RATIONAL BASIS FUNCTIONS FOR CURVED ELEMENTS

EUGENE L. WACHSPRESS
Knolls Atomic Power Laboratory, Schenectady, NY 12301, U.S.A.

(Received March 1979)

This is one of a series of papers presented at the TICOM 2nd International Conference on Computational Methods in Nonlinear Mechanics held in Austin, Texas, 26–30 March 1979. The survey by D. Apprato was also presented at this conference. Other work referred to by Apprato appears in the two papers by J. L. Gout.

Abstract—In this paper, construction of rational basis functions for curved elements is reviewed, some of the notation is described, and some applications to isoparametrics are developed.

1. RATIONAL BASIS FUNCTIONS FOR ALGEBRAIC ELEMENTS

Triangles and rectangles in global or isoparametric geometry suffice for most finite element computation. Theory has nevertheless been developed for more general elements, and this is a review of recent progress in one branch of finite element basis function research. The most general element considered is a "regular algebraic element", bounded by an algebraic curve with a simple component defining each side of the element. Regularity refers to a convexity-type of constraint on the boundary curve. The generality of partitioning a region into a collection of nonoverlapping algebraic elements is indicated by adoption of the term "retriangulation" in place of the more commonly used term "triangulation". In general, one cannot find polynomial basis functions having properties needed for interelement continuity and prescribed degree of approximation over each element. Foundations drawing heavily on classical algebraic geometry were laid for construction of rational basis functions for approximation over regular algebraic elements [1]. The construction was more precise for the reduced class of elements in which each boundary component is a rational algebraic curve. This is a curve that admits a rational parametrization, and this is equivalent to a curve of genus zero. The geometric consequence of this property is that each boundary component has the maximum allowable singular points for a curve of its order. These singular points play a crucial role in basis function construction.

Each element has a node at each of its vertices, just enough other nodes on each side to assure interelement continuity for the prescribed degree of approximation, and interior nodes needed to achieve the degree of approximation over the element. For degree k approximation, one chooses \((k - 1)(k - 2)/2\) interior nodes that do not all lie on any curve of order less than \(k - 2\).

The rational basis function associated with node \(i\) for degree \(k\) approximation over algebraic element \(e\) bounded by curve \(C_m\) of order \(m\) is of the form

\[ W_{i}^{k}(x, y) = \frac{N_{i} R_{i}}{Q_{r}}, \]

where \(N_{i}\) is the product of the components of \(C_m\) not containing \(i\), \(R_{i}\) is a polynomial whose degree depends on \(k\) and \(i\), and \(Q_{r}\) is the unique (except for a normalization constant) polynomial of maximal degree \(m - 3\) whose curve has multiplicity \(r - 1\) at all non-vertex points of curve \(C_m\) that have multiplicity \(r\).

The denominator polynomial, common to all basis functions for the element, is called the element "adjoint" polynomial, a term used in its algebraic geometry context.

The construction will now be illustrated for degree two approximation over a regular pentagon (Fig. 1), this being pertinent to subsequent discussion of numerical integration.
Points 1–10 are the elements nodes and A–E are the multiple points of the element boundary curve (excluding the vertices). Both the line through points p and q and the arbitrarily normalized linear form that vanishes on this line are denoted by \((p;q)\). Circle \(Q\) is the unique conic containing points A–E. This element is of order \(m = 5\) and \(Q\) is of order \(m - 3 = 2\). The basis functions associated with nodes 1 and 6 are:

\[
W_1(x, y) = k_1 \frac{(2;3)(3;4)(4;5)(6;10)}{Q} \\
W_6(x, y) = k_6 \frac{(2;3)(3;4)(4;5)(5;1)}{Q}
\]

with \(k_1\) and \(k_6\) chosen to normalize these functions to unity at nodes 1 and 6, respectively.

An algebraic geometry theorem asserts that if curves \(P\) and \(Q\) have a common intersection set with \(C\), then the ratio of the polynomials of least degree that vanish on \(P\) and \(Q\) respectively is a constant along \(C\) [1]. That \(W_1\) is quadratic on \((1;2)\) is demonstrated by noting that \((3;4)(4;5)\) and \(Q\) both intersect \((1;2)\) at \(B\) and \(E\). Rational basis function \(W_1\) thus varies as \(2;3)(6;10)\) on side \((1;2)\). Similar arguments provide the foundations for construction of rational basis functions in general. Complex character of intersections of curves that can occur in more abstruse situations requires a sophisticated algebraic geometry development.

The applicability of the construction to elements with curved sides may be seen by replacing sides \((2;3)\) and \((3;4)\) of the pentagon by a conic that intersects \((1;5)\) in \(A'\) and \(D'\), intersects \((1;2)\) in \(2\) and \(B'\), and intersects \((4;5)\) in \(4\) and \(C'\). The adjoint curve now becomes the unique \(Q'\) containing \(A', B', C', C', D'\), and \(E\). Vertex 3 disappears. However, three side nodes are required on each curved side of order two (conic side) for degree two approximation. Vertex node 3 is thus replaced by side node 3' to yield side nodes 3, 7' and 8' on the conic side. The product \((2;3)\) \((3;4)\) is replaced by the quadratic \((2;3';4;7';8')\) that vanishes on the conic side. A detailed analysis with many illustrative examples is given in [1]. The construction generalizes to three space dimensions.

Having found basis functions for algebraic elements, one next considers application to finite element computation. The major deterrent to use of these functions is difficulties associated with evaluation of integrals occurring during discretization. Formalism for relatively straightforward integration of polynomials over rectangles and triangles is well established. Such techniques are crucial in finite element computation and their applicability to isoparametric elements accounts for widespread use of these elements. Several researchers have attacked this
problem for more general elements, and difficulties associated with numerical integration are slowly being resolved [2, 3].

McLeod [4] treats this in depth in his article. An interesting possibility suggested by McLeod's work is a generalized isoparametric mapping which will now be discussed briefly.

2. ISOPARAMETRIC MAPPING

Two kinds of isoparametric elements are widely used. A straight-sided triangle is mapped into a triangle whose sides may be straight or arcs of parabolas (Fig. 2a) or a square is mapped into a four-sided element whose sides may be straight or arcs of a parabola (Fig. 2b).

These transformations are performed with the aid of well-known degree-two polynomial basis functions for continuous patchwork approximation over triangular and square grids. Let $B_i(p, q)$ be the basis function associated with node $i$. Then the mapping is defined by

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} = \sum_{i=1}^{I} \begin{bmatrix}
x_i \\
y_i
\end{bmatrix} B_i(p, q),
$$

(3)

with $I = 6$ for the triangle and $I = 8$ for the square. The $x, y$ coordinates of the nodes in the mapped element are specified, and these nodes together with (3) determine the boundary curves in the $x, y$ plane. These curves are either lines or "isoparametric parabolic arcs". Care must be exercised in node selection to assure a nonvanishing Jacobian determinant over the element. Geometric arguments first described by Jordan [5] provide a practical guide for node selection.

This technique generalizes in several directions. The degree-two $p, q$ basis may be replaced by functions associated with vertex nodes plus as many side nodes as required to yield a desired order curve for each boundary segment in the $x, y$ plane. This is illustrated in Fig. 3. Ingenuity must be exercised in determining the basis functions. They must have appropriate degree variation on the sides and must satisfy

$$
\begin{bmatrix}
P \\
Q
\end{bmatrix} = \sum_{i=1}^{P} \begin{bmatrix}
p_i \\
q_i
\end{bmatrix} B_i(p, q).
$$

(4)

Let $(6; 7)$ be the linear form that vanishes on the line through 6 parallel to (1,2) in Fig. 3a. Then one may select $B_i(p, q) = k_i(2; 3)(6; 7)$.

One may also generalize to permit more than four sides. The transformation may now be made from a regular $n$-gon to an element with $n$ sides, each of which may be of any prescribed order. Specification of element boundary nodes in the $x, y$ plane determines the element.
boundary through the isoparametric mapping. Primary motivation for this procedure is the existence of convenient quadrature rules for regular \( n \)-gons. The discrete finite element equations are obtained by numerical quadrature in the \( p, q \) plane.

By way of illustration, basis functions of the type shown in eqn (2) map the regular pentagon in Fig. 1 in the \( p,q \) plane into the isoparametric 5-con shown in Fig. 4. The mapping in eqn (3) is now applied with \( I = 10 \), \( x,y \) nodal coordinates determined from the points in Fig. 4, and basis functions \( B_i(p,q) \) for the regular pentagon in the \( p,q \) plane. The sides of the element in Fig. 4 are the isoparametric parabolic arcs determined by the nodes.

For finite element computation, basis functions of a prescribed degree are needed in the \( x,y \) plane. The lowest degree of approximation that guarantees interelement continuity is degree one. This may be achieved by using the basis functions that define the isoparametric element. The nonzero Jacobian over the element assures a 1-1 mapping from \( x,y \) to \( p,q \) so that

\[
B_i(x,y) = B_i(p(x,y),q(x,y)).
\]  
(5)

That this achieves degree one approximation is assured by eqns (3) and (4). Users are prone to overlook the fact that this basis is only of degree one in \( x \) and \( y \) even though the \( B_i(p,q) \) are a degree two basis (or perhaps higher) in \( p \) and \( q \). The natural isoparametric degree-one basis applies to generalized isoparametric elements like that shown in Fig. 4. The entire isoparametric methodology carries over.

It should be noted that the degree-one rational basis functions determined directly in the \( x,y \) plane for the \( n \)-con generated by the isoparametric transformation are not in general the functions of eqn (5). A particularly revealing situation occurs with straight-sided convex quadrilaterals. The 8-node isoparametric basis in (5) is only degree one in \( x \) and \( y \) in the general case whereas the 8-node rational basis for the quadrilateral is of degree two. It is not clear that the higher degree of approximation leads to better performance of the element. Numerical results do however seem to indicate higher accuracy for the higher degree basis [3].

Once the isoparametric element has been generated with a suitable \( p,q \) basis, another basis of any desired degree in \( x \) and \( y \) can be constructed for the transformed element in the \( x,y \) plane. That this is numerically feasible is seen from the following discussion of integration.

4. INTEGRATION

The \( B_i(p,q) \) and their derivatives with respect to \( p \) and \( q \) evaluated at quadrature points in the \( p,q \) plane remain fixed for an element and do not depend on the chosen \( x,y \) basis. Computational effort in discretization is greatly reduced when the transformation basis is used as the approximation basis as in eqn (5). This procedure is almost universally followed. If one uses an alternative basis, perhaps to achieve higher degree approximation or to model known behavior within an element by special basis functions, then a more time-consuming procedure must be followed. The procedure is not much more tedious and is quite practicable. This part of the overall finite element computation could be almost insignificant. One may organize the computation in the following manner:

1. Select quadrature nodes \( (p_s,q_s) \) for \( s = 1, 2, \ldots, S \) with the quadrature rule

\[
\iint f(p,q) \, dp \, dq = \sum_{s=1}^{S} w_s f(p_s,q_s).
\]
Rational basis functions for curved elements

2. Evaluate the x,y coordinates of the quadrature nodes and the absolute value of the determinant of the Jacobian of the isoparametric transformation using eqn (3). Let

\[ J_i = \left| \frac{\partial (x,y)}{\partial (p,q)} \right|_{x_i, y_i} \]

3. Evaluate the integrand in the x,y coordinate system at the S quadrature points: \( f_s = f(x_s, y_s) \).

4. Approximate the integral by

\[ \iint f(x,y) \, dx \, dy = \sum_{s=1}^{S} w_s f_s J_s \]

Step 3 contains the bulk of the computation added to conventional isoparametric quadrature. Rational basis functions and their derivatives of any order are easily evaluated for obtaining \( f_s \).

For the convex straight-sided quadrilateral, a projective transformation to a square facilitates integration with rational x,y basis functions in much the same manner as the isoparametric transformation. Both analytical and approximate quadrature have been used in numerical studies performed by various researchers with these quadrilateral elements [1-3].

Results have encouraged further studies with alternatives to conventional isoparametric elements, but application to actual engineering type problems has not yet been attempted with a "production program".

5. CONVERGENCE ANALYSIS

Considerable functional analytic error analysis has been applied to finite element methods. In particular, Ciarlet and Raviart [6] have established error bounds for isoparametric elements. Much of this analysis only applies to small distortion from rectangular elements. Successive refinement in the limiting process associated with the error analysis, though somewhat more difficult to visualize for irregular reticulations, leads eventually to partitions that approach regular triangulations.

This functional analytic approach has been applied to elements with rational basis functions by Arcangeli, Apprato, and Gout of the University of Pau[2], who have demonstrated that error bounds associated with polynomial basis functions generalize to rational functions of the type described here. A crucial parameter is the minimum distance of the adjoint curve from the element. Gradients of basis functions increase as the adjoint approaches the element.

7. CONCLUDING REMARKS

Rational basis functions can be constructed for a wide class of regular algebraic elements. Considerable progress has been made in evaluating integrals for finite element discretization with such elements. Functional analytic convergence analysis has established error bounds comparable to those obtained for computations using polynomial basis functions in global and isoparametric coordinates. Effort in this field is as yet confined to a few researchers. Although some benefit has been derived in a few cases from use of more varied elements than are usually treated by isoparametrics, development is still in a formative stage.

REFERENCES