View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Journal of Pure and Applied Algebra 78 (1992) 275–290
North-Holland

Algebraic functor slices

V. Koubek*

MFF KU, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

J. Sichler*

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

V. Trnková*

MÚUK, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

Communicated by F.W. Lawvere
Received 26 June 1991

Abstract

Koubek, V., J. Sichler and V. Trnková, Algebraic functor slices, Journal of Pure and Applied Algebra 78 (1992) 275–290.

Forgetful functors of any two categories of monadic algebras over $\mathcal{S}et$ for which the functor T in a monad $\mathbf{T} = (T, \eta, \mu)$ is not naturally equivalent to the identity or a constant functor or to their coproduct are slice equivalent to one another. In particular, any two forgetful functors of nondegenerate varieties of algebras (that is, varieties which possess a term which is neither a projection nor a constant) are slice equivalent.

Introduction

Classical results by Birkhoff [3] and de Groot [4] show that every group is isomorphic to the full automorphism group of a distributive lattice, and to the group of all autohomeomorphisms of a topological space. Following Isbell's ideas [9], the concept of a full embedding (that is, a full and faithful functor) has been investigated and used to generalize and substantially strengthen various classical representations of groups or monoids as automorphism groups or endomorphism monoids of given mathematical structures.

Early representation results of this kind were summarized in [15]. Following this monograph's terminology, we say that a category $\mathcal{K}$ is *algebraically universal* (or alg-universal) whenever any category $\mathcal{A}lg(\Delta)$ formed by all homomorphisms between universal algebras of an arbitrary set type Δ can be fully embedded into

* The support of the NSERC of Canada is gratefully acknowledged by all three authors.

0022-4049/92/$05.00 © 1992 – Elsevier Science Publishers B.V. All rights reserved

$\mathcal{K}$. Every small category can be fully embedded into $\mathcal{K}$ and, in particular, every monoid is isomorphic to the endomorphism monoid of an object from any alg-universal category $\mathcal{K}$.

Algebraically universal varieties of algebras are plentiful. For instance, the category $\mathcal{A}lg(\Delta)$ of all algebras of a unary type $\Delta = \{1, 1, \ldots\}$ is alg-universal exactly when Δ has at least two entries; see [8] and [24].

Early examples of alg-universal categories include the category $\mathcal{S}gp$ of all semigroups [7], and the category $\mathcal{L}att$ of all $(0, 1)$-lattices [6]. These examples of algebraic universality appear already in [15]; more recent results characterize the alg-universal varieties of $\mathcal{S}gp$ [12] as those satisfying only permutational identities while failing the identity $x^n y^n = (xy)^n$ for each $n > 1$, and the alg-universal subvarieties of $\mathcal{L}att$ [5] as those containing a lattice without a prime filter. The list of alg-universal varieties contains numerous other entries, such as De Morgan algebras [1], or varieties of distributive $(0, 1)$-lattices with two additional constants [11].

Other recent representation results such as [14], [21], [22], or [23] concern simultaneous representations of pairs of categories. An underlying idea of these is to investigate both a category $\mathcal{K}_1$ whose objects are structured in two different ways, and a category $\mathcal{K}_2$ obtained from $\mathcal{K}_1$ by deleting one of the two structures of $\mathcal{K}_1$. Any such deletion gives rise to a faithful functor $K : \mathcal{K}_1 \to \mathcal{K}_2$ and, in turn, leads to the question of simultaneous existence of a pair of full embeddings comparing such functors.

More precisely, an abstract simultaneous representation problem can be formulated as follows.

Given a functor

$$K : \mathcal{K}_1 \to \mathcal{K}_2 ,$$

for what functors $H : \mathcal{H}_1 \to \mathcal{H}_2$ do there exist full embeddings $\Phi_i : \mathcal{H}_i \to \mathcal{K}_i$ such that the diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{H} & \mathcal{H}_2 \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} \\ \mathcal{K}_1 & \xrightarrow[K]{} & \mathcal{K}_2 \end{array}$$

commutes?

Existing simultaneous representations already extend certain classical results. For instance, every group G and its subgroup H can be represented by a single Tychonoff space X so that H is isomorphic to the group of all autohomeomorphisms of X, while G is isomorphic to the group of all autohomeomorphisms of its β-compactification βX [23]. In another example, H is represented by the automorphism group of an algebra A with three unary operations, while G is isomorphic to the automorphism group of a reduct A^- of A obtained by the deletion of one of the operations of A [14].

In the latter example, at least two unary operations are needed to represent an arbitrary group G, see [15] or [16], while a single unary operation is all that is needed to select an arbitrary subgroup H of G. This illustrates a recurring, and perhaps even a general phenomenon: to represent a functor $H : \mathscr{H}_1 \to \mathscr{H}_2$ simultaneously in another one, say $K : \mathscr{K}_1 \to \mathscr{K}_2$, the structure of $\mathscr{K}_2$ must be considerably richer than the additional structure needed to produce $\mathscr{K}_1$ from $\mathscr{K}_2$.

We believe that the notion of a *functor slice*, introduced in [17] and investigated here, gauges the type and complexity of an additional structure needed to produce simultaneous representations from full embeddings. Various simultaneous representations were justified through an implicit application of functor slices [14], and functor slices also served explicitly as building blocks for several simultaneous representations in [13] and [17].

More recent investigations suggest that functor slices may well offer a classification of functors, particularly of faithful functors into the category $\mathscr{S}et$ of all sets and mappings [17]. The present paper aims to illustrate this idea by showing how functor slices describe concrete categories of algebraic nature.

1. Concepts and results

A commutative diagram of functors

$$\begin{array}{ccc} \mathscr{H}_1 & \xrightarrow{H} & \mathscr{H}_2 \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathscr{K}_1 & \xrightarrow[K]{} & \mathscr{K}_2 \end{array} \tag{1}$$

— in which we denote $\Omega = K \circ \Phi_1 = \Phi_2 \circ H$ — is called a *subpullback* if, for any two objects a and b of the category $\mathscr{H}_1$, the diagram (2) of hom-sets

$$\begin{array}{ccc} \mathscr{H}_1(a, b) & \xrightarrow{H} & \mathscr{H}_2(H(a), H(b)) \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathscr{K}_1(\Phi_1(a), \Phi_1(b)) & \xrightarrow[K]{} & \mathscr{K}_2(\Omega(a), \Omega(b)) \end{array} \tag{2}$$

is a pullback in $\mathscr{S}et$.

In other words, a commutative diagram (1) is a subpullback if and only if for any $a,b \in \operatorname{obj}\mathscr{H}_1$ and for any $h_2 \in \mathscr{H}_2(H(a), H(b))$, $k_1 \in \mathscr{K}_1(\Phi_1(a), \Phi_1(b))$ with $K(k_1) = \Phi_2(h_2)$ there is a unique morphism $h_1 \in \mathscr{H}_1(a, b)$ for which $\Phi_1(h_1) = k_1$ and $H(h_1) = h_2$.

Definition. We say that a functor $H : \mathcal{H}_1 \to \mathcal{H}_2$ is a *slice* of a functor $K : \mathcal{K}_1 \to \mathcal{K}_2$ whenever there exist faithful functors $\Phi_i : \mathcal{H}_i \to \mathcal{K}_i$ with $i \in \{1, 2\}$ completing the diagram (1) to a subpullback. If H is a slice of K and K is a slice of H, the functors H and K are said to be *slice-equivalent* (or *s-equivalent*).

To illustrate the terminology and the concept, let us consider the category $\mathcal{T}op$ of topological spaces and continuous maps and the category $\mathcal{P}os$ of partially ordered sets and monotone maps. From [17] we recall the fact that the forgetful functor $\mathcal{T}op \to \mathcal{S}et$ is a slice of the forgetful functor $\mathcal{P}os \to \mathcal{S}et$; hence there is a subpullback

$$\begin{array}{ccc} \mathcal{T}op & \longrightarrow & \mathcal{S}et \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{P}os & \longrightarrow & \mathcal{S}et \end{array}$$

of faithful functors. Furthermore, we recall a well-known fact that any faithful functor $F : \mathcal{S}et \to \mathcal{S}et$ contains a copy of the identity functor $I : \mathcal{S}et \to \mathcal{S}et$ in the sense that there exists a monotransformation $I \to F$. Thus if $\Phi_i(X_i, t_i) = (F(X_i), \leq_i)$ for objects (X_i, t_i) of $\mathcal{T}op$ and $i = 1,2$, then a mapping $f : (X_1, t_1) \to (X_2, t_2)$ is continuous exactly when $F(f) : (F(X_1), \leq_1) \to (F(X_2), \leq_2)$ is monotone and, since $f : X_1 \to X_2$ is the restriction of the mapping $F(f) : F(X_1) \to F(X_2)$ to $X_1 \subseteq F(X_1)$, it is natural to visualize the forgetful functor $\mathcal{T}op \to \mathcal{S}et$ as a 'sliced off' section of the forgetful functor $\mathcal{P}os \to \mathcal{S}et$.

It is easy to see that a functor G is a slice of K whenever G is a slice of H and H is a slice of K. Hence s-equivalence is a bona fide equivalence, and slices of any two s-equivalent functors form the same collection.

Convention. To avoid verbose statements, in cases of familiar concrete categories with standard forgetful functors into $\mathcal{S}et$ we shall say that such categories are s-equivalent whenever, in actual fact, the s-equivalence applies to their forgetful functors.

Under this convention, numerous familiar concrete categories fall into several 'baskets' determined by mutual s-equivalence of their members. Proofs of all s-equivalences below are quite straightforward and can be found in [17].

The basket **R** contains the following categories:

$S(F)$ for any faithful functor $F : \mathcal{S}et \to \mathcal{S}et$ of either variance; we recall that objects of $S(F)$ are all pairs (X, R) of sets with $R \subseteq F(X)$, and that a mapping $f : X \to X'$ is a morphism of $S(F)$ from (X, R) to (X', R') exactly when $[F(f)](R) \subseteq R'$ for a covariant functor F while $[F(f)](R') \subseteq R$ in the contravariant case; the forgetful functor $S(F) \to \mathcal{S}et$ sends (X, R) to the set X; in particular,

$\mathcal{R}el(\Delta^+)$ for an arbitrary *positive* type Δ^+, whose objects are relational systems of type Δ^+ and whose morphisms are all compatible maps; these are just categories $S(F)$ for which F is a coproduct of the appropriate covariant hom-functors;

$\mathcal{P}alg(\Delta^+)$ for an arbitrary *positive* type Δ^+; these are full subcategories of $\mathcal{R}el(\Delta^+)$ consisting of partial algebras and their homomorphisms;

$\mathcal{P}os$ – the category of all partially ordered sets and monotone mappings;

$\mathcal{T}op$ and all its full subcategories down to the category of all metrizable spaces;

$\mathcal{U}nif$ – the category of all uniform spaces and uniformly continuous maps, and all its full subcategories down to the category of all complete metrizable spaces;

$\mathcal{M}etr$ – the category of all metric spaces and maps which do not increase the distance, and all its full subcategories down to the category of all complete metric spaces of diameter at most one.

In the basket **A** we find:

$\mathcal{A}lg(\Delta^+)$ – the category of all universal algebras of a *positive* type Δ^+ and all their homomorphisms;

$\mathcal{C}omp$ – the category of all compact Hausdorff spaces and continuous maps, and all its full subcategories down to the category of all Boolean spaces.

The basket **P** contains:

$\mathcal{A}lg(\Delta^0)$ – the category of all universal algebras of a nonvoid *nullary* type Δ^0 and, in particular,

$\mathcal{S}et_*$ – the category of all pointed sets.

Let $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{P}$ denote the respective collection of slices of any, and hence each, forgetful functor of a category from a corresponding basket. It is easy to see that $\mathcal{S}et_* \to \mathcal{S}et$ is a slice of $\mathcal{A}lg(1) \to \mathcal{S}et$ and that $\mathcal{A}lg(1) \to \mathcal{S}et$ is a slice of $\mathcal{R}el(2) \to \mathcal{S}et$; hence $\mathcal{P}$ is a subcollection of $\mathcal{A}$, and $\mathcal{A}$ is a subcollection of $\mathcal{R}$. These three slice collections differ from one another already on the level of one-object categories and, for a functor $H : \mathcal{H}_1 \to \mathcal{H}_2$ between small categories, [17] demonstrates that:

(R) H is in $\mathcal{R}$ if and only if it is faithful,

(A) H is in $\mathcal{A}$ if and only if H is faithful and obeys Isbell's Zig Zag condition [10] and [17],

(P) H is in $\mathcal{P}$ if and only if it is faithful and satisfies the condition
(p) for $a,b,c \in \operatorname{obj} \mathcal{H}_1$, if $\mu \in \mathcal{H}_1(a, b)$ and $\beta \in \mathcal{H}_2(H(b), H(c))$, then $\beta = H(\nu)$ for some $\nu \in \mathcal{H}_1(b, c)$ if and only if $\beta \circ H(\mu) = H(\sigma)$ for some $\sigma \in \mathcal{H}_1(a, c)$.

Let $\mathcal{X}^{\mathrm{op}}$ denote the dual collection of all opposites $H^{\mathrm{op}} : \mathcal{H}_1^{\mathrm{op}} \to \mathcal{H}_2^{\mathrm{op}}$ to functors $H : \mathcal{H}_1 \to \mathcal{H}_2$ from a collection $\mathcal{X}$. While $\mathcal{R}$ and $\mathcal{A}$ are self-dual, that is, $\mathcal{R} = \mathcal{R}^{\mathrm{op}}$ and $\mathcal{A} = \mathcal{A}^{\mathrm{op}}$ [17], this is not the case for the collection $\mathcal{P}$, see (P) above. The collection $\mathscr{P}^{op}$ thus produces a fourth basket, denoted $\mathbf{P}^{op}$. This basket contains, for instance, the subcategory $\mathscr{R}el_*(1)$ of $\mathscr{R}el(1)$ whose morphisms, in addition to the single unary relation, preserve also its complement [17].

The purpose of this paper is to enrich the basket **A** in a following way.

Theorem 1.1. *Let* $\mathbf{T} = (T, \eta, \mu)$ *be a monad over* $\mathscr{S}et$. *If the functor* $T : \mathscr{S}et \to \mathscr{S}et$ *is not naturally equivalent to a constant functor or to the identity functor or to their coproduct, then the category* $\mathscr{S}et^{\mathbf{T}}$ *of all* **T**-*algebras is slice equivalent to* $\mathscr{A}lg(1)$.

What monads remain?

If T is naturally equivalent to a coproduct of the identity functor and a constant one with a nonvoid value, then, clearly, $\mathscr{S}et^{\mathbf{T}}$ falls into the basket **P**. If T is naturally equivalent to a constant functor, then $\mathscr{S}et^{\mathbf{T}}$ consists of one-element objects. Finally, for a functor T naturally equivalent to the identity endofunctor of $\mathscr{S}et$, the category $\mathscr{S}et^{\mathbf{T}}$ of monadic algebras is isomorphic to $\mathscr{S}et$. We call all these monads *degenerate*.

It is easily seen that the forgetful functor $U^{\mathbf{T}} : \mathscr{S}et^{\mathbf{T}} \to \mathscr{S}et$ is a slice of the forgetful functor $\mathscr{A}lg(1) \to \mathscr{S}et$; the converse requires more work.

To formulate the second result, we recall that a functor $U : \mathscr{K} \to \mathscr{L}$ is of *descent type* [2] if it has a left adjoint and if the associated comparison functor is full and faithful. We say that $U : \mathscr{K} \to \mathscr{L}$ has the *transfer property* if for any $\mathscr{L}$-isomorphism $\lambda : b \to U(a)$ there exists a $\mathscr{K}$-isomorphism $\kappa : \bar{a} \to a$ such that $U(\kappa) = \lambda$.

Theorem 1.1 admits a further generalization, Theorem 1.2, applicable, for instance, to the category of all Banach spaces and all linear operators of norm at most one, and to the category of all extremally disconnected compact Hausdorff spaces (see the remark at the conclusion of the article).

Theorem 1.2. *Let* $U : \mathscr{K} \to \mathscr{S}et$ *be a functor of a descent type with the transfer property. If the monad* **T** *arising from this adjunction is nondegenerate, then* U *is slice equivalent to the forgetful functor* $\mathscr{A}lg(1) \to \mathscr{S}et$.

We proceed as follows. First we show that the full subcategory $\mathscr{I}nv$ of $\mathscr{A}lg(1)$ formed by all monounary algebras (X, φ) satisfying $\varphi \circ \varphi = 1_X$ belongs to the basket **A**. The subsequent Section 3 establishes properties of set functors needed to prove Theorem 1.1. The proof of Theorem 1.1 is presented next; it makes an essential use of the fact that $\mathscr{I}nv$ determines the basket **A**. The proof of Theorem 1.2 concludes the article.

2. Involutory unary algebras

Let $\mathscr{A}lg(1)$ denote the category of all monounary algebras, and $\mathscr{I}nv$ its subvariety formed by all algebras $(X, \varphi) \in \mathscr{A}lg(1)$ satisfying $\varphi^2(x) = x$ for all $x \in X$.

We aim to show that $\mathcal{I}nv$ belongs to the basket **A**. Since $\mathcal{I}nv$ is a full subcategory of $\mathcal{A}lg(1)$, we need only the claim below to complete the task.

Proposition 2.1. *The underlying set functor $\mathcal{A}lg(1) \to \mathcal{S}et$ is a slice of the underlying set functor $\mathcal{I}nv \to \mathcal{S}et$.*

Proof. We must define functors Φ and F so that the diagram

$$\begin{array}{ccc} \mathcal{A}lg(1) & \longrightarrow & \mathcal{S}et \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{I}nv & \longrightarrow & \mathcal{S}et \end{array} \tag{3}$$

is a subpullback.

To this end, for any set X, let $B(X)$ denote the free Boolean group over X and, for every mapping $f : X \to Y$, let $B(f) : B(X) \to B(Y)$ denote its free extension. If $U : \mathcal{B}g \to \mathcal{S}et$ is the underlying set functor of the variety $\mathcal{B}g$ of Boolean groups, then the composite $G = U \circ B$ is a well-defined set functor.

Next we define a functor $F : \mathcal{S}et \to \mathcal{S}et$ by

$$F(X) = X \times G(X)$$

for any set X, and

$$F(f)(x, b) = (f(x), G(f)(b))$$

for any $(x, b) \in X \times G(X)$ and $f : X \to Y$.

Thus F is a set functor such that $F(f)(x, 0) = (f(x), 0)$, and $F(f)(x, x') = (f(x), f(x'))$ for all $x \in X$ and all x' in the generating set $X \subset G(X)$ of $B(X)$. Define a monounary algebra $\Phi(X, \varphi) = (F(X), \varphi^{\#})$ in which the mapping $\varphi^{\#} : F(X) \to F(X)$ is given by

$$\varphi^{\#}(x, b) = (x, \varphi(x) + b) \quad \text{for all } (x, b) \in F(X) = X \times G(X)\,.$$

Then $\varphi^{\#}(\varphi^{\#}(x, b)) = \varphi^{\#}(x, \varphi(x) + b) = (x, 2\varphi(x) + b) = (x, b)$, that is, $(\varphi^{\#})^2 = \mathrm{id}_{F(X)}$. Next we define

$$\Phi(f) = F(f) = f \times G(f) \quad \text{for all morphisms } f \text{ in } \mathcal{I}nv\,.$$

To verify that Φ is a functor, we observe that, for any morphism $f : (X, \varphi) \to (Y, \psi)$,

$$\begin{aligned} F(f)\varphi^{\#}(x, b) &= F(f)(x, \varphi(x) + b) = (f(x), G(f)(\varphi(x) + b)) \\ &= (f(x), f\varphi(x) + G(f)(b))\,, \\ \psi^{\#}F(f)(x, b) &= \psi^{\#}(f(x), G(f)(b)) = (f(x), \psi f(x) + G(f)(b))\,. \end{aligned}$$

From $f \circ \varphi = \psi \circ f$ it now follows that $F(f)$ is, indeed, a morphism of $\Phi(X, \varphi)$ into $\Phi(Y, \psi)$. Hence $\Phi : \mathcal{A}lg(1) \to \mathcal{I}nv$ is a well-defined faithful functor and the diagram (3) commutes.

Conversely, let $F(f) : (F(X), \varphi^{\#}) \to (F(Y), \psi^{\#})$ be a homomorphism in $\mathcal{I}nv$. Then

$$F(f)\varphi^{\#}(x, 0) = F(f)(x, \varphi(x)) = (f(x), f\varphi(x))$$

must coincide with

$$\psi^{\#}F(f)(x, 0) = \psi^{\#}(f(x), 0) = (f(x), \psi f(x))$$

for all $x \in X$, that is, $f : (X, \varphi) \to (Y, \psi)$ must be a homomorphism in $\mathcal{A}lg(1)$. The diagram (3) is, indeed, a subpullback. $\square$

Altogether, the variety $\mathcal{I}nv$ belongs to the basket **A**.

3. On set functors

First we recall some well-known facts about set functors $F : \mathcal{S}et \to \mathcal{S}et$ (see [19] and Proposition II.4 of [20]).

Let P be an arbitrary set, and let $\text{const}_P : \mathcal{S}et \to \mathcal{S}et$ denote the constant functor with the value P. We assume that const_P assigns P to every nonvoid set and the identity mapping 1_P to every mapping between nonvoid sets.

Every set functor F has a unique coproduct decomposition associated with the set of all natural transformations from the identity functor $I : \mathcal{S}et \to \mathcal{S}et$ into F. If $1 = \{0\}$ denotes the standard singleton, then F decomposes uniquely as a coproduct

$$F = \coprod \{F^a \mid a \in F(1)\},$$

of *components* F^a, given, for every set X and its unique mapping $c_X : X \to 1$, by $F^a(X) = F(c_X)^{-1}\{a\}$. The unique natural transformation $\eta^a : I \to F^a$ is then determined by $\eta^a_1(0) = a$.

Denote $2 = \{0, 1\}$, and let $v_j : 1 \to 2$ be the maps with $v_j(0) = j$ for $j \in 2$. If $F^a(v_0) \neq F^a(v_1)$, then F^a is faithful and η^a is a monotransformation. On the other hand, $F^a(v_0) = F^a(v_1)$ implies the existence of a unique $d^a_X \in F^a(X)$ such that $\eta^a_X(x) = d^a_X$ for every $x \in X$.

For $A_1, A_2 \subseteq X$, let $i_j : A_j \to X$ and $i : A_1 \cap A_2 \to X$ denote the corresponding inclusion maps. Then $A_1 \cap A_2 \neq \emptyset$ implies

$$[F^a(i_1)](F^a A_1) \cap [F^a(i_2)](F^a A_2) = [F^a(i)](F^a(A_1 \cap A_2)),$$

and hence this is also true for the original functor F. On the other hand, if $A_1 \neq \emptyset \neq A_2$ and $A_1 \cap A_2 = \emptyset$, then

$$[F^a(i_1)](F^a A_1) \cap [F^a(i_2)](F^a A_2) = \begin{cases} \emptyset & \text{when } F^a \text{ is faithful}, \\ \{d_X^a\} & \text{otherwise}. \end{cases}$$

These observations lead directly to the following claim:

Lemma 3.1. *Let $L : \mathcal{S}et \to \mathcal{S}et$ be a functor such that $L(1)$ is a singleton, and let $\eta : I \to L$ be the unique natural transformation. If L is not naturally equivalent to I or $\mathrm{const}_{\{0\}}$, then there exists a set P for which the functor $G = L \circ K$ with $K = I \times \mathrm{const}_P$ has a coproduct decomposition $G = G_0 \amalg G_1$ with insertions $\gamma^j : G_j \to G$ such that*

(a) *the functor G_0 is faithful, and*

(b) *there is a natural transformation $\rho : K \to G_1$ with $\eta_K = \gamma^1 \circ \rho$.*

Proof. If $\eta : I \to L$ is an epitransformation, then either L is faithful and hence η is a natural equivalence of I onto L, or else η consists of constant maps and L is naturally equivalent to a constant functor with a singleton value. Hence we can find a set P of the smallest cardinality such that $L(P) \backslash \eta_P(P) \neq \emptyset$; since $L(1)$ is a singleton, P must have at least two elements.

For any x, let $b_x : P \to \{x\} \times P$ denote the bijection defined by $b_x(p) = (x, p)$. Select some $q \in Q(P) = L(P) \backslash \eta_P(P)$ and denote $q_x = [L(b_x)](q)$.

Set $K = I \times \mathrm{const}_P$ and $G = L \circ K$. Then $G = \coprod \{G^a \mid a \in G(1)\}$, and we can define $G_0 = \coprod \{G^a \mid a \in Q(1 \times P)\}$ and $G_1 = \coprod \{G^a \mid a \in \eta_{1 \times P}(1 \times P)\}$. Hence $G = G_0 \amalg G_1$ as claimed. Next, let $\rho : K \to G_1$ be the natural transformation determined by $\rho_1(k) = \eta_{1 \times P}(k)$ for $k \in K(1) = 1 \times P$; it is clear that $\eta_K = \gamma^1 \circ \rho$.

To show that G_0 is faithful, we define a natural transformation $\sigma : I \to G_0$ by requiring that $\sigma_1(0) = q_0$, an element of $G_0(1) \subseteq L(1 \times P)$. To see that σ is a monotransformation, select distinct $x_1, x_2 \in X$ and observe that $A_1 = \{x_1\} \times P$ and $A_2 = \{x_2\} \times P$ are disjoint nonsingleton subsets of $X \times P$. The observation preceding this lemma implies that, in the case of a faithful functor L, the subsets $[L(i_j)](LA_j)$ of $L(X \times P)$ are disjoint, or else they intersect in the singleton $\{d_{X \times P}\} = \eta_{X \times P}(X \times P)$ and $G_0(X) \cap \eta_{X \times P}(X \times P) = \emptyset$. In either case, $\sigma_X(x_1) = [L(i_1)](q_{x_1})$ is distinct from $\sigma_X(x_2) = [L(i_2)](q_{x_2})$. □

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad over $\mathcal{S}et$, and let us assume that, as in Theorem 1.1, T is not naturally equivalent to a constant, identity or to their coproduct. We aim to show that then

(D) there is a set P such that the functor $H = T \circ K$ with $K = I \times \mathrm{const}_P$ has a coproduct decomposition $H = H_0 \amalg H_1$ such that H_0 is faithful, and $\eta_K = \gamma^1 \circ \rho$ for the coproduct insertion $\gamma^1 : H_1 \to H$ and some $\rho : K \to H_1$.

Let

$$T = \coprod \{T^a \mid a \in T(1)\}$$

be the decomposition of T into its components T^a. The natural transformation $\eta : I \to T$ then maps I into one of these components, say into $L = T^c$. From the fact that $\mu \circ T\eta : T \to T$ is the identity it easily follows that η is a monotransformation except when T is naturally equivalent to const_1; we may thus assume that L is a faithful component of T.

If L is not naturally equivalent to I, then Lemma 3.1 applies to L. Hence for some set P and $K = I \times \mathrm{const}_P$ there exists a coproduct decomposition $G_0 \amalg G_1$ of $L \circ K$ obeying Lemma 3.1(a) and 3.1(b). It follows that $T \circ K$ decomposes into $H_1 = G_1$ and $H_0 = \coprod \{T^a \circ K \mid a \neq c\} \amalg G_0$, and that these factors satisfy (D).

From now on, let us assume that L is naturally equivalent to I. Then T must have components other than L and, amongst these, at least one which is nonconstant.

If one of these components, say T^b, is faithful, then we choose $P = 1$. The functor $H = T \circ I$ can then be decomposed into a coproduct of $H_0 = T^b$ and H_1 which, in turn, is the coproduct of all components other than T^b. This decomposition also satisfies (D).

Finally, suppose that $T^b \neq L$ is a nonconstant component of T that is not faithful. Applying Lemma 3.1 to T^b, we obtain a set P and a decomposition $G = G_0 \amalg G_1$ of $G = T^b \circ K$ with $K = I \times \mathrm{const}_P$ that satisfies Lemma 3.1(a) and 3.1(b). But then $H_0 = G_0$ is a faithful factor of $H = T \circ K$ and, together with $H_1 = \coprod \{T^a \circ K \mid a \neq b\} \amalg G_1$, forms a coproduct decomposition of H for which (D) holds.

Since it exhausts all possibilities, the preceding argument shows that (D) holds for every nondegenerate monad.

4. Proof of Theorem 1.1

For future easy reference, we note that the natural transformations $\eta : I \to T$ and $\mu : T^2 \to T$ of a monad $\mathbf{T} = (T, \eta, \mu)$ satisfy

$$\mu \circ T\mu = \mu \circ \mu T \tag{4}$$

and

$$\mu \circ \eta T = 1_T = \mu \circ T\eta \,. \tag{5}$$

A pair (Y, h) with $h : T(Y) \to Y$ is a $\mathbf{T}$-algebra provided

$$h \circ T(h) = h \circ \mu_Y \quad \text{and} \quad h \circ \eta_Y = 1_Y \,. \tag{6}$$

A mapping $f : (Y, h) \to (Y', h')$ between **T**-algebras is a **T**-morphism if

$$f \circ h = h' \circ T(f) . \tag{7}$$

The concrete category $\mathcal{S}et^{\mathbf{T}}$ of all **T**-algebras and all **T**-morphisms has the natural forgetful functor $U^{\mathbf{T}}(Y, h) = Y$.

Proposition 4.1. *For any nondegenerate monad $\mathbf{T} = (T, \eta, \mu)$ there exist functors $\Psi : \mathcal{I}nv \to \mathcal{S}et^{\mathbf{T}}$ and $H : \mathcal{S}et \to \mathcal{S}et$ such that the diagram*

$$\begin{array}{ccc} \mathcal{I}nv & \longrightarrow & \mathcal{S}et \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{S}et^{\mathbf{T}} & \xrightarrow[U^{\mathbf{T}}]{} & \mathcal{S}et \end{array} \tag{8}$$

is a subpullback.

Proof. Let $A = (X, \varphi)$ be an object of $\mathcal{I}nv$, that is, let $\varphi : X \to X$ satisfy $\varphi^2 = 1_X$. Since **T** is nondegenerate, (D) of Section 3 is satisfied, and we define an auxiliary mapping $b_A : H(X) \to H(X)$ by

$$b_A = 1_{H_0(X)} \amalg H_1(\varphi) ; \tag{9}$$

clearly $b_A^2 = 1_{H(X)}$.

If (X, φ) and (X', φ') are objects of $\mathcal{I}nv$ and $f : X \to X'$ a mapping, then $f \circ \varphi = \varphi' \circ f$ implies that $b_{A'} \circ H(f) = H(f) \circ b_A$.

For any object $A = (X, \varphi)$ of $\mathcal{I}nv$ we set $\Psi(A) = (H(X), h_A)$, where

$$h_A = b_A \mu_{K(X)} T(b_A) . \tag{10}$$

Then $h_A : TH(X) \to H(X)$ is a well-defined mapping. To see that $\Psi(A) = (H(X), h_A)$ is a **T**-algebra, we must show that it obeys (6). Indeed, the second part of (6) is an easy consequence of (10), the naturality of η and of (5):

$$\begin{aligned} h_A \eta_{H(X)} &= b_A \mu_{K(X)} T(b_A) \eta_{TK(X)} \\ &= b_A \mu_{K(X)} \eta_{TK(X)} b_A = b_A^2 = 1_{H(X)} . \end{aligned}$$

A calculation verifying that $\Psi(A)$ obeys the first part of (6) uses (10), the fact that $b_A^2 = 1_{H(X)}$, the commutativity in (4), and the naturality of μ as follows:

$$\begin{aligned} h_A T(h_A) &= b_A \mu_{K(X)} T(b_A) T(b_A \mu_{K(X)} T(b_A)) \\ &= b_A \mu_{K(X)} T(\mu_{K(X)}) T^2(b_A) \end{aligned}$$

$$= b_A \mu_{K(X)} \mu_{TK(X)} T^2(b_A)$$
$$= b_A \mu_{K(X)} T(b_A) \mu_{TK(X)}$$
$$= h_A \mu_{H(X)} .$$

Thus $\Psi(A) = (H(X), h_A)$ is an $\mathbf{T}$-algebra for each object $A = (X, \varphi)$ of $\mathcal{I}nv$.

Let $A' = (X', \varphi')$ be another object of $\mathcal{I}nv$ and let $f : (X, \varphi) \to (X', \varphi')$ be a morphism in $\mathcal{I}nv$. If $\Psi(A') = (H(X'), h_{A'})$, then

$$H(f)h_A = H(f)b_A \mu_{K(X)} T(b_A)$$

and

$$h_{A'} TH(f) = b_{A'} \mu_{K(X')} T(b_{A'} H(f)) .$$

Recalling that $b_{A'} H(f) = H(f) b_A$ in conjunction with the naturality of μ and the fact that $H = TK$, we obtain

$$H(f)h_A = b_{A'} TK(f) \mu_{K(X)} T(b_A)$$
$$= b_{A'} \mu_{K(X')} T^2 K(f) T(b_A)$$
$$= b_{A'} \mu_{K(X')} T(H(f) b_A) ,$$

and hence also $H(f)h_A = h_{A'} TH(f)$. According to (7), the mapping $H(f)$ is a morphism from $\Psi(A)$ to $\Psi(A')$ in $\mathcal{S}et^{\mathbf{T}}$.

Therefore $\Psi : \mathcal{I}nv \to \mathcal{S}et^{\mathbf{T}}$ is a well-defined faithful functor and the diagram (8) commutes.

To show that (8) is a subpullback, let A and A' be objects of $\mathcal{I}nv$ as before, and let $f : X \to X'$ be a mapping such that $H(f) : \Psi(A) \to \Psi(A')$ is an $\mathbf{T}$-morphism, that is, a mapping satisfying

$$H(f)h_A = h_{A'} TH(f) .$$

Clearly, any such f also satisfies

$$H(f)h_A T(\eta_{K(X)})\gamma^0_X = h_{A'} TH(f) T(\eta_{K(X)})\gamma^0_X . \tag{11}$$

Next we claim that

$$T(b_A) T(\eta_{K(X)}) = T(\eta_{K(X)}) TK(\varphi) = T(\eta_{K(X)}) H(\varphi) . \tag{12}$$

Indeed, since $\gamma^1 : H_1 \to TK$ is the insertion of H_1 into $TK = H = H_0 \amalg H_1$, from (9) it follows that $b_A \gamma^1_X = \gamma^1_X H_1(\varphi)$. Using the natural transformation $\rho : K \to H_1$

with $\eta_{K(X)} = \gamma^1_X \rho_X$ we get

$$b_A \eta_{K(X)} = b_A \gamma^1_X \rho_X = \gamma^1_X H_1(\varphi)\rho_X = \gamma^1_X \rho_X K(\varphi) = \eta_{K(X)} K(\varphi)\,,$$

and (12) follows immediately.

Thus

$$\begin{aligned} h_A T(\eta_{K(X)}) &= b_A \mu_{K(X)} T(b_A) T(\eta_{K(X)}) \\ &= b_A \mu_{K(X)} T(\eta_{K(X)}) H(\varphi) = b_A H(\varphi) \end{aligned}$$

by (10), (12) and (5). Hence

$$h_A T(\eta_{K(X)})\gamma^0_X = b_A H(\varphi)\gamma^0_X = b_A \gamma^0_X H_0(\varphi) = \gamma^0_X H_0(\varphi)\,,$$

where the last equality follows from the definition of b_A. The left-hand side of (11) thus becomes

$$H(f) h_A T(\eta_{K(X)})\gamma^0_X = H(f)\gamma^0_X H_0(\varphi) = \gamma^0_{X'} H_0(f\varphi)\,. \tag{13}$$

Since η is natural, we have $H(f)\eta_{K(X)} = \eta_{K(X')} K(f)$ and hence also

$$\begin{aligned} h_{A'} TH(f) T(\eta_{K(X)}) &= h_{A'} T(\eta_{K(X')}) H(f) \\ &= b_{A'} \mu_{K(X')} T(b_{A'}) T(\eta_{K(X')}) H(f)\,. \end{aligned}$$

But $T(b_{A'})T(\eta_{K(X')}) = T(\eta_{K(X')})H(\varphi')$ by (12), and $\mu_{K(X')}T(\eta_{K(X')}) = 1_{H(X')}$ by (5). Therefore,

$$h_{A'} TH(f) T(\eta_{K(X)}) = b_{A'} H(\varphi') H(f)\,,$$

and the definition of $b_{A'}$ implies that the right-hand side of (11) takes on the form

$$\begin{aligned} &h_{A'} TH(f) T(\eta_{K(X)})\gamma^0_X \\ &\quad = b_{A'} H(\varphi' f)\gamma^0_X = b_{A'}\gamma^0_{X'} H_0(\varphi' f) = \gamma^0_{X'} H_0(\varphi' f)\,. \end{aligned} \tag{14}$$

Since γ^0 is a monotransformation and the functor H_0 is faithful, from (11), (13) and (14) we conclude that $\varphi' f = f\varphi$. Therefore, $f : A \to A'$ is a morphism of $\mathcal{I}nv$, and the diagram (8) is a subpullback as claimed. □

To see that, for any nontrivial monad $\mathbf{T}$, the forgetful functor $U^{\mathbf{T}} : \mathcal{S}et^{\mathbf{T}} \to \mathcal{S}et$ is a slice of $\mathcal{A}lg(1) \to \mathcal{S}et$, we simply define a functor $\Gamma : \mathcal{S}et^{\mathbf{T}} \to \mathcal{A}lg(1)$ by $\Gamma(Y, h) = (T(Y), \eta_Y \circ h)$ on the objects of $\mathcal{S}et^{\mathbf{T}}$, and by $\Gamma(f) = T(f)$ on its morphisms. If (Y, h) and (Y', h') are $\mathbf{T}$-algebras, then the naturality and injectivity of η imply that a mapping $g : Y \to Y'$ satisfies $T(g) \circ \eta_Y \circ h = \eta_{Y'} \circ h' \circ T(g)$ if and only if $g \circ h = h' \circ T(g)$. But the latter statement is equivalent to the claim that the diagram

$$\begin{array}{ccc} \mathcal{S}et^{\mathbf{T}} & \xrightarrow{U^{\mathbf{T}}} & \mathcal{S}et \\ {\scriptstyle \Gamma}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{A}lg(1) & \longrightarrow & \mathcal{S}et \end{array}$$

is a subpullback.

In conjunction with Proposition 4.1, this shows that, for any nondegenerate monad $\mathbf{T}$, the category $\mathcal{S}et^{\mathbf{T}}$ belongs to the basket $\mathbf{A}$.

Corollary 4.2. *If V is a nontrivial variety of finitary or infinitary algebras that is not polynomially equivalent to the variety of sets, then either*

(a) *V is essentially nullary and belongs to the basket* $\mathbf{P}$, *or*

(b) *V is not essentially nullary and belongs to the basket* $\mathbf{A}$. $\square$

5. Proof of Theorem 1.2

Let $U : \mathcal{K} \to \mathcal{S}et$ be a functor of descent type. Then U has a left adjoint $F : \mathcal{S}et \to \mathcal{K}$; let $\eta : I \to U \circ F$ and $\varepsilon : F \circ U \to I$ respectively denote the unit and the counit of this adjunction. The comparison functor

$$\Phi : \mathcal{K} \to \mathcal{S}et^{\mathbf{T}}$$

into the category of all $\mathbf{T}$-algebras over the monad $\mathbf{T} = (T, \eta, \mu) = (UF, \eta, U\varepsilon F)$, determined by $\Phi(a) = (U(a), U(\varepsilon_a))$ on objects of $\mathcal{K}$ is full and faithful. Recall that $U = U^{\mathbf{T}} \circ \Phi$, where $U^{\mathbf{T}} : \mathcal{S}et^{\mathbf{T}} \to \mathcal{S}et$ is given by $U^{\mathbf{T}}(Y, h) = Y$ for any $\mathbf{T}$-algebra (Y, h). Since Φ is full and faithful, the diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{U} & \mathcal{S}et \\ {\scriptstyle \Phi}\downarrow & & \| \\ \mathcal{S}et^{\mathbf{T}} & \xrightarrow[U^{\mathbf{T}}]{} & \mathcal{S}et \end{array}$$

is a subpullback. For a nondegenerate monad $\mathbf{T}$, let $\Psi : \mathcal{I}nv \to \mathcal{S}et^{\mathbf{T}}$ be the functor from the proof of Theorem 1.1 for which the diagram

$$\begin{array}{ccc} \mathcal{I}nv & \longrightarrow & \mathcal{S}et \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle TK} \\ \mathcal{S}et^{\mathbf{T}} & \xrightarrow[U^{\mathbf{T}}]{} & \mathcal{S}et \end{array}$$

is a subpullback. Should there be a functor $\Sigma : \mathcal{I}nv \to \mathcal{K}$ for which $\Psi = \Phi \circ \Sigma$, then, clearly, the forgetful functor $\mathcal{I}nv \to \mathcal{S}et$ is a slice of $U : \mathcal{K} \to \mathcal{S}et$. To complete the proof of Theorem 1.2, we need to show that this is the case whenever U has the transfer property.

Let $A = (X, \varphi)$ be an object of $\mathcal{I}nv$. Recall that $\Psi(X, \varphi) = (TK(X), h_A)$ is a $\mathbf{T}$-algebra with $h_A = b_A \mu_{K(X)} T(b_A)$ for some involution $b_A : TK(X) \to TK(X)$. For the object $a = FK(X)$ of $\mathcal{K}$ we have $\Phi(a) = (TK(X), U(\varepsilon_{FK(X)})) = (TK(X), \mu_{K(X)})$. It is clear that the diagram

$$\begin{array}{ccc} T^2K(X) & \xrightarrow{T(b_A)} & T^2K(X) \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle \mu_{K(X)}} \\ TK(X) & \xrightarrow[b_A]{} & TK(X) \end{array}$$

commutes; hence $b_A : \Psi(A) \to \Phi(a)$ is an isomorphism of these $\mathbf{T}$-algebras. Therefore, $U^{\mathbf{T}}(b_A) : U^{\mathbf{T}}\Psi(A) \to U(a)$ is an isomorphism in $\mathcal{S}et$ and, because U has the transfer property, there exists a $\mathcal{K}$-isomorphism $\sigma : \bar{a} \to a$ such that $U(\bar{a}) = U^{\mathbf{T}}\Psi(A) = TK(X)$ and $U(\sigma) = U^{\mathbf{T}}(b_A)$. It follows that the composite $b_A^{-1} \circ \Phi(\sigma) : \Phi(\bar{a}) \to \Psi(A)$ is a $\mathbf{T}$-algebra isomorphism with $U^{\mathbf{T}}(b_A^{-1} \circ \Phi(\sigma)) = U^{\mathbf{T}}(b_A^{-1}) \circ U^{\mathbf{T}}(b_A) = 1_{TK(X)}$. Thus the diagram

$$\begin{array}{ccc} T^2K(X) & = & T^2K(X) \\ {\scriptstyle U(\varepsilon_{\bar{a}})}\downarrow & & \downarrow{\scriptstyle h_A} \\ TK(X) & = & TK(X) \end{array}$$

commutes and, consequently, $\Phi(\bar{a}) = \Psi(A)$. We set $\Sigma(A) = \bar{a}$ and extend this assignment to a functor $\Sigma : \mathcal{I}nv \to \mathcal{K}$ in a standard manner.

The comparison functor $\Phi : \mathcal{K} \to \mathcal{S}et^{\mathbf{T}}$ is a full embedding and $U^{\mathbf{T}} \circ \Phi = U$, so that U is a slice of $U^{\mathbf{T}}$. As noted at the conclusion of the preceding section, the functor $U^{\mathbf{T}}$ is, in turn, a slice of the forgetful functor $\mathcal{A}lg(1) \to \mathcal{S}et$. Combined with the argument above, this observation completes the proof of Theorem 1.2. □

Remark. (a) Let $\mathcal{B}an_1$ be the category of all Banach spaces and all linear operators of norm at most one, and let $U : \mathcal{B}an_1 \to \mathcal{S}et$ denote its natural unit-ball forgetful functor. Since U is of descent type [18], has a left adjoint leading to a nondegenerate monad, and enjoys the transfer property, Theorem 1.2 applies to U to show that $\mathcal{B}an_1$ belong to the basket $\mathbf{A}$.

(b) It is well known (cf. [2, p. 114]) that the category $\mathcal{C}omp$ of all compact Hausdorff spaces and all their continuous maps is a category $\mathcal{S}et^{\mathbf{T}}$ of monadic algebras over a monad $\mathbf{T} = (T, \eta, \mu)$ whose functor part T assigns the set βX of all ultrafilters to any given set X. It follows that the forgetful functor $U : \mathcal{C} \to \mathcal{S}et$

of the full subcategory $\mathscr{C}$ of $\mathscr{C}omp$ consisting of all β-compactifications of discrete spaces is of descent type and has the transfer property. Hence any full subcategory of $\mathscr{C}omp$ down to the category $\mathscr{C}$ (e.g., the category of all extremally disconnected spaces) belongs to the basket **A**.

References

[1] M.E. Adams and H.A. Priestley, De Morgan algebras are universal, Discrete Math. 66 (1987) 1–13.

[2] M. Barr and G. Wells, Toposes, Triples and Theories (Springer, Berlin, 1985).

[3] G. Birkhoff, On groups of automorphisms, Revista Un. Mat. Argentina 11 (1949) 155–157 (in Spanish).

[4] J. De Groot, Groups represented by homeomorphism groups, Math. Ann. 138 (1959) 80–102.

[5] P. Goralčík, V. Koubek and J. Sichler, Universal varieties of (0, 1)-lattices, Canad. J. Math. 42 (1990) 470–490.

[6] G. Grätzer and J. Sichler, On the endomorphism semigroup (and category) of bounded lattices, Pacific J. Math. 35 (1970) 639–647.

[7] Z. Hedrlín and J. Lambek, How comprehensive is the category of semigroups?, J. Algebra 11 (1969) 195–212.

[8] Z. Hedrlín and A. Pultr, On full embeddings of categories of algebras, Illinois J. Math. 10 (1966) 392–406.

[9] J.R. Isbell, Subobjects, adequacy, completeness and categories of algebras, Rozprawy Matematyczne 36 (1964).

[10] J.R. Isbell, Epimorphisms and dominions, in: Proceedings of the Conference on Categorical Algebra, La Jolla, 1965 (Springer, Berlin, 1966) 232–246.

[11] V. Koubek, Infinite image homomorphisms of distributive bounded lattices, in: Lectures in Universal Algebra (Szeged, 1983), Colloq. Math. Soc. János Bolyai 43 (North-Holland, Amsterdam, 1985) 241–281.

[12] V. Koubek and J. Sichler, Universal varieties of semigroups, J. Austral. Math. Soc. Ser. A 36 (1984) 143–152.

[13] W. Lampe, J. Sichler and V. Trnková, Homomorphisms of unary algebras and of their expansions, to appear.

[14] M. Petrich, J. Sichler and V. Trnková, Simultaneous representations in categories of algebras, Algebra Universalis 27 (1990) 426–453.

[15] A. Pultr and V. Trnková, Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories (North-Holland, Amsterdam, 1980).

[16] J. Sichler, Group universal unary varieties, Algebra Universalis 11 (1980) 12–21.

[17] J. Sichler and V. Trnková, Functor slices and simultaneous representations, in: Contributions to General Algebra (Hölder–Pichler–Tempsky/Teubner, Wien/Stuttgart, 1991) 299–320.

[18] W. Tholen, Relative Bildzerlegungen und algebraische Kategorien, Thesis, Univ. Münster, 1973.

[19] V. Trnková, Some properties of set functors, Comment. Math. Univ. Carolin. 10 (1969) 323–352.

[20] V. Trnková, On descriptive classification of set functors I, Comment. Math. Univ. Carolin. 12 (1971) 143–174.

[21] V. Trnková, Simultaneous representations in discrete structures, Comment. Math. Univ. Carolin. 27 (1986) 633–649.

[22] V. Trnková, Simultaneous representations in categories, in: Categorical Algebra and its Applications, Proceedings, Louvain-La-Neuve, 1987, Lecture Notes in Mathematics 1348 (Springer, Berlin, 1987) 342–363.

[23] V. Trnková, Non-constant continuous maps of spaces and of their β-compactifications, Topology Appl. 33 (1989) 47–62.

[24] P. Vopěnka, A. Pultr and Z. Hedrlín, A rigid relation exists on any set, Comment. Math. Univ. Carolin. 6 (1965) 149–155.