) \leftrightarrow \theta(J)$$

as required. \( \square \)

**Note.** Of course, every subset of the poset $(\mathcal{P}(\mathbb{R}), \subseteq)$ is similarly realizable within $(\mathcal{P}(\mathbb{R}), \leftrightarrow)$. An immediate consequence is:

**Corollary 4.** Every poset $E$ of cardinality not exceeding $\mathfrak{c}$ can be realized within $(\mathcal{P}(\mathbb{R}), \leftrightarrow)$.

**Proof.** First, augment $E$ if necessary to have exactly $c$ elements. Then represent $E$ within $\mathcal{P}(E)$ in the standard way by defining, for each $x \in E$,

$$e(x) = \{ y \in E : y \leq x \}.$$ 

\( \square \)

**References**


