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On the structure of split Lie color algebras

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ABSTRACT

The class of Lie color algebras contains the one of Lie superalgebras and so the one of Lie algebras. In order to begin an approach to the structure of arbitrary Lie color algebras, (with no restrictions neither on the dimension nor on the base field), we introduce the class of split Lie color algebras as the natural extension of the classes of split Lie algebras and split Lie superalgebras. By developing techniques of connections of roots for this kind of algebra, we show that any such algebra L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian (graded) subalgebra H and each I_j a well described (graded) ideal of L satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of L is characterized and it is shown that L is the direct sum of the family of its minimal (graded) ideals, each one being a simple split Lie color algebra.

1. Introduction and previous definitions

Lie color algebras were introduced as 'generalized Lie algebras' in 1960 by Ree [8], being also called color Lie superalgebras (see [1]). Since then, this kind of algebra has been an object of constant interest in mathematics, (see [9,10] and [6,7,14–16] for recent references), being also remarkable for the important role played in theoretical physic, especially in conformal field theory and supersymmetries [12,13].

In the present paper we begin an approach to the structure of infinite dimensional Lie color algebras by introducing the class of split Lie color algebras of arbitrary dimension as the natural extension of the class of split Lie superalgebras studied in [4], which in turn extends the class of split Lie algebras [2,11].

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Throughout this paper, Lie color algebras L are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is no restriction on $\dim L$, the products $[L_\alpha, L_{-\alpha}]$, or $\{k \in \mathbb{K} : k\alpha \in \Lambda\}$, where L_α denotes the root space associated to the root α, and Λ is the set of nonzero roots of L.

In §2 we develop techniques of connections of roots in the framework of split Lie color algebras so as to show that L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian (graded) subalgebra H and each I_j a well described (graded) ideal of L satisfying $[I_j, I_k] = 0$ if $j \neq k$. In §3 and under certain conditions, the simplicity of L is characterized and it is shown that L is the direct sum of the family of its minimal (graded) ideals, each one being a simple split Lie color algebra.

Definition 1.1. Let $\mathbb{K}$ be an arbitrary field and Γ an abelian group. A skew-symmetric bicharacter of Γ is a map $\epsilon : \Gamma \times \Gamma \longrightarrow \mathbb{K} \setminus \{0\}$ satisfying

$$\epsilon(g_1, g_2) = \epsilon(g_2, g_1)^{-1},$$

$$\epsilon(g_1, g_2 + g_3) = \epsilon(g_1, g_2)\epsilon(g_1, g_3),$$

for any $g_1, g_2, g_3 \in \Gamma$.

It is clear that $\epsilon(g, 0) = 1$ for any $g \in \Gamma$, where 0 denotes the identity element of Γ.

Definition 1.2. Let $L = \bigoplus_{g\in\Gamma} L_g$ be a Γ-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\bar{v}$ the unique group element in Γ such that $v \in L_{\bar{v}}$, which will be called the homogeneous degree of v. We shall say that L is a Lie color algebra if it is endowed with a $\mathbb{K}$-bilinear map

$$[\cdot, \cdot] : L \times L \longrightarrow L$$

satisfying:

$$[v, w] = -\epsilon(\bar{v}, \bar{w})[w, v], \quad \text{(Skew-symmetry)}$$

$$[v, [w, t]] = [[v, w], t] + \epsilon(\bar{v}, \bar{w})[w, [v, t]], \quad \text{(Jacobi identity)}$$

for all homogeneous elements $v, w, t \in L$.

Lie superalgebras are examples of Lie color algebras with $\Gamma = \mathbb{Z}_2$ and $\epsilon(i, j) = (-1)^{ij}$, for any $i, j \in \mathbb{Z}_2$. We also note that L_0 is a Lie algebra.

The usual regularity concepts will be understood in the graded sense. For instance, an *ideal* I of L is a graded subspace $I = \bigoplus_{g\in\Gamma} I_g$ of L such that $[I, L] \subset I$. A Lie color algebra L will be called *simple* if $[L, L] \neq 0$ and its only (graded) ideals are $\{0\}$ and L.

Let us introduce the class of split algebras in the framework of Lie color algebras. We recall that given an element x of a Lie algebra $\mathfrak{L}$; the adjoint mapping is denoted by ad_x and defined as $ad_x(y) = [x, y]$ for any $y \in \mathfrak{L}$. A splitting Cartan subalgebra $\mathfrak{H}$ of a Lie algebra $\mathfrak{L}$ is defined as a maximal abelian subalgebra, (MASA), of $\mathfrak{L}$ such that the adjoint mappings ad_h for $h \in \mathfrak{H}$ are simultaneously diagonalizable. If $\mathfrak{L}$ contains a splitting Cartan subalgebra $\mathfrak{H}$, then $\mathfrak{L}$ is called a split Lie algebra. This means that we have a root space decomposition $\mathfrak{L} = \mathfrak{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$ where $\mathfrak{L}_\alpha = \{v_\alpha \in \mathfrak{L} : [h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in \mathfrak{H}\}$ for a linear functional $\alpha \in \mathfrak{H}^*$, $\Lambda := \{\alpha \in \mathfrak{H}^* \setminus \{o\} : \mathfrak{L}_\alpha \neq 0\}$ and $o \in \mathfrak{H}^*$ denotes the zero linear functional. The subspaces $\mathfrak{L}_\alpha$ for $\alpha \in \mathfrak{H}^*$ are called root spaces of $\mathfrak{L}$ and the elements $\alpha \in \Lambda \cup \{o\}$ are called roots of $\mathfrak{L}$.

We introduce the concept of split Lie color algebra in an analogous way. We begin by considering a maximal abelian graded subalgebra $H = \bigoplus_{g\in\Gamma} H_g$ among the abelian graded subalgebras of L. Observe that H is necessarily a maximal abelian subalgebra of L as the following lemma shows.

Lemma 1.1. *Let $H = \bigoplus_{g\in\Gamma} H_g$ be a maximal abelian graded subalgebra of a Lie color algebra L. Then H is a maximal abelian subalgebra of L.*

Proof. Consider an abelian subalgebra K of L such that $H \subset K$. For any $x \in K$ we have $[x, H_g] = 0$ for each $g \in \Gamma$, and so by writing $x = \sum_{i=1}^{n} x_{g_i}$ with $x_{g_i} \in L_{g_i}$ for $i = 1, \ldots, n$, being $g_i \in \Gamma$ and $g_i \neq g_j$ if $i \neq j$, we get by the grading $[x_{g_i}, H_g] = 0$. Hence, for any g_i, $i = 1, \ldots, n$, we have $(H_{g_i} + \mathbb{K}x_{g_i}) \oplus (\bigoplus_{g\in\Gamma\setminus\{g_i\}} H_g)$ is an abelian graded subalgebra of L containing H and so $x_{g_i} \in H_{g_i}$. From here we get $x \in H$ and then $K = H$. $\square$

Definition 1.3. Denote by $H = \bigoplus_{g\in\Gamma} H_g$ a maximal abelian (graded) subalgebra, (MAGSA), of a Lie color algebra L. For a linear functional $\alpha : H_0 \longrightarrow \mathbb{K}$, we define the root space of L, (respect to H), associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [h_0, v_\alpha] = \alpha(h_0)v_\alpha \ \ \textit{for any} \ \ h_0 \in H_0\}.$$

The elements $\alpha \in (H_0)^*$ satisfying $L_\alpha \neq 0$ are called roots of L respect to H and we denote $\Lambda := \{\alpha \in (H_0)^*\setminus\{\circ\} : L_\alpha \neq 0\}$ where $\circ \in (H_0)^*$ is the zero lineal functional. We say that L is a split Lie color algebra, respect to H, if

$$L = H \oplus \left(\bigoplus_{\alpha\in\Lambda} L_\alpha\right).$$

We also say that Λ is the *root system* of L.

Split Lie algebras and split Lie superalgebras are examples of split Lie color algebras. Hence, the present paper extends the results in [2,4].

It is clear that the root space associated to the zero root $L_\circ$ satisfies $H \subset L_\circ$. Conversely, given any $v_\circ \in L_\circ$ we can write $v_\circ = h + \sum_{i=1}^{n} v_{\alpha_i}$ with $h \in H$ and $v_{\alpha_i} \in L_{\alpha_i}$ for $i = 1, \ldots, n$, being $\alpha_i \in \Lambda$ with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence $0 = [h_0, h + \sum_{i=1}^{n} v_{\alpha_i}] = \sum_{i=1}^{n} \alpha_i(h_0)v_{\alpha_i}$ for any $h_0 \in H_0$. So, taking into account the direct character of the sum and that $\alpha_i \neq 0$, we have that any $v_{\alpha_i} = 0$ and then $v_\circ \in H$. Consequently

$$H = L_\circ. \tag{1}$$

Lemma 1.2. *Let $L = \bigoplus_{g\in\Gamma} L_g$ be a split Lie color algebra with corresponding root space decomposition $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$. If we denote by $L_{\alpha,g} = L_\alpha \cap L_g$, then the following assertions hold.*

1. *$L_\alpha = \bigoplus_{g\in\Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{\circ\}$.*
2. *$H_g = L_{\circ,g}$. In particular $H_0 = L_{\circ,0}$.*
3. *L_0 is a split Lie algebra, respect to H_0, with root space decomposition $L_0 = H_0 \oplus (\bigoplus_{\alpha\in\Lambda} L_{\alpha,0})$*

Proof

1. By the Γ-grading of L we may express any $v_\alpha \in L_\alpha$, $\alpha \in \Lambda \cup \{\circ\}$, in the form $v_\alpha = v_{\alpha,g_1} + \cdots + v_{\alpha,g_n}$ with $v_{\alpha,g_i} \in L_{g_i}$ for distinct $g_1, \ldots, g_n \in \Gamma$. If $h_0 \in H_0$ then $[h_0, v_{\alpha,g_i}] = \alpha(h_0)v_{\alpha,g_i}$ for $i = 1, \ldots, n$. Hence $L_\alpha = \bigoplus_{g\in\Gamma}(L_\alpha \cap L_g)$ and we can write $L_\alpha = \bigoplus_{g\in\Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{\circ\}$.
2. Consequence of Eq. (1) and item 1.
3. We also have $L_g = H_g \oplus (\bigoplus_{\alpha\in\Lambda} L_{\alpha,g})$ for any $g \in \Gamma$. By considering $g = 0$ we get $L_0 = H_0 \oplus (\bigoplus_{\alpha\in\Lambda} L_{\alpha,0})$. Hence, the direct character of the sum and the fact that $\alpha \neq 0$ for any $\alpha \in \Lambda$ give us that H_0 is a MASA of the Lie algebra L_0. Hence L_0 is a split Lie algebra respect to H_0. $\square$

Lemma 1.3 is an immediate consequence of the Jacobi identity and the fact $\epsilon(0, g) = 1$ for any $g \in \Gamma$. Together, Lemmas 1.2 and 1.3 show the root space decomposition provides a refinement (see [5, Definition 3.1.4]) of the Γ-grading of L.

Lemma 1.3. *If $[L_\alpha, L_\beta] \neq 0$ with $\alpha, \beta \in \Lambda \cup \{o\}$, then $\alpha + \beta \in \Lambda \cup \{o\}$ and $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.*

From Lemmas 1.3 and 1.2-1 we can assert that

$$[L_{\alpha,g_1}, L_{\beta,g_2}] \subset L_{\alpha+\beta,g_1+g_2}$$

for any $g_1, g_2 \in \Gamma$.

Definition 1.4. A root system of a split Lie color algebra L is called symmetric if $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$ for any linear functional $\alpha : H_0 \to \mathbb{K}$.

2. Connections of Roots. Decompositions

In the following, L denotes a split Lie color algebra with a symmetric root system Λ and $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ the corresponding root space decomposition.

Definition 2.1. Let α and β be two nonzero roots. We say that α is connected to β if there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

1. $\alpha_1 = \alpha$.
2. $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots.., \alpha_1 + \cdots + \alpha_{n-1}\} \subset \Lambda$.
3. $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \in \{\pm\beta\}$.

We also say that $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to β.

Observe that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$ and so α is connected to $\pm\alpha$.

The next result shows the connection relation is of equivalence. The proof is analogous to the one for split Lie algebras given in [2, Proposition 2.1] and virtually identical to the one for split Lie superalgebras given in [4, Proposition 2.1].

Proposition 2.1. *The relation $\sim$ in Λ defined by $\alpha \sim \beta$ if and only if α is connected to β is of equivalence.*

For any $\alpha \in \Lambda$, we denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \beta \sim \alpha\}.$$

Clearly if $\beta \in \Lambda_\alpha$ then $-\beta \in \Lambda_\alpha$ and, by Proposition 2.1, if $\gamma \notin \Lambda_\alpha$ then $\Lambda_\alpha \cap \Lambda_\gamma = \emptyset$.

Our next goal is to associate an (adequate) ideal L_{Λ_α} of L to any Λ_α. For $\Lambda_\alpha, \alpha \in \Lambda$, we define $H_{\Lambda_\alpha} := span_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in \Lambda_\alpha\}$. Then H_{Λ_α} is the direct sum of

$$\sum_{\beta \in \Lambda_\alpha, g \in \Gamma} [L_{\beta,g}, L_{-\beta,-g}] \subseteq H_0$$

and

$$\sum_{\substack{\beta \in \Lambda_\alpha; \\ g,g' \in \Gamma, g+g' \neq 0}} [L_{\beta,g}, L_{-\beta,g'}] \subseteq \bigoplus_{g \in \Gamma \setminus \{0\}} H_g.$$

We also define

$$V_{\Lambda_\alpha} := \bigoplus_{\beta \in \Lambda_\alpha} L_\beta = \bigoplus_{\beta \in \Lambda_\alpha, g \in \Gamma} L_{\beta,g}.$$

Finally, we denote by L_{Λ_α} the following (graded) subspace of L,

$$L_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.$$

Proposition 2.2. *Let $\alpha \in \Lambda$. Then the following assertions hold.*

1. $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$.
2. *If $\gamma \notin \Lambda_\alpha$ then $[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = 0$.*

Proof.

1. Taking into account $H = L_\circ$ and Lemma 1.3, we have

$$[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha} + \sum_{\beta,\delta \in \Lambda_\alpha} [L_\beta, L_\delta]. \quad (2)$$

If $\delta = -\beta$ then

$$[L_\beta, L_\delta] \subset H_{\Lambda_\alpha}. \quad (3)$$

If $\delta \neq -\beta$, by Lemma 1.3 we have that in case $[L_\beta, L_\delta] \neq 0$ then $\beta + \delta \in \Lambda$. From here, if $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to β then $\{\alpha_1, \ldots, \alpha_n, \delta\}$ is a connection from α to $\beta+\delta$ in case $\alpha_1 + \cdots + \alpha_n = \beta$ and $\{\alpha_1, \ldots, \alpha_n, -\delta\}$ in case $\alpha_1 + \cdots + \alpha_n = -\beta$. Hence $\beta + \delta \in \Lambda_\alpha$ and so

$$[L_\beta, L_\delta] \subset V_{\Lambda_\alpha}. \quad (4)$$

From Eqs. (2)–(4) we conclude $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$.

2. We have

$$[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\gamma} \oplus V_{\Lambda_\gamma}] \subset [H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]. \quad (5)$$

Consider the above third summand $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ and suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[L_\beta, L_\eta] \neq 0$. As necessarily $\beta \neq -\eta$, then $\beta + \eta \in \Lambda$. So $\{\beta, \eta, -\beta\}$ is a connection between β and η. By the transitivity of the connection relation we have $\gamma \in \Lambda_\alpha$, a contradiction. Hence $[L_\beta, L_\eta] = 0$ and so

$$[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0. \quad (6)$$

Consider now the first summand $[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ in (5) and suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[[L_\beta, L_{-\beta}], L_\eta] \neq 0$. Then

$$[[L_{\beta,g}, L_{-\beta,g'}], L_\eta] \neq 0$$

for some $g, g' \in \Gamma$. By Jacobi identity, either $[L_{-\beta,g'}, L_\eta] \neq 0$ or $[L_{\beta,g}, L_\eta] \neq 0$ and so $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] \neq 0$ in any case, what contradicts Eq. (6). Hence

$$[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0.$$

Finally, we note that the same above argument shows

$$[V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] = 0.$$

By Eq. (5) we conclude $[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = 0$. $\square$

Proposition 2.2-1 let us assert that for any $\alpha \in \Lambda$, L_{Λ_α} is a Lie color subalgebra of L that we call the Lie color subalgebra of L *associated* to Λ_α.

Theorem 2.1. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the Lie color subalgebra*

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$$

of L associated to Λ_α is an ideal of L.

2. *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha \in \Lambda}[L_\alpha, L_{-\alpha}]$.*

Proof

1. Since $[L_{\Lambda_\alpha}, H] = [L_{\Lambda_\alpha}, L_\circ] \subset V_{\Lambda_\alpha}$, taking into account Proposition 2.2 we have

$$[L_{\Lambda_\alpha}, L] = \left[L_{\Lambda_\alpha}, H \oplus \left(\bigoplus_{\beta \in \Lambda_\alpha} L_\beta\right) \oplus \left(\bigoplus_{\gamma \notin \Lambda_\alpha} L_\gamma\right)\right] \subset L_{\Lambda_\alpha}.$$

2. The simplicity of L implies $L_{\Lambda_\alpha} = L$. Therefore $\Lambda_\alpha = \Lambda$ and $H = \sum_{\alpha \in \Lambda}[L_\alpha, L_{-\alpha}]$. $\square$

Theorem 2.2. *For a vector space complement $\mathcal{U}$ of $\mathrm{span}_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}] : \alpha \in \Lambda\}$ in H, we have*

$$L = \mathcal{U} + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals L_{Λ_α} of L described in Theorem 2.1-1, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. By Proposition 2.1, we can consider the quotient set $\Lambda/\sim := \{[\alpha] : \alpha \in \Lambda\}$. Let us denote by $I_{[\alpha]} := L_{\Lambda_\alpha}$. We have $I_{[\alpha]}$ is well defined and, by Theorem 2.1-1, an ideal of L. Therefore

$$L = \mathcal{U} + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

By applying Proposition 2.2-2 we also obtain $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. $\square$

Let us denote by $\mathcal{Z}(L) = \{v \in L : [v, L] = 0\}$ the *center* of L.

Corollary 2.1. *If $\mathcal{Z}(L) = 0$ and $[L, L] = L$, then L is the direct sum of the ideals given in Theorem 2.1,*

$$L = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Proof. Taking into account Theorem 2.2, from $[L, L] = L$ it is clear that $L = \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$. The direct character of the sum now follows from the facts $[I_{[\alpha]}, I_{[\beta]}] = 0$, if $[\alpha] \neq [\beta]$, and $\mathcal{Z}(L) = 0$. $\square$

3. The simple components

In this section we study if any of the components in the decomposition given in Corollary 2.1 is simple. Under certain conditions we give an affirmative answer. From now on $\mathrm{char}(\mathbb{K}) = 0$.

Lemma 3.1. *Let $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$ be a split Lie color algebra. If I is an ideal of L then $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha))$.*

Proof. We may view $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$ as a weight module respect to the split Lie algebra L_0 with maximal abelian subalgebra H_0, (see Lemma 1.2-3), in the natural way. The characteristic property of ideals gives us that I is a submodule of L. It is well-known that a submodule of a weight module is again a weight module. From here, I is a weight module respect to L_0, (and H_0), and so $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha))$. □

Taking into account the above lemma, observe that the grading of I and Lemma 1.2-1 let us write

$$I = \bigoplus_{g\in\Gamma} I_g = \bigoplus_{g\in\Gamma}\left((I_g \cap H_g) \oplus \left(\bigoplus_{\alpha\in\Lambda}(I_g \cap L_{\alpha,g})\right)\right). \tag{7}$$

Lemma 3.2. *Let L be a split Lie color algebra with $\mathcal{Z}(L) = 0$. If I is an ideal of L such that $I \subset H$, then $I = \{0\}$.*

Proof. Suppose there exists a nonzero ideal I of L such that $I \subset H$. We have $[I, H] \subset [H, H] = 0$. We also have that the fact $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] \subset I \subset H$ implies $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] \subset H \cap (\bigoplus_{\alpha\in\Lambda} L_\alpha) = 0$. From here $I \subset \mathcal{Z}(L) = 0$, a contradiction. □

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split Lie color algebras, in a similar way to the ones for split Lie algebras, split Lie superalgebras and split Lie triple systems, (see [2–4] for these notions and examples). For each $g \in \Gamma$, we denote by $\Lambda_g := \{\alpha \in \Lambda : L_{\alpha,g} \neq 0\}$, (see Lemma 1.2-1).

Definition 3.1. We say that a split Lie color algebra L is root-multiplicative if given $\alpha \in \Lambda_{g_i}$ and $\beta \in \Lambda_{g_j}$, with $g_i, g_j \in \Gamma$, such that $\alpha + \beta \in \Lambda$, then $[L_{\alpha,g_i}, L_{\beta,g_j}] \neq 0$.

Definition 3.2. We say that a split Lie color algebra L is of maximal length if for any $\alpha \in \Lambda_g$, $g \in \Gamma$, we have $\dim L_{\kappa\alpha,\kappa g} = 1$ for $\kappa \in \{\pm 1\}$.

Observe that if L is of maximal length, then Eq. (7) let us assert that given any nonzero ideal I of L then

$$I = \bigoplus_{g\in\Gamma}\left((I_g \cap H_g) \oplus \left(\bigoplus_{\alpha\in\Lambda_g^I} L_{\alpha,g}\right)\right) \tag{8}$$

where $\Lambda_g^I := \{\alpha \in \Lambda : I_g \cap L_{\alpha,g} \neq 0\}$ for each $g \in \Gamma$.

Theorem 3.1. *Let L be a split Lie color algebra of maximal length, root multiplicative and with $\mathcal{Z}(L) = 0$. Then L is simple if and only if it has all its nonzero roots connected and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$.*

Proof. The first implication is Theorem 2.1-2. To prove the converse, consider I a nonzero ideal of L. By Lemma 3.2 and Eq. (8) we can write $I = \bigoplus_{g\in\Gamma}((I_g \cap H_g) \oplus (\bigoplus_{\alpha\in\Lambda_g^I} L_{\alpha,g}))$ with $\Lambda_g^I \subset \Lambda_g$ for any $g \in \Gamma$ and some $\Lambda_g^I \neq \emptyset$. Hence, we may choose $\alpha_0 \in \Lambda_g^I$ being so

$$0 \neq L_{\alpha_0,g} \subset I. \tag{9}$$

For any $\beta \in \Lambda \backslash \{\pm\alpha_0\}$, the fact that α_0 and β are connected gives us a connection $\{\gamma_1, \ldots., \gamma_r\}$ from α_0 to β such that

$$\gamma_1 = \alpha_0,$$
$$\gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3, ..., \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_{r-1} \in \Lambda$$

and

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_r \in \{\pm\beta\}.$$

Consider $\alpha_0 = \gamma_1$, γ_2 and $\gamma_1 + \gamma_2$. Since $\gamma_2 \in \Lambda$ there exists $g_1 \in \Gamma$ such that $L_{\gamma_2,g_1} \neq 0$. From here, the root-multiplicativity and maximal length of L show $0 \neq [L_{\alpha_0,g}, L_{\gamma_2,g_1}] = L_{\alpha_0+\gamma_2,g+g_1}$, and by Eq. (9)

$$0 \neq L_{\alpha_0+\gamma_2,g+g_1} \subset I.$$

We can argue in a similar way from $\alpha_0 + \gamma_2$, γ_3 and $\alpha_0 + \gamma_2 + \gamma_3$ to get

$$0 \neq L_{\alpha_0+\gamma_2+\gamma_3,g_2} \subset I$$

for some $g_2 \in \Gamma$. Following this process with the connection $\{\gamma_1, ..., \gamma_r\}$ we obtain that

$$0 \neq L_{\alpha_0+\gamma_2+\gamma_3+\cdots\gamma_r,g_3} \subset I$$

and so either $0 \neq L_{\beta,g_3} \subset I$ or $0 \neq L_{-\beta,g_3} \subset I$ for some $g_3 \in \Gamma$. That is,

$$0 \neq L_{\epsilon\beta,g_3} \subset I \text{ for some } \epsilon \in \{\pm 1\}, \text{ some } g_3 \in \Gamma \tag{10}$$

and for any $\beta \in \Lambda$.

Taking into account $H = \sum_{\gamma\in\Lambda}[L_\gamma, L_{-\gamma}]$, the grading of L gives us $H_0 = \sum_{\gamma\in\Lambda,g\in\Gamma}[L_{\gamma,g}, L_{-\gamma,-g}]$. From here, there exists $\gamma \in \Lambda$ and $g_4 \in \Gamma$ such that

$$[[L_{\gamma,g_4}, L_{-\gamma,-g_4}], L_{\epsilon\beta,g_3}] \neq 0. \tag{11}$$

By the Jacobi identity either $[L_{\gamma,g_4}, L_{\epsilon\beta,g_3}] \neq 0$ or $[L_{-\gamma,-g_4}, L_{\epsilon\beta,g_3}] \neq 0$ and so $L_{\gamma+\epsilon\beta,g_4+g_3} \neq 0$ or $L_{-\gamma+\epsilon\beta,-g_4+g_3} \neq 0$. That is (see Eq. (10))

$$0 \neq L_{\kappa\gamma+\epsilon\beta,\kappa g_4+g_3} \subset I \tag{12}$$

for some $\kappa \in \{\pm 1\}$. Since $\epsilon\beta \in \Lambda_{g_3}$ we have by the maximal length of L that $-\epsilon\beta \in \Lambda_{-g_3}$. By Eq. (12) and the root-multiplicativity and maximal length of L we obtain

$$0 \neq [L_{\kappa\gamma+\epsilon\beta,\kappa g_4+g_3}, L_{-\epsilon\beta,-g_3}] = L_{\kappa\gamma,\kappa g_4} \subset I. \tag{13}$$

Taking into account Eq. (13) and that Eq. (11) gives us

$$\beta([L_{\gamma,g_4}, L_{-\gamma,-g_4}]) \neq 0,$$

we have that for any $g_5 \in \Gamma$ such that $L_{\epsilon\beta,g_5} \neq 0$ necessarily

$$0 \neq [[L_{\gamma,g_4}, L_{-\gamma,-g_4}], L_{\epsilon\beta,g_5}] = L_{\epsilon\beta,g_5} \subset I$$

and so $L_{\epsilon\beta} \subset I$. That is, we can assert that

$$L_{\epsilon\beta} \subset I \tag{14}$$

for any $\beta \in \Lambda$ and some $\epsilon \in \{\pm 1\}$. Since $H = \sum_{\beta\in\Lambda}[L_\beta, L_{-\beta}]$ we get

$$H \subset I. \tag{15}$$

Now, given any $-\epsilon\beta \in \Lambda$, by the facts $-\epsilon\beta \neq 0$, $H \subset I$ and the maximal length of L we have

$$[H_0, L_{-\epsilon\beta}] = L_{-\epsilon\beta} \subset I. \tag{16}$$

From Eqs. (14)–(16) we conclude $I = L$. Consequently L is simple. □

Theorem 3.2. *Let L be a split Lie color algebra of maximal length, root multiplicative, and satisfying $\mathcal{Z}(L) = 0$, $[L, L] = L$. Then L is the direct sum of the family of its minimal ideals, each one being a simple split Lie color algebra having all its nonzero roots connected.*

Proof. By Corollary 2.1, $L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ is the direct sum of the ideals $I_{[\alpha]} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha} = (\sum_{\beta\in[\alpha]}[L_\beta, L_{-\beta}]) \oplus (\bigoplus_{\beta\in[\alpha]} L_\beta)$ having any $I_{[\alpha]}$ its root system, Λ_α, with all of its roots connected. It is easy to check that Λ_α has all of its roots Λ_α-connected, (connected through roots in Λ_α). We also have that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of L. Clearly $I_{[\alpha]}$ is of maximal length, and finally $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$, (where $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Theorem 2.2), and $\mathcal{Z}(L) = 0$. We can apply Theorem 3.1 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. □

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