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Turing degrees of hypersimple relations on computable structures

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Abstract

Let $\mathscr{A}$ be an infinite computable structure, and let R be an additional computable relation on its domain A. The syntactic notion of formal hypersimplicity of R on $\mathscr{A}$, first introduced and studied by Hird, is analogous to the computability-theoretic notion of hypersimplicity of R on A, given the definability of certain effective sequences of relations on A. Assuming that R is formally hypersimple on $\mathscr{A}$, we give general sufficient conditions for the existence of a computable isomorphic copy of $\mathscr{A}$ on whose domain the image of R is hypersimple and of arbitrary nonzero computably enumerable Turing degree.

MSC: 03C57; 03D45

Keywords: Computable structure; Hypersimple relation; Turing degree

1. Introduction

In an attempt to construct an incomplete computably enumerable (abbreviated by c.e.) set, Post [13] introduced various classes of c.e. sets with "thin" complements. These sets include hypersimple sets. Hypersimple sets form a proper subclass of the class of simple sets. A c.e. set X is simple if its complement $\bar{X}$ is immune, that is, $\bar{X}$ is infinite but does not contain any infinite c.e. set. An infinite set is hyperimmune if no computable function majorizes its principal function. Hypersimple sets are c.e. sets with hyperimmune complements. Dekker [3] showed that every nonzero c.e. Turing degree contains a hypersimple set. Jockusch [10] introduced the class of

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semirecursive sets. Semirecursive sets coincide with the initial segments of computable linear orders. Clearly, they are closed under complements. Jockusch showed that every immune semirecursive set is hyperimmune. Hence every simple semirecursive set is hypersimple. Hird [8] also studied immunity and hyperimmunity of intervals of computable linear orders. Metakides, Nerode, Downey and Remmel (see [11,12,4]) extensively studied various computability-theoretic properties of c.e. substructures and c.e. relations on computable algebraic structures. In particular, Remmel [14] studied simplicity and hypersimplicity, as well as Turing degrees, of the sets of all nonatoms of computable Boolean algebras.

Ash and Nerode [2] initiated the study of computability-theoretic properties and their syntactic counterparts of c.e. relations on general computable structures. We [7] also considered Turing degrees of these relations. In [9], Hird gave syntactic definitions of the so-called quasi-simple relations, and of hypersimple relations on computable structures. He also established the first existence results for these relations. In [6], we introduced a syntactic definition and established an existence result for nowhere simple relations on computable structures. Ash et al. [1] further studied quasi-simple relations by considering their Turing degrees. The class of quasi-simple relations on a computable structure does not always coincide with the class of its simple relations. In [5], we investigated immunity and simplicity of relations on computable structures, and relative immunity and relative simplicity of relations on countable structures.

In this paper, we continue Hird's study of hypersimple relations on general computable structures. We consider Turing degrees of these relations. In Section 2 we specify notation and definitions. In Section 3, we review Hird's syntactic definition and the result on the existence of hyperimmune relations on computable structures. In Section 4 we establish our main theorem, which for a computable relation R on the domain of a computable structure $\mathcal{A}$, gives sufficient conditions for the existence of a computable isomorphic copy of $\mathcal{A}$ such that the corresponding isomorphic image of R is a hypersimple relation of an arbitrary nonzero c.e. Turing degree. The method of proof is a variant of the priority method developed by Ash et al. In Section 5, we give some applications of the main theorem. As corollaries we obtain several old results on certain relations on particular mathematical structures.

2. Notation and definitions

We consider only countable structures for computable languages. We will denote structures by script letters, and their domains by the corresponding capital Latin letters. Let $\mathcal{A}$ be a structure whose language is L. Then L_A is the language $L \cup \{\mathbf{a} : a \in A\}$, L expanded by adding a constant $\mathbf{a}$ for every $a \in A$. The expansion $(\mathcal{A}, a)_{a\in A}$ of $\mathcal{A}$ to the language L_A such that for every $a \in A$, $\mathbf{a}$ is interpreted by a is also denoted by $\mathcal{A}_A$. If $\mathcal{A}$ is an infinite computable structure, we may assume that $A = \omega$. If for a function f we have $f(b) = a$, then we also assume that $f(\mathbf{b}) = \mathbf{a}$. The *atomic diagram* of $\mathcal{A}$ is the set of all atomic and negated atomic sentences of L_A that are true in $\mathcal{A}_A$. Similarly, the *existential diagram* of $\mathcal{A}$ is the set of all existential sentences of L_A that are true in $\mathscr{A}_A$. A structure is *computable* if its domain is computable and its atomic diagram is computable. A structure is 1-*decidable* if its domain is computable and its existential diagram (equivalently, its universal diagram) is computable.

Let $\mathscr{A}$ be a fixed computable structure for language L, and let R be an additional computable relation on A. Without loss of generality, we may assume that R is unary. By $\bar{R}$ we denote the complement of R with respect to A. Let P be a new unary relation symbol, and let L^P be the expanded language $L \cup \{P\}$. If we are interested in the case when the image of R on a computable copy of $\mathscr{A}$ is c.e., then the first-order finitary existential ($\sum_1^0$) formulae in L^P with only positive occurrences of P play a special role. A $\sum_1^0$ formula in L^P, possibly with individual constants (parameters), in which P occurs only positively will also be called a $\sum_1^{0,P^+}$ formula.

If f is a partial function, then $dom(f)$ is the domain of f, $ran(f)$ is the range of f, and $f(a)\downarrow$ denotes that $a \in dom(f)$. The range of a sequence $\vec{x}$ will also be denoted by $\{\vec{x}\}$, and its length by $lh(\vec{x})$. If the elements of a sequence $\vec{x}$ are linearly ordered, then by $\max(\vec{x})$ we denote its largest element and by $\min(\vec{x})$ its smallest element. If $\vec{x} = (x_0, \ldots, x_{m-1})$ and f is a function, then $f(\vec{x}) =_{\mathrm{def}} (f(x_0), \ldots, f(x_{m-1}))$. The concatenation of sequences is sometimes denoted by $\hat{}$. We use the symbol $\subseteq$ both for the subset and the subsequence relation. By $\leqslant_T$ we denote Turing reducibility, and by $\equiv_T$ Turing equivalence of sets. Let $W_0, W_1, W_2, \ldots$ be a standard computable enumeration of all c.e. sets. Hence, a set $X \subseteq \omega$ is *simple* if X is c.e., $\bar{X}$ is infinite and for every $n \in \omega$,

$$W_n \text{ is infinite} \Rightarrow W_n \cap X \neq \emptyset.$$

We now define the canonical index m of a finite set D_m. Let $D_0 = \emptyset$. For $m > 0$, let $D_m = \{a_0, \ldots, a_{k-1}\}$, where $a_0 < \cdots < a_{k-1}$ and $m = 2^{a_0} + \cdots + 2^{a_{k-1}}$. A sequence $(U_i)_{i\in\omega}$ of finite sets is a *strong array* if there is a unary computable function g such that for every $i \in \omega$, $U_i = D_{g(i)}$. An infinite set S is *hyperimmune*, abbreviated by *h-immune*, if there is no strong array $(U_i)_{i\in\omega}$ of pairwise disjoint sets such that for every $i \in \omega$, we have $U_i \cap S \neq \emptyset$. A c.e. set is *hypersimple*, abbreviated by *h-simple*, if its complement is h-immune. For more information on h-simple sets of natural numbers see [15].

3. Formally *h*-immune relations and Hird's result

Let $\mathscr{A}$ be an infinite computable structure for language L. Let S be an additional infinite co-infinite unary relation on A. The following definition introduces a syntactic property, due to Hird, that corresponds to the semantic property of S being h-immune on A. We will term this syntactic property being formally h-immune on $\mathscr{A}$.

Definition 3.1 (Hird [9]). (i) A *formal strong array on* $\mathscr{A}$ is a computable sequence of existential formulae in L with finitely many parameters $\vec{\mathbf{c}}$, $(\psi_i(\vec{\mathbf{c}}, \vec{x_i}))_{i\in\omega}$, such that for every finite set $F \subseteq A$, there is $i \in \omega$ and a sequence $\vec{a_i} \in A^{lh(\vec{x_i})}$ satisfying

$$[\mathscr{A}_A \models \psi_i(\vec{\mathbf{c}}, \vec{\mathbf{a}}_i)] \wedge [\{\vec{a_i}\} \cap F = \emptyset].$$

(ii) We say that the relation S is *formally h-immune on* $\mathscr{A}$ if there is no formal strong array $(\psi_i(\vec{\mathbf{c}},\vec{x_i}))_{i\in\omega}$ on $\mathscr{A}$ such that for every $i\in\omega$,

$$(\forall \vec{a_i} \in A^{lh(\vec{x_i})})[(\mathscr{A}_A \models \psi_i(\vec{\mathbf{c}},\vec{\mathbf{a}_i})) \Rightarrow (\{\vec{a_i}\} \cap S \neq \emptyset)].$$

Being formally h-immune on $\mathscr{A}$ turns out to be a necessary condition for the existence of a computable copy $\mathscr{B}$ of $\mathscr{A}$ such that the corresponding image of S is h-immune on B.

Proposition 3.1. *Let $\mathscr{A}$ be a computable structure and let S be a new unary relation on its domain. Assume that f is an isomorphism from a computable structure $\mathscr{B}$ onto $\mathscr{A}$. Let $Y =_{\text{def}} f^{-1}(S)$.*

(i) *If $(\psi_i(\vec{\mathbf{c}},\vec{x_i}))_{i\in\omega}$ is a formal strong array on $\mathscr{A}$, then $(\psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{x_i}))_{i\in\omega}$ is a formal strong array on $\mathscr{B}$.*
(ii) ([9]) *If Y is h-immune on B, then S is formally h-immune on $\mathscr{A}$.*

Proof. (i) This is true because for every finite set $F\subseteq B$, there is $i\in\omega$ and $\vec{a_i}\in A^{lh(\vec{x_i})}$ such that

$$[\mathscr{A}_A \models \psi_i(\vec{\mathbf{c}},\vec{\mathbf{a}_i})] \wedge [\{\vec{a_i}\} \cap f(F) = \emptyset].$$

Hence

$$[\mathscr{B}_B \models \psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{\mathbf{b}_i})] \wedge [\{\vec{b_i}\} \cap F = \emptyset],$$

where $\vec{b_i} = f^{-1}(\vec{a_i})$.

(ii) Assume that S is not formally h-immune on $\mathscr{A}$. Let $(\psi_i(\vec{\mathbf{c}},\vec{x_i}))_{i\in\omega}$ be a corresponding formal strong array on $\mathscr{A}$. It follows by (i) that $(\psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{x_i}))_{i\in\omega}$ is a formal strong array on $\mathscr{B}$. Since for every $\vec{a_i}\in A^{<\omega}$ we have

$$(\{\vec{a_i}\} \cap S \neq \emptyset) \Rightarrow (f^{-1}\{\vec{a_i}\} \cap Y \neq \emptyset),$$

it follows that

$$(\forall \vec{b_i} \in B^{lh(\vec{x_i})})[(\mathscr{B}_B \models \psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{\mathbf{b}_i})) \Rightarrow (\{\vec{b_i}\} \cap Y \neq \emptyset)].$$

We now show that Y is not h-immune by enumerating a corresponding strong array. We simultaneously enumerate finite sets $\{\vec{b_i}\}$ whose sequences $\vec{b_i}$ satisfy the formulae in the sequence $(\psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{x_i}))_{i\in\omega}$ such that none of these sets intersects any of the previously enumerated sets. This is possible by the main property of a formal strong array. For every such set $\{\vec{b_i}\}$ with $\mathscr{B}_B \models \psi_i(f^{-1}(\vec{\mathbf{c}}),\vec{\mathbf{b}_i})$, it follows that $\{\vec{b_i}\}\cap Y \neq \emptyset$, as required. $\square$

To prove the converse of Proposition 3.1(ii), we need the following extra decidability condition (H) on $(\mathscr{A},S)$:

There is an algorithm that decides for a given sequence $\vec{c}\in A^{<\omega}$ and an existential formula $\psi(\vec{y},\vec{x})$ in L with $lh(\vec{y})=lh(\vec{c})$, whether

$$(\forall \vec{a} \in A^{lh(\vec{x})})[(\mathscr{A}_A \models \psi(\vec{\mathbf{c}},\vec{\mathbf{a}})) \Rightarrow (\{\vec{a}\} \cap S \neq \emptyset)].$$

Condition (H) implies that S is a computable relation, since we can choose for $c \in A$ the formula $\psi_c(\mathbf{c}, x)$ to be $x = \mathbf{c}$. The decidability condition (H) also implies that $\mathscr{A}$ is 1-decidable by choosing $\vec{a}$ to always be some fixed $a \in S$.

Theorem 3.2 (Hird [9]). *Let a computable structure $\mathscr{A}$ in L and a new unary relation S on its domain A satisfy the decidability condition (H). Assume that S is formally h-immune on $\mathscr{A}$. Then there is a computable structure $\mathscr{B}$ and an isomorphism f from $\mathscr{B}$ onto $\mathscr{A}$ such that the set $f^{-1}(S)$ is h-immune on B.*

4. Formally *h*-simple relations

Let $\mathscr{A}$ be a computable structure for L, and let R be a new unary infinite co-infinite relation on A. Let P be a new unary relation symbol.

Definition 4.1. (i) A $\sum_1^{0,P^+}$ *formal strong array on* $(\mathscr{A}, R)$ is a computable sequence of existential $\sum_1^{0,P^+}$ formulae $(\psi_i(\vec{\mathbf{c}}, \vec{x_i}))_{i\in\omega}$ with finitely many parameters $\vec{\mathbf{c}}$ such that for every finite set $F \subseteq A$, there is $i \in \omega$ and a sequence $\vec{a_i} \in A^{lh(\vec{x_i})}$ satisfying

$$[(\mathscr{A}_A, R) \models \psi_i(\vec{\mathbf{c}}, \vec{\mathbf{a}_i})] \wedge [\{\vec{a_i}\} \cap F = \emptyset].$$

(ii) We say that the relation R is *formally h-simple on* $\mathscr{A}$ if R is c.e. and there is no $\sum_1^{0,P^+}$ formal strong array $(\psi_i(\vec{\mathbf{c}}, \vec{x_i}))_{i\in\omega}$ on $(\mathscr{A}, R)$ such that for every $i \in \omega$,

$$(\forall \vec{a_i} \in A^{lh(\vec{x_i})})[((\mathscr{A}_A, R) \models \psi_i(\vec{\mathbf{c}}, \vec{\mathbf{a}_i})) \Rightarrow (\{\vec{a_i}\} \cap \bar{R} \neq \emptyset)].$$

Clearly, a formally h-simple relation on $\mathscr{A}$ has a formally h-immune complement on $\mathscr{A}$. Hird [9] established that, under a suitable decidability condition, R is formally h-simple on $\mathscr{A}$ iff there is a computable copy $\mathscr{B}$ of $\mathscr{A}$ such that the corresponding image of R is h-simple on B. We will give sufficient conditions on $(\mathscr{A}, R)$ for the existence of a computable copy $\mathscr{B}$ of $\mathscr{A}$ such that the corresponding image of R is h-simple on B and of arbitrary nonzero c.e. Turing degree. First we need some definitions.

Let $\vec{c}, \vec{a} \in A^{<\omega}$. We say that $\vec{a}$ is *free over* $\vec{c}$ if $\vec{a} \notin R^{<\omega}$ and for every $\sum_1^{0,P^+}$ formula $\psi(\vec{\mathbf{c}}, \vec{x})$ with $lh(\vec{x}) = lh(\vec{a})$, if

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}})$$

then

$$(\exists \vec{a'} \in R^{lh(\vec{a})})[(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}'})].$$

Let the set of all sequences that are free over $\vec{c}$ be denoted by $fr(\vec{c})$. (A similar relation of freeness was used in [7,1].) Clearly, if $\vec{a} \in fr(\vec{c})$ and $\vec{c'}$ is a subsequence of $\vec{c}$, then $\vec{a} \in fr(\vec{c'})$. We will now present our main result.

Theorem 4.1. *Let R be a new computable unary relation on the domain A of a computable structure $\mathscr{A}$. Assume that R is formally h-simple on $\mathscr{A}$. Consider the following four conditions*:
(1) *It is decidable for a given $\vec{c} \in A^{<\omega}$ and $\vec{a} \in A^{<\omega}$ whether $\vec{a} \in fr(\vec{c})$.*
(2) *For every $\vec{c} \in A^{<\omega}$, there is an element a such that $a \in fr(\vec{c})$.*
(3) *It is decidable for a given $\vec{c} \in A^{<\omega}$ and a $\sum_1^{0,P^+}$ formula $\psi(\vec{\mathbf{c}}, \vec{x})$ whether*

$$(\forall \vec{a} \in A^{lh(\vec{x})})[((\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}})) \Rightarrow (\{\vec{a}\} \cap \bar{R} \neq \emptyset)].$$

(4) *Let $\vec{c} \in A^{<\omega}$; $\hat{a} \in A^{<\omega} - R^{<\omega}$; $a_0, a_1, \ldots, a_k \in \bar{R}$; $\vec{c_0}, \ldots, \vec{c_t} \in A^{<\omega}$, and let*

$$\psi(\vec{\mathbf{c}}, \hat{\mathbf{a}}, \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_k, \vec{\mathbf{c}_0}, \ldots, \vec{\mathbf{c}_j}, \ldots, \vec{\mathbf{c}_t})$$

be a $\sum_1^{0,P^+}$ sentence in $(L^P)_A$ such that

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \hat{\mathbf{a}}, \mathbf{a}_0, \ldots, \mathbf{a}_k, \vec{\mathbf{c}_0}, \ldots, \vec{\mathbf{c}_t}).$$

In addition, assume that

$$\hat{a} \in fr(\vec{c}),$$
$$a_0 \in fr(\vec{c}),$$

for every $i \in \{1, \ldots, k\}$, there is a sequence $\alpha_i \in A^{<\omega}$ such that

$$(\vec{c}\,\hat{}\,a_0\hat{}\ldots\hat{}\,a_{i-1} \subseteq \alpha_i) \wedge a_i \in fr(\alpha_i) \wedge \hat{a} \notin fr(\alpha_i),$$

and for every $j \in \{0, \ldots, t\}$, there is a sequence $\gamma_j \in A^{<\omega}$ such that

$$(\gamma_j \subseteq \vec{c}) \wedge \vec{c_j} \in fr(\gamma_j) \wedge \vec{c_j} \notin fr(\vec{c}).$$

Then there are $\widehat{a'} \in R^{lh(\hat{a})}$, $a'_0 \in A$, $a'_1, \ldots, a'_k \in R$ and sequences $\vec{c'_j}$ with $lh(\vec{c'_j}) = lh(\vec{c_j})$, $0 \leqslant j \leqslant t$, such that

$$\vec{c'_j} \in fr(\gamma_j)$$

and

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \widehat{\mathbf{a}'}, \mathbf{a}'_0, \ldots, \mathbf{a}'_k, \vec{\mathbf{c}'_0}, \ldots, \vec{\mathbf{c}'_t}).$$

Let C be a noncomputable c.e. set. If conditions (1)–(4) *hold for $\mathscr{A}$ and R, then there is a computable copy $\mathscr{B}$ of $\mathscr{A}$ and an isomorphism $f : \mathscr{B} \to \mathscr{A}$ such that $f^{-1}(R)$ is h-simple on B, and*

$$f^{-1}(R) \equiv_T C.$$

Proof. Let $(C_s)_{s \in \omega}$ be a computable enumeration of the set C such that $C_0 = \emptyset$ and for every $s \in \omega$, $|C_{s+1} - C_s| = 1$. Let $C_{\text{at }(s+1)} =_{\text{def}} C_{s+1} - C_s$. Without loss of generality, we assume that $A = \omega$. We will construct a computable structure $\mathscr{B}$ with domain $B = \omega$.

We assume the usual ordering on A and B. We will also construct an isomorphism f from $\mathscr{B}$ onto $\mathscr{A}$ such that $f^{-1}(R)$ is h-simple and $f^{-1}(R) \equiv_T C$. Condition (4) is a general condition allowing an h-simplicity requirement to act, while preserving higher priority coding requirements, and the possibility for unsatisfied higher priority h-simplicity requirements to act later if permitted by C.

Let $\mathscr{F}$ be the set of all finite 1-1 functions from B to A. Let $p \in \mathscr{F}$ and let $\theta = \theta(\mathbf{b}_0, \ldots, \mathbf{b}_{i-1})$ be a sentence in L_B. We say that p *makes* θ *true in* $\mathscr{A}$ if $p(b_0)\downarrow, \ldots, p(b_{i-1})\downarrow$ and

$$\mathscr{A}_A \models \theta(p(\mathbf{b}_0), \ldots, p(\mathbf{b}_{i-1})).$$

Let $\Delta(p)$ consist of all atomic and negated atomic sentences θ in L_B such that the Gődel number of θ is smaller than $\mu n[n \notin dom(p)]$, and p makes θ true in $\mathscr{A}$; as well as of all sentences of the form $P(\mathbf{b})$, where $b \in dom(p)$ and $p(b) \in R$.

At the end of every stage s of the construction, we will have a sequence l_s of odd length, whose last term is a finite version of the isomorphism f at s. That is, l_s is of the form

$$l_s = (p_0^s, \beta_0^s, p_1^s, \beta_1^s, p_2^s, \ldots, \beta_n^s, p_{n+1}^s, \ldots, \beta_{r_s-1}^s, p_{r_s}^s),$$

where

$$\emptyset = p_0^s \subseteq p_1^s \subseteq \cdots \subseteq p_{r_s}^s$$

and $\beta_0^s, \ldots, \beta_{r_s-1}^s \in B^{<\omega}$. By construction, for every $n \in \{1, \ldots, r_s\}$, we will have that p_n^s is a 1-1 function that maps an initial segment of B into A such that $\{0, \ldots, n-1\} \subseteq ran(p_n^s)$. Hence, $\{0, \ldots, n-1\} \subseteq dom(p_n^s)$. Let $f_s =_{\mathrm{def}} p_{r_s}^s$. The construction will guarantee that $\lim_{s\to\infty} f_s$ exists. Let $f =_{\mathrm{def}} \lim_{s\to\infty} f_s$. At every stage s, we will have $\Delta(f_s) \subseteq \Delta(f_{s+1})$. The atomic diagram $\Delta(\mathscr{B})$ of the structure $\mathscr{B}$ will be

$$\Delta(\mathscr{B}) = \bigcup_{s\in\omega} \Delta(f_s).$$

The sequence β_n^s is designated to code at stage s whether $n \in C_s$. For every $n \in \{0, \ldots, r_s - 1\}$, we have that $lh(\beta_n^s) \leqslant n+1$ and $\min(\beta_n^s) \notin dom(p_n^s)$. We will show that for $n \in \omega$, $\lim_{s\to\infty} \beta_n^s$ exists. Let $\beta_n =_{\mathrm{def}} \lim_{s\to\infty} \beta_n^s$.

The construction will satisfy the following requirements for every $n \in \omega$ and the corresponding sequence $\beta_n \in B^{<\omega}$:

$$R_{2n}\colon\ f(\beta_n) \cap R \neq \emptyset \ \Leftrightarrow\ n \in C,$$
$$R_{2n+1}\colon\ (\exists m)[m \in W_n \wedge f(D_m) \subseteq R].$$

The requirements R_{2n}, $n \in \omega$, code C into $f^{-1}(R)$. We will call them the *coding requirements*. In a requirement R_{2n+1}, we think of W_n as a possible set of canonical indices of a strong array. Hence, the requirements R_{2n+1}, $n \in \omega$, ensure that $f^{-1}(\bar{R})$ is an h-immune subset of B. We will call them the *h-simplicity requirements*.

Assume that β_n^s is defined. The sequence β_n^s consists of successive elements of B such that $lh(\beta_n^s) \leqslant n+1$, $\{\beta_n^s\} \cap dom(p_n^s) = \emptyset$ and $\{\beta_n^s\} \subseteq dom(p_{n+1}^s)$. Also,

$$n \in C_s \Rightarrow [p_{n+1}(\{\beta_n^s\}) \cap R \neq \emptyset],$$
$$n \notin C_s \Rightarrow (\forall b > 0)[b \in \{\beta_n^s\} \Rightarrow p_{n+1}^s(b) \in fr(p_{n+1}^s(0,\ldots,b-1))].$$

In particular, if $n \notin C_s$, then $(p_{n+1}^s(\{\beta_n^s\})) \subseteq \bar{R}$.

Let $\vec{d} \in B^{<\omega}$. Let s be a stage. If $f_s(\vec{d}) \notin fr(f_s(0))$, then we set $frseg_s(\vec{d}) =_{\mathrm{def}} 0$. Otherwise, let

$$frseg_s(\vec{d}) =_{\mathrm{def}} \max\{b \in B\colon f_s(\vec{d}) \in fr(f_s(0)\hat{}\ldots\hat{}f_s(b-1))\}.$$

That is, $frseg_s(\vec{d})$ determines the largest initial segment $[0,\ldots,b-1]$ of B such that $f_s(\vec{d})$ is free over $f_s(0,\ldots,b-1)$. Clearly, $frseg_s(\vec{d}) \leqslant \min(\{\vec{d}\} \cap f_s^{-1}(\bar{R}))$. Let D be a finite subset of B. Then $frseg_s(D) =_{\mathrm{def}} frseg_s(\vec{d})$, where $\vec{d} \in B^{<\omega}$ is such that $D = \{\vec{d}\}$. Now, we define $frseg(\vec{d})$ and $frseg(D)$ similarly as $frseg_s(\vec{d})$ and $frseg_s(D)$, using f instead of f_s.

The requirement R_{2n} *requires attention* at stage $(s+1)$ if $n = r_s$ or

$$n < r_s \wedge n \in C_{\mathrm{at}\ (s+1)} \wedge f_s(\{\beta_n^s\}) \cap R = \emptyset.$$

By construction, we will have that $r_{s+1} \leqslant r_s + 1$.

The requirement R_{2n+1} *requires attention* at stage $(s+1)$ if

$$(\forall z \in W_{n,s+1})\neg[D_z \subseteq dom(f_s) \wedge f_s(D_z) \subseteq R],$$

and there is $m \in W_{n,s+1}$ such that

$$f_s(D_m \cap dom(f_s)) \subseteq R$$

or

$$\begin{aligned} & f_s(D_m \cap dom(f_s)) \nsubseteq R \wedge \\ & \qquad [frseg_s(D_m) > \max(\beta_n^s)] \wedge \\ & \qquad (\exists c \in C_{\mathrm{at}\ (s+1)})[frseg_s(D_m) \geqslant c]. \end{aligned}$$

The condition $\neg[D_z \subseteq dom(f_s) \wedge f_s(D_z) \subseteq R]$ is equivalent to either $D_z \nsubseteq dom(f_s)$ or $[D_z \subseteq dom(f_s) \wedge f_s(D_z) \cap \bar{R} \neq \emptyset]$. The condition $[frseg_s(D_m) > \max(\beta_n^s)]$ ensures that the higher priority coding requirements will not be injured, while the condition $(\exists c \in C_{\mathrm{at}\ (s+1)})[frseg_s(D_m) \geqslant c]$ allows "D_m to be permitted by C."

Construction. *Stage* 0. Define $l_0 = (p_0^0)$, where $p_0^0 = \emptyset$.

Stage $s+1$. Assume that $l_s = (p_0^s, \beta_0^s, p_1^s, \beta_1^s, \ldots, \beta_{r_s-1}^s, p_{r_s}^s)$. We choose the least i_0 such that R_{i_0} requires attention at $(s+1)$. Note that $i_0 \leqslant 2r_s$. The requirement R_{i_0} will *receive attention* as follows.

Case (i). Assume that $R_{i_0} = R_{2r_s}$.

Let $\beta_{r_s}^{s+1}$ be the sequence of the first (r_s+1) elements of B that are not in $dom(p_{r_s}^s)$. We choose $p_{r_s+1}^{s+1}$ to be a 1-1 function from an initial segment of B into A such that

$\{0,\ldots,r_s\}\subseteq ran(p^{s+1}_{r_s+1})$, $p^{s+1}_{r_s+1}\supseteq p^s_{r_s}$ and $\{\beta^{s+1}_{r_s}\}\subseteq dom(p^{s+1}_{r_s+1})$. In addition, if $r_s\in C_{s+1}$, then

$$(\exists b\in\{\beta^{s+1}_{r_s}\})[p^{s+1}_{r_s+1}(b)\in R],$$

and if $r_s\notin C_{s+1}$, then

$$(\forall b\in\{\beta^{s+1}_{r_s}\})[p^{s+1}_{r_s+1}(b)\in fr(p^{s+1}_{r_s+1}(0,\ldots,b-1))].$$

Notice that if $r_s\notin C_{s+1}$, then $p^{s+1}_{r_s+1}(\{\beta^{s+1}_{r_s}\})\subseteq\bar{R}$. We can effectively choose $p^{s+1}_{r_s+1}$ as described, using conditions (1) and (2). We now set

$$l_{s+1}=_{\text{def}}\ l_s\hat{\ }(\beta^{s+1}_{r_s},p^{s+1}_{r_s+1}).$$

Hence $(\forall i\leqslant r_s)[p^{s+1}_i=p^s_i]$ and $(\forall i<r_s)[\beta^{s+1}_i=\beta^s_i]$. Obviously, the requirement R_{2r_s} does not injure any requirement.

Case (ii). Assume that $R_{i_0}=R_{2n}$ for some $n<r_s$.

Thus, we have $n\in C_{\text{at }(s+1)}$ and $f_s(\beta^s_n)\cap R=\emptyset$. We will not change the subsequence $(p^s_0,\beta^s_0,p^s_1,\ldots,p^s_n,\beta^s_n)$ of l_s at stage $(s+1)$. That is, we will have

$$(\forall i\leqslant n)[p^{s+1}_i=p^s_i\wedge\beta^{s+1}_i=\beta^s_i].$$

We will define p^{s+1}_{n+1} to be some new function q in $\mathscr{F}$ such that $q\supseteq p^s_n$, $\{\beta^s_n\}\subseteq dom(q)$, $\{0,\ldots,n\}\subseteq ran(q)$ and

$$(\forall b\in\{\beta^s_n\})[q(b)\in R].$$

For every $i\in\{n+1,\ldots,r_s-1\}$, we will injure the requirement R_{2i} by *abandoning* the sequence β^s_i. That is, the sequence β^{s+1}_i will not be defined, but $\{\beta^s_i\}\subseteq dom(p^{s+1}_{n+1})$ and $(\forall b\in\{\beta^s_i\})[p^{s+1}_{n+1}(b)\in R]$. The function p^{s+1}_{n+1} will be determined using condition (4). Let

$$\vec{c^*}=dom(p^s_n)$$

and let

$$\vec{c}=ran(p^s_n)=f_s(\vec{c^*}).$$

Hence

$$\beta^s_0\hat{\ }\ldots\hat{\ }\beta^s_{n-1}\subseteq\vec{c^*}.$$

Let

$$\vec{b}=\beta^s_n\hat{\ }\ldots\hat{\ }\beta^s_{r_s-1}$$

and

$$\vec{a}=f_s(\vec{b}).$$

We now consider all h-simplicity requirements of higher priority than R_{2n}, which may act later if permitted by C. That is, we consider each $e<n$ such that for every $z\in W_{e,s}$,

$$\neg[D_z \subseteq dom(f_s) \wedge f_s(D_z) \subseteq R],$$

while for some $j \in W_{e,s+1}$,

$$frseg_s(D_j) > \max(\beta_e^s).$$

The condition $frseg_s(D_j) > \max(\beta_e^s)$ ensures that possible future action of R_{2e+1} will not injure its higher priority coding requirements. Hence, R_{2e+1} may act later when some element c such that $c \leqslant frseg_s(D_j)$ gets enumerated in C. Note that

$$frseg_s(D_j) < n$$

since, otherwise, R_{2e+1} would be a requirement of higher priority than R_{2n}, requiring attention at $(s+1)$. For every e as above, choose $j_e \in W_{e,s+1}$ with the largest $frseg_s(D_{j_e})$. Define the increasing sequence $\vec{d_e}$ such that $\{\vec{d_e}\} = D_{j_e}$. Let $\vec{c_e} = f_s(\vec{d_e})$. Let $\vec{d}$ be the concatenation of all sequences $\vec{d_e}$, where $e<n$ and $\vec{d_e}$ is defined:

$$\vec{d} = \vec{d_{e_0}}\hat{}\ldots\hat{}\vec{d_{e_t}}$$

for some $e_0 < \cdots < e_t < n$.

Let $\psi(\vec{\mathbf{c}^*}, \vec{\mathbf{b}}, \vec{\mathbf{d}})$ be a $\sum_1^{0,P^+}$ sentence in $(L^P)_B$ obtained by forming the conjunction $\theta(\vec{\mathbf{c}^*}, \vec{\mathbf{b}}, \vec{\mathbf{d}}, \vec{\mathbf{u}})$ of all sentences in $\Delta(f_s)$, where $\vec{\mathbf{u}}$ does not contain any of the constants in $\vec{\mathbf{c}^*}\hat{}\vec{\mathbf{b}}\hat{}\vec{\mathbf{d}}$, and then setting $\psi(\vec{\mathbf{c}^*}, \vec{\mathbf{b}}, \vec{\mathbf{d}})$ to be $(\exists \vec{y})\theta(\vec{\mathbf{c}^*}, \vec{\mathbf{b}}, \vec{\mathbf{d}}, \vec{y})$. Clearly,

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}}, f_s(\vec{\mathbf{d}})).$$

Assume that

$$\vec{a} = (a_0, a_1, \ldots, a_k).$$

Let

$$\hat{a} = a_0.$$

We will now use condition (4). Let $\vec{a'} \in R^{lh(\vec{a})}$ and $\vec{c'} \in A^{lh(\vec{d})}$ be the least sequences such that

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}'}, \vec{\mathbf{c}'})$$

and

$$\vec{c'} = \vec{c_{e_0'}}\hat{}\ldots\hat{}\vec{c_{e_t'}},$$

where $\vec{c_{e_0'}}, \ldots, \vec{c_{e_t'}}$ are chosen as in condition (4). Hence, if $f_{s+1}(\vec{c^*}) = \vec{c}$ and $f_{s+1}(\vec{d}) = \vec{c'}$, then for every $j \in \{0, \ldots, t\}$,

$$frseg_{s+1}(\vec{d_{e_j}}) \geqslant frseg_s(\vec{d_{e_j}}).$$

Let $\vec{v} \in A^{lh(\vec{u})}$ be the least sequence such that

$$(\mathscr{A}_A, R) \models \theta(\vec{\mathbf{c}}, \vec{\mathbf{a'}}, \vec{\mathbf{c'}}, \vec{\mathbf{v}}).$$

Find the least q in $\mathscr{F}$ such that $n \in ran(q)$, $q(\vec{c^*}) = \vec{c}$, $q(\vec{b}) = \vec{a'}$, $q(\vec{d}) = \vec{c'}$ and $q(\vec{u}) = \vec{v}$. Hence $q \supseteq p_n^s$. Let

$$l_{s+1} =_{\text{def}} (p_0^s, \beta_0^s, p_1^s, \dots, p_n^s, \beta_n^s, q).$$

Hence, $f_{s+1} = p_{n+1}^{s+1} = q$ and

$$f_{s+1}(\vec{c^*}) = \vec{c},$$
$$f_{s+1}(\{\beta_n^s \hat{} \dots \hat{} \beta_{r_s-1}^s\}) \subseteq R,$$
$$f_{s+1}(\vec{d}) = \vec{c'}.$$

Case (iii). Assume that $R_{i_0} = R_{2n+1}$ and for some $m \in W_{n,s+1}$,

$$f_s(D_m \cap dom(f_s)) \subseteq R,$$

while $D_m \nsubseteq dom(f_s)$. Fix the least m as above. We now effectively extend $p_{r_s}^s$ into the first 1-1 function q from an initial segment of B into A such that $\{0, \dots, r_s\} \subseteq ran(q)$, $D_m \subseteq dom(q)$ and $q(D_m) \subseteq R$. Then we set

$$l_{s+1} =_{\text{def}} (p_0^s, \beta_0^s, p_1^s, \dots, p_{r_s-1}^s, \beta_{r_s-1}^s, q).$$

Hence $f_{s+1} = p_{r_s}^{s+1} = q$, and for every $i < r_s$, $p_i^{s+1} = p_i^s$ and $\beta_i^{s+1} = \beta_i^s$. Clearly, the requirement R_{2n+1} does not injure any requirement.

Case (iv). Assume that $R_{i_0} = R_{2n+1}$ and for some $m \in W_{n,s+1}$, we have

$$f_s(D_m \cap dom(f_s)) \nsubseteq R,$$
$$frseg_s(D_m) > \max(\beta_n^s)$$

and

$$(\exists c \in C_{\text{at }(s+1)})[frseg_s(D_m) \geqslant c].$$

Choose such m with the largest $frseg_s(D_m)$. Let j' be such that

$$frseg_s(D_m) \in \beta_{j'}.$$

We have that $j' > n$ because $frseg_s(D_m) > \max(\beta_n^s)$. Since the requirement R_{2n+1} requires attention, it follows that

$$\neg(\exists z \in W_{n,s+1})[D_z \subseteq dom(f_s) \wedge f_s(D_z) \subseteq R].$$

We will use condition (4) to satisfy R_{2n+1}. Assume that the increasing sequence $\vec{d_m}$ is such that $\{\vec{d_m}\} = D_m$. Let

$$\hat{a} = f_s(\vec{d_m}).$$

We also set

$$\vec{c} = f_s(\vec{c^*}),$$

where

$$\vec{c^*} = 0\hat{}\ldots\hat{}(frseg_s(D_m) - 1),$$

and

$$\vec{a} = f_s(\vec{b}),$$

where

$$\vec{b} = (frseg_s(D_m)\hat{}\ldots\hat{}\beta^s_{j'}(lh(\beta^s_{j'}) - 1))\hat{}\beta^s_{j'+1}\hat{}\ldots\hat{}\beta^s_{r_s-1}.$$

Thus, $\vec{b}(0) = frseg_s(D_m)$ and, hence,

$$a_0 = f_s(frseg_s(D_m)).$$

We now consider all higher priority h-simplicity requirements R_{2e+1} that may act later if permitted by C. That is, we consider every $e < n$ such that

$$(\forall z \in W_{e,s+1})\neg[D_z \subseteq dom(f_s) \wedge f_s(D_z) \subseteq R],$$

while for some $j \in W_{e,s+1}$, we have

$$frseg_s(D_j) > \max(\beta^s_e).$$

Note that $frseg_s(D_j) < frseg_s(D_m)$ since, otherwise, $frseg_s(D_j) \geqslant c$, and R_{2e+1} would be a requirement of higher priority than R_{2n+1}, requiring attention at $(s+1)$. For every e as above, let $j_e \in W_{e,s+1}$ be the corresponding canonical index with the largest $frseg_s(D_{j_e})$. Let the increasing sequence $\vec{d_e}$ be such that $\{\vec{d_e}\} = D_{j_e}$. Set $\vec{c_e} = f_s(\vec{d_e})$. Let $\vec{d}$ be the concatenation of all sequences $\vec{d_e}$, where $e < n$ and $\vec{d_e}$ is defined:

$$\vec{d} = \vec{d_{e_0}}\hat{}\ldots\hat{}\vec{d_{e_t}}$$

for some $e_0 < \cdots < e_t < n$.

Let $\psi(\vec{\mathbf{c}^*}, \vec{\mathbf{d}_m}, \vec{\mathbf{b}}, \vec{\mathbf{d}})$ be a $\sum_1^{0,P^+}$ sentence in $(L^P)_B$ obtained by forming the conjunction $\theta(\vec{\mathbf{c}^*}, \vec{\mathbf{d}_m}, \vec{\mathbf{b}}, \vec{\mathbf{d}}, \vec{\mathbf{u}})$ of all sentences in $\Delta(f_s)$, where $\vec{\mathbf{u}}$ does not contain any of the constants in $\vec{\mathbf{c}^*}\hat{}\vec{\mathbf{d}_m}\hat{}\vec{\mathbf{b}}\hat{}\vec{\mathbf{d}}$, and then taking the formula $(\exists\vec{y})\theta(\vec{\mathbf{c}^*}, \vec{\mathbf{d}_m}, \vec{\mathbf{b}}, \vec{\mathbf{d}}, \vec{y})$. Clearly,

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \hat{\mathbf{a}}, \vec{\mathbf{a}}, f_s(\vec{\mathbf{d}})).$$

We now use condition (4). Let $\widehat{a'} \in R^{lh(\hat{a})}$, $\vec{a'} \in A^{lh(\vec{a})}$ and $\vec{c'} \in A^{lh(\vec{d})}$ be the least sequences such that

$$(\mathscr{A}_A, R) \models \psi(\vec{\mathbf{c}}, \widehat{\mathbf{a}'}, \vec{\mathbf{a}'}, \vec{\mathbf{c}'}),$$

$$(\forall i > 0)[i \in dom(\vec{a'}) \Rightarrow \vec{a'}(i) \in R]$$

and

$$\vec{c'} = \vec{c'_{e_0}}\hat{}\ldots\hat{}\vec{c'_{e_t}},$$

where, $\vec{c'_{e_0}},\ldots,\vec{c'_{e_t}}$ are chosen as in (4). Hence, if $f_{s+1}(\vec{c^*}) = \vec{c}$ and $f_{s+1}(\vec{d}) = \vec{c'}$, then for every $j \in \{0,\ldots,t\}$,

$$frseg_{s+1}(\vec{d_{e_j}}) \geqslant frseg_s(\vec{d_{e_j}}).$$

Choose the least $\vec{v} \in A^{lh(\vec{u})}$ such that $(\mathscr{A}_A, R) \models \theta(\vec{\mathbf{c}}, \widehat{\mathbf{a}'}, \vec{\mathbf{a}'}, \vec{\mathbf{c}'}, \vec{\mathbf{v}})$. Find the least q in $\mathscr{F}$ such that $\{0,\ldots j'+1\} \subseteq ran(q)$, $q(\vec{c^*}) = \vec{c}$, $q(\vec{d_m}) = \widehat{a'}$, $q(\vec{b}) = \vec{a'}$, $q(\vec{d}) = \vec{c'}$ and $q(\vec{u}) = \vec{v}$.

If $frseg_s(D_m)$ is not the last (greatest) element in $\beta_{j'}$, then we set

$$l_{s+1} =_{\mathrm{def}} (p_0^s, \beta_0^s, \ldots, p_{j'-1}^s, \beta_{j'-1}^s, q).$$

That is, for every $i \in \{j',\ldots,r_s-1\}$, we *abandon* the sequence β_i^s and injure the requirement R_{2i}.

If $frseg_s(D_m)$ is the last element in $\beta_{j'}^s$, then $\beta_{j'}^s = \beta\hat{}frseg_s(D_m)$ for some $\beta \neq \emptyset$. This is true because the construction guarantees that $lh(\beta_{j'}^s) \geqslant 2$. We set

$$l_{s+1} =_{\mathrm{def}} (p_0^s, \beta_0^s, \ldots, p_{j'-1}^s, \beta_{j'-1}^s, p_{j'}^s, \beta, q).$$

That is, $\beta_{j'}^s = \beta_{j'}^{s+1}\hat{}frseg_s(D_m)$, and we say that we *reduce* the sequence $\beta_{j'}^s$ (by one element). For every $i \in \{j'+1,\ldots,r_s-1\}$, we *abandon* the sequence β_i^s and injure the requirement R_{2i}. We do not injure the requirement $R_{2j'}$ because

$$f_s(\beta_{j'}^s) \cap R = \emptyset \quad \Leftrightarrow \quad f_{s+1}(\beta_{j'}^{s+1}) \cap R = \emptyset.$$

End of the *Construction*

Lemma 4.2. *Let $m \in \omega$. For every s, if the sequence β_m^s is defined, then $\beta_m^s \neq \emptyset$. There is s' such that $(\forall s \geqslant s')[\beta_m^s = \beta_m^{s'}]$. There is s'' such that $(\forall s \geqslant s'')[p_m^s = p_m^{s''}]$. Every requirement receives attention only finitely many times.*

Proof. We first notice that each h-simplicity requirement receives attention at most once. Assume that for some s_0, the sequence $\beta_m^{s_0}$ is undefined, while $\beta_m^{s_0+1}$ is defined. Then we have Case (i) of the construction at (s_0+1) and $lh(\beta_m^{s_0+1}) = m+1$. If the sequence β_m^s is reduced at stage $(s+1)$, then some h-simplicity requirement R_{2n+1} for $n < m$ receives attention at $(s+1)$. However, there are at most m such requirements, so the sequence β_m^s will never become empty.

If the sequence β_m^s is abandoned at stage $(s+1)$, then the requirement R_{2m} has been injured at $(s+1)$. The requirement R_{2m} is injured by a coding requirement R_{2n} at stage $(s+1)$ only when $n \in C_{\mathrm{at}\ (s+1)}$ and $n < m$. Hence there are only finitely many such stages. The requirement R_{2m} is injured by a h-simplicity requirement R_{2n+1} at stage $(s+1)$ only when $n < m$ and R_{2n+1} receives attention at $(s+1)$. Hence there are only finitely many such stages. In any case, β_m^s settles at some stage $s = s'$.

Similarly, we can show that p_m^s also settles at some stage $s = s''$. Clearly, by construction, every requirement receives attention only finitely often. $\square$

Let

$$\beta_m =_{\text{def}} \lim_{s\to\infty} \beta_m^s \wedge p_m =_{\text{def}} \lim_{s\to\infty} p_m^s.$$

The construction implies that $\{\beta_m\} \subseteq dom(p_{m+1})$ and $p_m \subseteq p_{m+1}$. Thus, $f = \bigcup_{m\geqslant 0} p_m$. It follows by construction that $\mathscr{B}$ is a computable structure, and that f is an isomorphism from $\mathscr{B}$ onto $\mathscr{A}$. Let

$$X = f^{-1}(R).$$

Lemma 4.3. $X \leqslant_T C$

Proof. Let $b \in B$. Using oracle C, we find the least stage s such that $f_s(b)\downarrow$ and $(\forall c \leqslant b)[c \in C \Rightarrow c \in C_s]$. Then

$$b \in X \quad \Leftrightarrow \quad f_s(b) \in R.$$

This is true because the construction guarantees that if $[f_t(b) \in \bar{R} \wedge f_{t+1}(b) \in R]$ then $(\exists c \leqslant b)[c \in C_{\text{at }(t+1)}]$. $\square$

Lemma 4.4. $C \leqslant_T X$.

Proof. Let $c \in \omega$. We will show how to decide, computably in X, whether $c \in C$. Find the least stage s_0 such that l_{s_0} has length $(2c+3)$, and hence is of the form $l_{s_0} = (p_0^{s_0}, \beta_0^{s_0}, \ldots, \beta_{c-1}^{s_0}, p_c^{s_0}, \beta_c^{s_0}, p_{c+1}^{s_0})$. The sequence of coding elements designated for c at stage s_0 is $\beta_c^{s_0}$. If $\beta_c^{s_0} \cap X = \emptyset$ or, equivalently, $f(\beta_c^{s_0}) \cap R = \emptyset$, then $c \notin C$. Now assume that $f(\beta_c^{s_0}) \cap R \neq \emptyset$. Find the least stage s $(s > s_0)$ at which some member of the sequence $\beta_c^{s_0} = \beta_c^{s-1}$ is enumerated in R. If $c \in C_s$, then, obviously, $c \in C$. Now assume that $c \notin C_s$. Then β_c^{s-1} is reduced or abandoned at s because a higher priority requirement receives attention at s. Let s_1 be the least stage such that $s_1 > s$ and a new sequence $\beta_c^{s_1}$ is defined. We now continue in this manner until we conclude, using oracle X, that $c \notin C$ or find a stage s_i such that $c \in C_{s_i}$. This conclusion will eventually happen because the requirements of priority higher than R_{2c} can receive attention only finitely often. $\square$

Lemma 4.5. *The relation X is h-simple on B.*

Proof. If follows by construction that X is c.e. Now, assume that $\bar{X}$ is not h-immune on B. Let n be the least number such that W_n witnesses the existence of a strong array $(D_m)_{m\in W_n}$ of pairwise disjoint sets satisfying $(\forall m \in W_n)[D_m \cap \bar{X} \neq \emptyset]$. Equivalently, for every $m \in W_n$, we have that $f(D_m) \cap \bar{R} \neq \emptyset$. Hence the requirement R_{2n+1} is not satisfied. We will show that C is computable, or that R is not formally h-simple on $\mathscr{A}$, contradicting the assumptions of the theorem.

Case (i). Assume that the following condition holds:

$$(\forall b)(\exists m \in W_n)[\mathit{frseg}(D_m) > b],$$

where $\mathit{frseg}(D_m)$ is defined as before, using f. Let s_0 be a stage by which all requirements R_e for $e \leqslant 2n+1$ have received attention for the last time. Hence, we can show that C is computable as follows. Given $k \in \omega$, find the least $s > s_0$ such that for some m, we have that $m \in W_{n,s}$, $\mathit{frseg}_s(D_m) > \max(\beta_n^s)$ and $\mathit{frseg}_s(D_m) > k$. Then C_s must be correct on $[0, \ldots, \mathit{frseg}(D_m)]$ since, otherwise, C would permit R_{2n+1} to act after stage s_0. Thus,

$$k \in C \quad \Leftrightarrow \quad k \in C_s.$$

Case (ii). Now, assume that the following condition holds:

$$(\exists b)(\forall m \in W_n)[\mathit{frseg}(D_m) \leqslant b].$$

For $m \in W_n$, let $\vec{d_m}$ be a sequence of elements in B such that $\{\vec{d_m}\} = D_m$, and let $\vec{a_m} =_{\text{def}} f(\vec{d_m})$. Hence, there is a sequence $\vec{c} \in A^{<\omega}$ such that

$$(\forall m \in W_n)[\vec{a_m} \notin \mathit{fr}(\vec{c})].$$

Thus, for every $m \in W_n$, there is a $\sum_1^{0,P^+}$ formula $\psi_m(\vec{\mathbf{c}}, \vec{x_m})$ such that

$$\mathscr{A}_A \models \psi_m(\vec{\mathbf{c}}, \vec{\mathbf{a}_m})$$

and

$$(\forall \vec{a} \in A^{lh(\vec{x_m})})[(\mathscr{A}_A \models \psi_m(\vec{\mathbf{c}}, \vec{\mathbf{a}})) \Rightarrow (\{\vec{a}\} \cap \bar{R} \neq \emptyset)].$$

We can use the decidability condition (3) to effectively find ψ_m. Let $F \subseteq A$ be a finite set. Since the family $\{D_m : m \in W_n\}$ consists of pairwise disjoint sets, there is $m_0 \in W_n$ such that $D_{m_0} \cap f^{-1}(F) = \emptyset$. Therefore, we obtain a $\sum_1^{0,P^+}$ formal strong array which witnesses that R is not formally h-simple on $\mathscr{A}$. $\square$

5. Examples

Let $\mathscr{A}$ be $(\omega, =)$ and let $R \subseteq A$ be a computable infinite co-infinite subset. Conditions (1)–(4) of Theorem 4.1 are satisfied because for $\vec{c} \in A^{<\omega}$ and $\vec{a} \notin R^{<\omega}$, we can prove, similarly as in [1], that

$$[\vec{a} \in \mathit{fr}(\vec{c})] \quad \Leftrightarrow \quad [\{\vec{a}\} \cap \bar{R} \cap \{\vec{c}\} = \emptyset].$$

Hence, we obtain Dekker's result.

Corollary 5.1 (Dekker [3]). *Every nonzero c.e. Turing degree contains an h-simple set.*

Let $\mathscr{A} = (\omega, \prec)$ be a computable linear order of type $(\omega + \omega^*)$ with the computable ω-part R. Conditions (1)–(4) of Theorem 4.1 are satisfied because for $\vec{c} \in A^{<\omega}$ and $\vec{a} \notin R^{<\omega}$, we can prove, similarly as in [7], that

$$\vec{a} \in fr(\vec{c}) \quad \Leftrightarrow \quad a_r \prec c_l,$$

where $\{\vec{a}\} \cap \bar{R} = \{a_0 \prec \cdots \prec a_r\}$ and $\{\vec{c}\} \cap \bar{R} = \{c_l \prec \cdots \prec c_0\}$.

Corollary 5.2. *For every noncomputable c.e. set C, there is a computable linear order of type $(\omega + \omega^*)$ such that its ω-part X is h-simple and $X \equiv_T C$.*

In a similar fashion, we can apply Theorem 4.1 to strengthen Hird's result in [8] on h-immune co-c.e. intervals of computable linear orders.

Now, let η be the order type of the rationals.

Let $\mathscr{A}$ be a computable linear order of type η and let R be a computable dense co-dense subset of A. Conditions (1)–(4) of Theorem 4.1 are satisfied because for $\vec{c} \in A^{<\omega}$ and $\vec{a} \notin R^{<\omega}$, we can prove, similarly as in [1], that

$$[\vec{a} \in fr(\vec{c})] \quad \Leftrightarrow \quad [\{\vec{a}\} \cap \bar{R} \cap \{\vec{c}\} = \emptyset].$$

Thus, we have the following corollary.

Corollary 5.3. *For every noncomputable c.e. set C, there is a computable linear order of type η with a dense co-dense h-simple subset X such that $X \equiv_T C$.*

Let $\mathscr{A} = (\omega, <)$ and let $R = 2\omega$. Conditions (1)–(4) of Theorem 4.1 are satisfied because for $\vec{c} \in A^{<\omega}$ and $\vec{a} \notin R^{<\omega}$, we can prove, similarly as in [7], that

$$\vec{a} \in fr(\vec{c}) \Leftrightarrow (\forall a \in \{\vec{a}\} \cap \bar{R})[a > \max(\vec{c})].$$

Corollary 5.4. *For every noncomputable c.e. set C, there is a computable isomorphic copy of $(\omega, <)$ such that its subset X of all "even numbers" is h-simple and $X \equiv_T C$.*

Let $\mathscr{A}$ be a computable Boolean algebra isomorphic to the Boolean algebra consisting of all finite and co-finite subsets of ω. By $At(\mathscr{A})$ we denote the set of all atoms of $\mathscr{A}$. Since $\mathscr{A}$ is computable, $At(\mathscr{A})$ is a co-c.e. set. For a subset $M \subseteq A$, by M^* we denote the Boolean subalgebra of $\mathscr{A}$ generated by M. For $c \in A$, let $(c] =_{\text{def}} \{a \in A : a \leqslant c\}$.

The theory of Boolean algebras admits elimination of quantifiers in terms of the relations $(A_n)_{n \in \omega}$, where $A_n(x)$ states that x is the join of n atoms. There is a computable structure isomorphic to $\mathscr{A}$, in which the sequence $(A_n)_{n \in \omega}$ is uniformly computable. Thus, $\mathscr{A}$ can be chosen to be decidable with the definable set $At(\mathscr{A})$. Let $R =_{\text{def}} \overline{At(\mathscr{A})}$. Hence, $(\mathscr{A}, R)$ satisfies the decidability condition (3) of Theorem 4.1. (See also [9].)

Let $\vec{c} \in A^{<\omega}$. Then the subalgebra $\{\vec{c}\}^*$ has a finite set of atoms, $At(\{\vec{c}\}^*) = \{b_0, \ldots, b_{m-1}\}$. The sets

$$At(\mathscr{A}) \cap (b_0], \ldots, At(\mathscr{A}) \cap (b_{m-1}]$$

form a partition of $At(\mathscr{A})$. Without loss of generality, assume that there is $k \leqslant m$ such that $b_0, \ldots, b_{k-1}$ are exactly those atoms in $At(\{\vec{c}\}^*)$ that have infinitely many atoms of $\mathscr{A}$ below them. That is, the set $At(\mathscr{A}) \cap (b_i]$ is infinite exactly for $i \in \{0, \dots, k-1\}$.

Lemma 5.5 (Hird [9]). *Let $\psi(\vec{\mathbf{c}}, \vec{x})$ be an existential formula in $L(\mathscr{A})$, and let $\vec{a} \in A^{<\omega}$ be such that*

$$\mathscr{A}_A \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a}})$$

and for some $i \in \{0, \dots, k-1\}$,

$$\{\vec{a}\} \cap At(\mathscr{A}) \subseteq (b_i].$$

Then there exists $\vec{a'} \in A^{lh(\vec{c})}$ such that

$$\mathscr{A}_A \models \psi(\vec{\mathbf{c}}, \vec{\mathbf{a'}})$$

and

$$\{\vec{a'}\} \cap At(\mathscr{A}) = \emptyset.$$

Hence, for $\vec{c} \in A^{<\omega}$ and $\vec{a} \notin R^{<\omega}$, we can prove that

$$[\vec{a} \in fr(\vec{c})] \quad \Leftrightarrow \quad [(\{\vec{a}\} \cap At(\mathscr{A}) \subseteq (b_0]) \vee \cdots \vee (\{\vec{a}\} \cap At(\mathscr{A}) \subseteq (b_{k-1}])].$$

Thus, Conditions (1), (2) and (4) of Theorem 4.1 are satisfied, and as the next corollary we obtain Remmel's result.

Corollary 5.6 (Remmel [14]). *Let C be a noncomputable c.e. set. There is a computable Boolean algebra $\mathscr{B}$ isomorphic to the Boolean algebra of all finite and cofinite subsets of ω, such that the set X of all nonatoms of $\mathscr{B}$ is h-simple and $X \equiv_T C$.*

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