CHARACTERIZATION OF UNIGRAPHIC AND UNIDIGRAPHIC INTEGER-PAIR SEQUENCES

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Given a graph (digraph) $G$ with edge (arc) set $E(G) = \{(u_1, v_1), (u_2, v_2), \ldots, (u_q, v_q)\}$, where $q = |E(G)|$, we can associate with it an integer-pair sequence $S_G = ((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ where $a_i, b_i$ are the degrees (indegrees) of $u_i, v_i$ respectively. An integer-pair sequence $S$ is said to be graphic (digraphic) if there exists a graph (digraph) $G$ such that $S_G = S$. In this paper we characterize unigraphic and unidigraphic integer-pair sequences.

1. Introduction and definitions

All graphs (digraphs) considered here are finite, without isolated vertices and without loops or multiple edges (arcs). Given a graph (digraph) $G$ we denote its vertex set by $V(G)$ and edge (arc) set by $E(G)$. The edge (arc) joining vertex $u$ to vertex $v$ is denoted by $uv$.

Let $G$ be a graph (digraph) with $E(G) = \{u_1v_1, u_2v_2, \ldots, u_qv_q\}$ where $q = |E(G)|$. Then by the integer-pair sequence $S_G$ of $G$ we mean the sequence $((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ where $a_i, b_i$ are the degrees (indegrees) of $u_i, v_i$ respectively.

Let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q))$ be a sequence of ordered pairs of positive (non-negative) integers. We say $S' \equiv S$ if $S'$ can be obtained from $S$ by a permutation of its members, and $S' \Delta S$ if $S' \equiv S''$ where $S''$ is obtained from $S$ by interchanging $a_i$ and $b_i$ in some of the members of $S$. Then a graph (digraph) $G$ is said to be a realization of $S$ if $S_G \Delta S$ (or $S_G \equiv S$). If $S$ has a graph (digraph) as a realization, then $S$ is said to be graphic (digraphic). Further, if any two graph (digraph) realizations of $S$ are isomorphic, then $S$ is said to be unigraphic (unidigraphic).

Integer-pair sequences were first introduced by Hakimi and Patrinos in [3], where it was considered to extend the concepts and results of degree sequences. Characterizations of graphic (digraphic) integer-pair sequences were also obtained in the same paper. In this paper we characterize unigraphic (unidigraphic) integer-pair sequences, thus solving a problem posed in [3]. For degree sequences the corresponding problem of characterizing unigraphic degree sequences has been solved in the case of graphs by Koren in [6]; and in Das [2] in the case of digraphs. Further results on integer-pair sequences have been obtained in [1, 2, 7].
We now introduce some definitions and notations. For definitions not given here and notations not explained the reader is referred to [4].

Let \( G \) be a graph (digraph) and \( A, B \subseteq V(G) \). Then \( G[A, B] \) is defined by the following:

\[ V(G[A, B]) = A \cup B \text{ and } E(G[A, B]) = \{ uv \in E(G) : u \in A, v \in B \}. \]

\( G[A, A] \) is denoted by \( G[A] \) sometimes. If \( x \in V(G) \), then the degree (out-degree and in-degree respectively) of \( x \) in \( G \) is denoted by \( d_G(x) \) and \( d_G^{-1}(x) \) respectively. By \( G \rightarrow I(xuyv) \rightarrow H \) we mean that we get \( H \) from \( G \) by replacing \( xu, yv \) by \( xv, yu \) in \( E(G) \) such that \( xu, yv \in E(G) \); \( yu, xv \notin E(G) \) and \( d_G(u) = d_G(v) \) \( (d_G^{-1}(u) = d_G^{-1}(v)) \). Then, clearly, \( H \) is also a graph (digraph) and \( S_I \equiv S_G \).

Now let \( S = ((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) \) be a sequence of ordered pairs of positive integers.

We then have (following [3]): \( S_1 \) is the sequence \((a_1, b_1, a_2, b_2, \ldots, a_n, b_n); S_2 = \{d_1, d_2, \ldots, d_n\} \) is the set of distinct members of \( S_1 \); \( k'(r, s) \) is the number of times the ordered integer-pair \((r, s)\) occurs in \( S \);

\[ k'(r, s) = \begin{cases} k'(r, s) + k'(s, r), & \text{if } r \neq s, \\ k'(r, s), & \text{if } r = s; \end{cases} \]

\[ n(r) \text{ is the number of times } r \text{ occurs in } S_1; \]

\[ l_i = \frac{n(d_i)}{d_i} \text{ for } 1 \leq i \leq n. \]

Note that if \( S \) is graphic and \( G \) is a graph realization of \( S \) then there are \( l_i \) vertices of degree \( d_i \) in \( G \) (as shown in [3]) and hence \( l_i \) is a positive integer.

Hence for graphic \( S \) we have the following: For \( 1 \leq i \leq n \) we define:

\[ X_i = (2k(d_i, d_i))(mod l_i) + \sum_{j \neq i} (l_i(d_i, d_j)(mod l_i)), \]

\[ Y_i = (-2k(d_i, d_i))(mod l_i) + \sum_{j \neq i} ((-k(d_i, d_j))(mod l_i)). \]

\( \Pi_{ii} \) is the following sequence of length \( l_i \): \((2k(d_i, d_i))(mod l_i)\) of the members are \( \{2k(d_i, d_i)/l_i\} \) and the remaining, if any, are \([2k(d_i, d_i)/l_i]\). (As usual, \( \{x\} \) denotes the least integer not less than \( x \) and \( [x] \) denotes the greatest integer not greater than \( x \).)

For \( 1 \leq i \neq j \leq n \) we define \( \Pi_{ij} \) to be the following pair of sequences \([r_1, r_2, \ldots, r_{l_i}], (r_1, r_2, \ldots, r_{l_j})\) where \( k(d_i, d_j)(mod l_m) \) of the \( r_i \)'s are \( \{k(d_i, d_j)/l_m\} \) and the remaining, if any, are \([k(d_i, d_j)/l_m]\) for \( m = i, j \).

Also, when \( S \) is graphic, any graph realization \( G \) of \( S \) is taken to be on the vertex set \( V = \bigcup_{i=1}^{n=1} V_i \) where \( |V_i| = l_i \) and \( d_G(x) = d_i \) for all \( x \in V_i \), \( 1 \leq i \leq n \). \( G[V_i, V_j] \) is denoted by \( G_{ij} \) and \( G[V_i] \) by \( G_i \) or \( G_{ii} \).

For the bipartite graph \( G_{ij}, i \neq j \) the bipartition is always taken to be \( V_i \cup V_j \) and \( \Delta_{ij}(G), \delta_{ij}(G) \) denote respectively the maximum, minimum degree in \( G_{ij} \) of a
vertex in $V_i$. $G_{ij}$ is said to be semiregular on both sides if

$$\Delta_{ij}(G) - \delta_{ij}(G) \leq 1 \quad \text{and} \quad \Delta_{ji}(G) - \delta_{ji}(G) \leq 1.$$ 

Given a pair of sequences $[\phi_1, \phi_2]$ we say it has a realization by bipartite graph if there is a bipartite graph $G$, with bipartition $V_1 \cup V_2$, such that the degrees in $G$ of the vertices in $V_m$ are given by $\phi_m$ for $m = 1, 2$. (Pairs of sequences with unique realization by bipartite graphs have been characterized by Koren in [5].) Then we also write $\Pi(G) = [\phi_1, \phi_2].$

A graph $G$ is said to be semiregular if $\Delta(G) - \delta(G) \leq 1$ where $\Delta(G), \delta(G)$ denote respectively the maximum, minimum degree of a vertex in $G$.

$x \mid y$ means $x$ divides $y$ and $x \nmid y$ means $x$ does not divide $y$.

The degree sequence of a graph $G$, written $\Pi(G)$, is the sequence of the degrees of the vertices of $G$. Two sequences $\Pi_1, \Pi_2$ are equal if $\Pi_1$ can be obtained by a permutation of the members of $\Pi_2$. Similarly $[\Pi_1, \Pi_2] = [\phi_1, \phi_2]$ if $\Pi_1 = \phi_1$ and $\Pi_2 = \phi_2$.

Further definitions, required only in the case of digraphs, will be given in the section on digraphs.

2. Unigraphic integer-pair sequences

We first give two structural results which will be used repeatedly in the proof of the necessity of the characterizing theorem.

**Lemma 2.1.** Let $G$ be a semiregular graph with $q$ edges and $n$ vertices. If $2 < q \leq \frac{1}{2}n(n-1)-2$, then there exist distinct $x, y, v \in V(G)$ such that $xy \notin E(G), yv \in E(G)$ and $d_G(x) \leq d_G(y)$.

**Proof.** Let $a$ be the maximum degree of a vertex in $V(G)$. If $a = 1$, then there are four vertices of degree 1. Two of these, which are nonadjacent, may be taken to be $x$ and $y$ and $v$ the only vertex adjacent to $y$. Similarly, if $a = n - 1$, then there are four vertices of degree $n - 2$ and two of these which are adjacent may be taken as $x$ and $y$ and $v$ the only vertex non-adjacent to $x$.

So we suppose $2 \leq a \leq n - 2$. Let $x \in V(G)$ be such that $d_G(x) = a$. So there is $v \in V(G)$ such that $xv \notin E(G)$. As $G$ is semiregular so $d_G(v) \geq 1$. Hence there is $y \in V(G)$ such that $yv \in E(G)$. This choice of $x, y$ and $v$ serves. Hence the lemma is proved.

**Lemma 2.2.** Let $G$ be a bipartite graph with bipartition $V_1 \cup V_2$, and with $q$ edges, which is semiregular on both sides. If $2 < q \leq mn - 2$ where $|V_1| = n \geq 2$ and $|V_2| = m \geq 2$, then there exist $x, y \in V_1$ and $u, v \in V_2$ such that $xu, yv \in E(G)$ and $xv, yu \notin E(G)$. 
Proof. Let $a_i$ be the maximum degree of a vertex in $V_i$, $i = 1, 2$. As before we can see that if $a_1 = 1$ or $m$ ($a_2 = 1$ or $n$) the lemma is true. So we suppose that $2 \leq a_1 \leq m - 1$ and $2 \leq a_2 \leq n - 1$. Let $x \in V_1$ be such that $d_G(x) = a_1$. So there is $v \in V_2$ such that $xv \notin E(G)$. Now $\gamma(v) \geq 1$ and so there is $y \in V_1$ such that $yu \in E(G)$. As $d_G(x) > d_G(y)$ so $y$ is not joined to all the vertices that $x$ is joined to. Hence there is $u \in V_2$ such that $yu \notin E(G)$, $xu \in E(G)$. This proves the lemma.

We will use the following result on unigraphic degree sequences:

Lemma 2.3 (Koren [6]). Let $\Pi = a^n$ be a graphic degree sequence. Then $\Pi$ is unigraphic if and only if $a \in \{1, n - 2, n - 1\}$ or $\Pi = 2^5$. ($a^n$ is the sequence of $n$ $a$'s.)

We will use the following canonical realization of a graphic integer-pair sequence.

Lemma 2.4 (Rao and Taneja [7]). If $S$ is a graphic integer-pair sequence, there is a graph realization $G$ of $S$ such that for every $i, j, 1 \leq i \neq j \leq n$, $G_i$ is semiregular and $G_{ij}$ is semiregular on both sides.

In what follows till the statement of the theorem in this section, we take $S$ to be a unigraphic integer-pair sequence and $G$ to be the canonical realization of the following notational simplifications: we write $\Delta_{ij}$ for $\Delta_{ij}(G)$, $\delta_{ij}$ for $\delta_{ij}(G)$, $d_i(x)$ for $d_{ci}(x)$ and $d(x)$ for $d_G(x)$.

We now state and prove a series of assertions about unigraphic $S$. These will be required to prove the necessity in the characterizing theorem.

Assertion 1. Either $X_i = Y_i = 0$ or $X_i = l_i$ or $Y_i = l_i$ for $1 \leq i \leq n$.

Proof. For $l_i = 1$ it is clearly true. So let $l_i \geq 2$ and suppose that the assertion does not hold. Now as $S$ is graphic so $l_i \mid X_i$ and $l_i \mid Y_i$. Also $X_i = 0$ if and only if $Y_i = 0$. So we have that $X_i, Y_i \equiv 2l_i$.

So there exist $x, y \in V_i$ and $j, 1 \leq j \leq n$, such that $d_i(x) > d_i(y)$. As $d(x) = d(y) = d_i$ so there is $m$, $1 \leq m \leq n$, such that $d_{im}(y) > d_{im}(x)$. As $X_i, Y_i \equiv 2l_i$ so each vertex of $V_i$ has degree $\Delta_{ik}(\delta_{ik})$ in $G_{ik}$, where $\Delta_{ik} \neq \delta_{ik}$, for at least two different values of $k$, $1 \leq k \leq n$. So let $p, r$ be such that $d_{ip}(x) = \Delta_{ip} \neq \delta_{ip}$ and $d_{ir}(x) = \delta_{ir} \neq \Delta_{ir}$ and $\{p, r\} \cap \{j, m\} = \emptyset$.

Claim. $d_{ip}(y) = \Delta_{ip}$ and $d_{ir}(y) = \delta_{ir}$.

If not, then by interchanging the neighbourhoods of $x$ and $y$ in both $G_{ij}$ and $G_{im}$ we will get a realization $H$ of $S$ which has one vertex less than $G$ with the property that it is joined to $\Delta_{ij}$ vertices in $V_i$, $\Delta_{ip}$ in $V_p$, $\delta_{im}$ in $V_m$ and $\delta_{ir}$ in $V_r$. This proves the necessity in the characterizing theorem.
This implies that $S$ is not unigraphic. Hence the claim.

Now there is $z \in V_r$ such that $d_{ir}(z) = \delta_{ir}$.

Suppose $d_{im}(z) = \delta_{im}$. Then there are two cases. First if there exists $s \notin \{j, p, m, r\}$ such that $d_s(z) = \Delta_{ts} \neq \delta_{ts}$, $d_t(y) = \delta_{ts}$, then interchanging neighbourhoods of $y$ and $z$ in $G_{ip}$ and $G_{is}$ we get $H$ and then as above we can get a realization which is not isomorphic to $H$. So this case cannot occur and hence we have $d_{ij}(z) = \Delta_{ij}$ and $d_{ir}(z) = \Delta_{ir}$ as $X_i \geq 2l_i$. Interchanging neighbourhoods of $y$ and $z$ in $G_{im}$ and $G_{ij}$ we get a realization which has in $V_i$ one more vertex, than $G$ has, which is joined to $\Delta_p$ points in $V_r$ and $\delta_{ij}$ points in $V_r$.

So we have $d_{im}(z) = \Delta_{im}$. Then as before by comparing $z$ with $x$ we see that $d_{ij}(z) = \Delta_{ij}$ and $d_{ir}(z) = \delta_{ir}$. So there is $w \in V_i$ such that $d_u(w) = \Delta_{ui}$. Hence there is $u \in V_r$ such that $w \in E(G), z \notin E(G)$. Similarly there is $t$ ($t$ may be same as $j$ or $m$) such that $d_t(w) = \delta_{ti} < \Delta_{ui} = d_u(z)$. Hence there is $u \in V_r$ such that $w \in E(G), z \notin E(G), G \rightarrow I(wu, zu) \rightarrow G'$. Now in $G'$ we interchange the neighbourhoods of $z$ and $y$ in both $G[V_i, V_j]$ and $G[V_i, V_j]$ to get realization in which $z$ is adjacent to $\delta_{ip}$ points in $V_p$ and $\delta_{ij}$ points in $V_r$, contradicting the claim.

Hence Assertion 1 is proved.

Note that $X_i = l_i$ ($Y_i = l_i$) implies that each vertex of $V_i$ has degree $\Delta_{ij}(\delta_{ij})$ in some $G_{ij}$, where $\Delta_{ij} \neq \delta_{ij}$, exactly once. Also $X_i = Y_i = l_i$ implies that there are exactly two distinct values of $i$ such that $\Delta_{ij} \neq \delta_{ij}$.

Assertion 2. If $i \neq j; l, \geq 2$ and $l_1 \leq k(d_i, d_j) \leq l_1 - 1$, then $\Pi_{ij}$ has unique realization by bipartite graph; $X_i, Y_i = l_i$; for all $m \neq j, i$, with $l_m \geq 2$, either

$$0 \leq k(d_i, d_m) \leq 1 \quad \text{or} \quad l_m - 1 \leq k(d_i, d_m) \leq l_m;$$

and either

$$0 \leq k(d_i, d_i) \leq 1 \quad \text{or} \quad \frac{1}{2}l_i(l_i - 1) - 1 \leq k(d_i, d_i) \leq \frac{1}{2}l_i(l_i - 1).$$

Further if $l_i \mid k(d_i, d_i)$, then for all $m \neq j, i$, $k(d_i, d_m) = l_i$ or 0; and $k(d_i, d_i) = \frac{1}{2}l_i(l_i - 1)$ or 0.

Proof. Clearly $\Pi_{ij}$ has unique realization by bipartite graph. We require the following claim to prove the rest of the assertion.

Claim. Let $x, y \in V_i$, then there is a bipartite realization of $\Pi_{ij}$ with bipartition $V_i \cup V_j$ such that there is a $w \in V_j$ which is joined to $x$ but not to $y$.

Let $V_j = \{v_1, \ldots, v_{l_i}\}$ where $d_{ii}(v_k) = d_{ii}(v_{k+1})$. Then $y$ can be joined to $v_1$ to $v_2$ and $x$ to $v_2$. Then $G_{ij}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. Then $G_{im}$ satisfies conditions of Lemma 2.2 and hence there exist $x, y \in V_i, v \in V_n$. 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such that: \(d_{im}(x) \geq d_{im}(y)\) and \(xv \notin E(G_{im})\), \(yw \in E(G_{im})\). Also by Claim there is \(w \in V_i\) such that \(xw \notin E(G), yw \in E(G)\). \(G \rightarrow I(xwvy) \rightarrow H\). Then \(II(H[V_i, V_m]) \neq II_{im}\) as \(\Delta_{im}(H) - \delta_{im}(H) \geq 2\). Contradiction to the fact that \(S\) is unigraphic.

Similarly we get a contradiction using Lemma 2.1 if

\[2 \leq k(d_i, d_j) \leq \frac{1}{2}l_i(l_i - 1) - 2.\]

Now let \(l_i \mid k(d_i, d_j)\). Then for all \(x, y \in V_i\), \(d_{ij}(x) = d_{ij}(y)\). Suppose there is \(\nu \in (V(G) - V_i)\) such that \(x\nu \notin E(G), y\nu \in E(G)\) for some \(x, y \in V_i\). Then we get a \(w\) as in Claim. \(G \rightarrow I(xwyv) \rightarrow H\) and \(II(H[V_i, V_j]) \neq II_{ij}\). Contradiction. Hence there is no such \(w\). Hence the second paragraph of Assertion 2 holds.

Finally to show that \(X_i, Y_i < 2l_i\). Suppose not. Suppose \(X_i > 2l_i\). Then by the second paragraph of Assertion 2, just proved, we get that \(l_i \mid k(d_i, d_j)\). So let \(x \in V_i\) be such that \(d_{ij}(x) = \Delta_{ij} \neq \delta_{ij}\). As \(X_i > 2l_i\) so there is distinct \(r\) such that \(d_{ir}(x) = \Delta_{ir} \neq \delta_{ir}\). Let \(y \in V_i\) be such that \(d_{ir}(y) = \delta_{ir}\). Then there is \(\nu \in V_i\) such that \(x\nu \in E(G), y\nu \notin E(G)\). By Claim there is a \(w \in V_i\) such that \(xw \notin E(G), yw \in E(G)\). \(G \rightarrow I(xwvy) \rightarrow H\). Then \(II(H[V_i, V_j]) \neq II_{ij}\). Hence \(X_i < l_i\). Similarly \(Y_i < l_i\).

Hence Assertion 2 is proved.

**Assertion 3.** If \(i \neq j, l_i, l_j \geq 2\) and \(0 < k(d_i, d_j) < l_i,\) then \(X_i = l_i\).

**Proof.** Suppose not. Then \(X_i > 2l_i\) and by Assertion 1 \(Y_i = l_i\). If \(k(d_i, d_j) = 1\), then all but one vertex of \(V_i\) have degree \(\delta_{ij}\) in \(G_{ij}\) where \(\delta_{ij} \neq \Delta_{ij}\). Hence there is just one more \(k\) such that \(\Delta_{ik} \neq \delta_{ik}\), and only the remaining vertex of \(V_i\) has degree \(\delta_{ik}\) in \(G_{ik}\). Then we get that \(X_i = l_i\) also as the only non-zero contribution to \(X_i\) are from \(k(d_i, d_j)\) and \(k(d_i, d_k)\).

So \(k(d_i, d_j) \geq 2\). Hence \(G_{ij}\) satisfies conditions of Lemma 2.2 and so we can get \(x, y \in V_i\) and \(u, v \in V_j\) such that \(xu, yv \in E(G)\) and \(xv, yu \notin E(G)\). Also as \(\delta_{ij} = 0\) so \(d_{ij}(x) = d_{ij}(y) = \Delta_{ij} = 1\).

As \(X_i > 2l_i\) so there is \(m \neq j\) such that \(d_{im}(x) = \Delta_{im} \neq \delta_{im}\). Hence there exist \(z \in V_i, w \in V_m\) such that \(xw \in E(G), zw \notin E(G)\) and \(d_{im}(z) = \delta_{im}\). If \(yw \notin E(G)\), then \(G \rightarrow I(xwvy) \rightarrow H\). Then in \(H, x\) has degree 2 in \(H[V_i, V_j]\) but there is no vertex of \(V_i\) of degree 2 in \(G_{ij}\) implying \(S\) is not unigraphic. Hence \(yw \in E(G)\).

As \(Y_i = l_i\) so \(d_{ij}(z) = \Delta_{ij} = 1\). Let \(p \in V_j\) such that \(zp \in E(G)\). So at least one of \(x, y\) is nonadjacent to \(p\). Let \(xp \notin E(G)\). \(G \rightarrow I(xwzp) \rightarrow H\). Then as above we get a contradiction.

Hence Assertion 3 is proved.

**Assertion 4.** If \(i \neq j, l_i, l_j \geq 2\) and \(l_i - l_i - l < k(d_i, d_j) < l_i\), then \(Y_i = l_i\).

**Proof.** Let \(\bar{G}\) be the complement of \(G\). Then \(S_{\bar{G}}\) is unigraphic. So Assertion 3 holds for \(S_{\bar{G}}\). This implies that Assertion 4 holds for \(S\).
Assertion 5. If \( i \neq j \), \( l_i, l_j \geq 2 \) and \( 1 < k(d_i, d_j) < l_i \), then for all \( m \neq i, j \) either \( k(d_i, d_m) = l_m \) or \( 0 \leq k(d_i, d_m) < l_i \); and either
\[
k(d_i, d_i) \leq \frac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geq \frac{11}{2} (l_i - 1) - 1.
\]

Proof. Now by Lemma 2.2 there exist \( x, y \in V_i \) and \( u, v \in V_j \) such that \( xu, yv \in E(G) \) and \( xv, yu \notin E(G) \). Hence \( d_i(x) = d_i(y) = \Delta_i = 1 \). Also applying Assertion 3 to the hypothesis we get that \( X_i = l_i \).

Now suppose there is \( m \neq i, j \) such that \( l_i < k(d_i, d_m) < l_m \). Then clearly \( l_m > 2 \). Hence using Assertion 2 we get that \( k(d_i, d_m) > l_m - l_i \) or \( k(d_i, d_m) < l_i \). So \( l_m - l_i < k(d_i, d_m) < l_m \). As \( \Delta_i = l_i \) so \( d_{im}(x) = d_{im}(y) = \Delta_m = l_m - 1 \geq 1 \). Hence without loss of generality we can take \( G_{im} \) (constructed as in proof of Claim in Assertion 2) to be so that there is \( w \in V_m \) such that \( yw \in E(G), xw \notin E(G) \). \( G \to I(xuyw) \to H \) and as before \( H \) is not isomorphic to \( G \). Contradiction.

Now if
\[
\frac{1}{2} l_i \leq k(d_i, d_i) \leq \frac{1}{2} l_i (l_i - 1) - 2.
\]
Then \( l_i \geq 4 \). Then also \( d_i(x) = d_i(y) = \delta_i \) where \( 1 \leq \delta_i \leq l_i - 2 \). (\( \delta_i \) may be equal to \( \Delta_i \)). Now we let \( V_i = \{v_1, v_2, \ldots, v_{l_i - 1}, x = v_{l_i - 1}, y = v_l \} \) such that \( d_i(v_k) > d_i(v_{k + 1}) \) for \( k = 1, 2, \ldots, l_i - 1 \). If \( \delta_i = l_i - 3 \), then \( G_{ii} \) is constructed as follows: \( x \) is joined to \( v_1 \) to \( v_{d_i(x)} \). Then \( y \) is joined to \( v_{d_i(x) + 1} = w \). If \( \delta_i = l_i - 2 \), then there are at least \( 4 \) vertices of degree \( l_i - 2 \) as \( k(d_i, d_i) \leq \frac{1}{2} l_i (l_i - 1) - 2 \); and we take \( G_{ii} \) such that \( xy \in E(G_{ii}) \). So there is \( w \in V_i \) such that \( w \notin x \) and \( xw \notin E(G_{ii}) \). So in any case we can get a \( G_{ii} \) such that there is a \( w \in V_i \), \( w \neq x \) and \( yw \in E(G_{ii}), xw \notin E(G_{ii}) \). \( G \to I(xuyw) \to H \) and \( I(H[V_i]) \neq \Pi_i \). A contradiction.

Hence Assertion 5 is proved.

Assertion 6. If \( i \neq j \), \( l_i, l_j \geq 2 \) and \( l_j - l_i < k(d_i, d_j) < l_j - 1 \), then for all \( m \neq i, j \) either \( k(d_i, d_m) = 0 \) or \( l_m - l_i < k(d_i, d_m) < l_m \); and either
\[
k(d_i, d_i) > \frac{1}{2} (l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leq 1.
\]

Proof. Apply Assertion 5 to \( S_G \).

Assertion 7. If \( i, j, r \) are all distinct, \( X_i = l_i \); \( 0 < k(d_i, d_j) < l_i \) and \( 0 < k(d_i, d_r) < l_i \), then for all \( m \neq i, j, r \) either \( k(d_i, d_m) = l_m \) or \( 0 \leq k(d_i, d_m) < l_i \); and either
\[
k(d_i, d_i) \leq \frac{1}{2} l_i \quad \text{or} \quad k(d_i, d_i) \geq \frac{11}{2} (l_i - 1) - 1.
\]

Proof. Note \( X_i = l_i \) implies \( l_i \geq 2 \). Also if Assertion 2 could be applied to either \( i, j \) or \( i, r \) then we are done. Hence, as Assertion 2 cannot be applied, we can get \( x, y \in V_i, u \in V_j, v \in V_j \) such that \( xu, yv \in E(G) \) and \( xv, yu \notin E(G) \). The rest of the proof is similar to that of Assertion 5.

Assertion 8. If \( i, j, r \) are all distinct; \( Y_i = l_i \); \( 0 < k(d_i, d_j) < l_i \) and \( 0 < k(d_i, d_r) < l_i \),
then for all \( m \neq i, j, r \) either \( k(d_i, d_m) = 0 \) or \( l_lm - l_i < k(d_i, d_m) \leq l_lm \); and either
\[
k(d_i, d_i) > \frac{1}{2}l_i(l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leq 1.
\]

**Proof.** Apply Assertion 7 to \( S_G \).

**Assertion 9.** If \( k(d_i, d_i) \neq 0 \) or \( \frac{1}{2}l_i(l_i - 1) \) and \( l_i \mid 2k(d_i, d_i) \), then either
\[
k(d_i, d_i) = \frac{1}{2}l_i \quad \text{or} \quad \frac{1}{2}l_i(l_i - 2) \quad \text{or} \quad k(d_i, d_i) = l_i = 5;
\]
and for all \( m \neq i \), \( k(d_i, d_m) = l_lm \) or 0.

**Proof.** Since \( \Pi_{ui} \) is unigraphic so the values of \( k(d_i, d_i) \) are obtained using Lemma 2.3. If for some \( m \neq i \) we have \( 0 < k(d_i, d_m) < l_lm \), then we can get \( x, y \in V_i, w \in V_m \) such that \( xw \in E(G) \), \( yw \notin E(G) \) as \( G_{im} \) is semiregular on both sides. Also, whichever of the permitted values \( k(d_i, d_i) \) takes, we can get a \( G_{ui} \) such that there is \( u \in V_i, u \neq x \) and \( ux \notin E(G_{ui}) \), \( uy \in E(G_{ui}) \). \( G \rightarrow I(xuwv) \rightarrow H \) and \( \Pi(H[V_i]) \neq \Pi_{ui} \). A contradiction. Hence Assertion 9 is proved.

**Assertion 10.** If \( l_i \nmid 2k(d_i, d_i) \), then \( \Pi_{ui} \) is unigraphic.

**Proof.** The proof is immediate.

Now we state and prove the theorem characterizing unigraphic integer-pair sequences.

**Theorem 2.1.** An integer-pair sequence \( S \) is unigraphic if and only if \( S \) is graphic and the following conditions are satisfied.

**C1.** Either \( X_i = Y_i = 0 \) or \( X_i = l_i \) or \( Y_i = l_i \) for \( 1 \leq i \leq n \).

**C2.** If \( i \neq j; l_i, l_j \geq 2 \) and \( l_i \leq k(d_i, d_j) \leq l_l - l_i \), then \( \Pi_{ij} \) has unique realization by bipartite graph \( X_i, Y_i \leq l_i \); for all \( m \neq i, j \), with \( l_m \geq 2 \), either \( 0 \leq k(d_i, d_m) \leq 1 \) or \( \frac{1}{2}l_m - 1 \leq k(d_i, d_m) \leq l_lm \); and either
\[
0 \leq k(d_i, d_i) \leq 1 \quad \text{or} \quad \frac{1}{2}l_i(l_i - 1) - 1 \leq k(d_i, d_i) \leq \frac{1}{2}l_i(l_i - 1).
\]
Further, if \( l_i \mid k(d_i, d_i) \), then for all \( m \neq j, i \), \( k(d_i, d_m) = l_lm \) or 0; and \( k(d_i, d_i) = \frac{1}{2}l_i(l_i - 1) \) or 0.

**C3.** If \( i \neq j; l_i, l_j \geq 2 \) and \( 0 < k(d_i, d_j) < l_i \), then \( X_i = l_i \).

**C4.** If \( i \neq j; l_i, l_j \geq 2 \) and \( l_i - l_j \leq k(d_i, d_j) \leq l_l - l_i \), then \( Y_i = l_i \).

**C5.** If \( i \neq j; l_i, l_j \geq 2 \) and \( 1 < k(d_i, d_j) < l_i \), then for all \( m \neq j, i \) either \( k(d_i, d_m) = l_lm \) or \( 0 \leq k(d_i, d_m) \leq l_lm \); and either
\[
k(d_i, d_i) < \frac{1}{2}l_i \quad \text{or} \quad k(d_i, d_i) > \frac{1}{2}l_i(l_i - 1) - 1.
\]

**C6.** If \( i \neq j; l_i, l_j \geq 2 \) and \( l_i - l_j < k(d_i, d_j) \leq l_l - 1 \), then for all \( m \neq j, i \) either
\[
k(d_i, d_m) = 0 \quad \text{or} \quad \frac{1}{2}l_i(l_i - 1) - 1 \leq k(d_i, d_m) \leq l_lm \); and either
\[
k(d_i, d_i) > \frac{1}{2}l_i(l_i - 2) \quad \text{or} \quad k(d_i, d_i) \leq 1.
\]
C7. If $i, j, r$ are all distinct; $X_i = l_i$; $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_r) < l_i l_r$, then for all $m \neq i, j, r$ either $k(d_i, d_m) = l_i l_m$ or $0 \leq k(d_i, d_m) < l_i$; and either

$$k(d_i, d_j) < \frac{1}{2} l_i$$

or

$$k(d_i, d_j) \geq \frac{1}{2} l_i (l_i - 1) - 1.$$

C8. If $i, j, r$ are all distinct; $Y_i = l_i$; $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_r) < l_i l_r$, then for all $m \neq i, j, r$ either $k(d_i, d_m) = 0$ or $0 \leq k(d_i, d_m) < l_i$; and either

$$k(d_i, d_j) > \frac{1}{2} l_i (l_i - 2)$$

or

$$k(d_i, d_j) \leq 1.$$

C9. If $k(d_i, d_j) \neq 0$ or $\frac{1}{2} l_i (l_i - 1)$ and $l_i | 2k(d_i, d_j)$, then either

$$k(d_i, d_j) = \frac{1}{2} l_i$$

or

$$\frac{1}{2} l_i (l_i - 2) \leq k(d_i, d_j) \leq \frac{1}{2} l_i (l_i - 1)$$

and for all $m \neq i$, $k(d_i, d_m) = l_i l_m$ or $0$.

C10. If $l_i \neq 2k(d_i, d_i)$, then $\Pi_{ii}$ is unigraphic.

**Proof.** The necessity of conditions C1 to C10 follow from Assertions 1 to 10 respectively.

Sufficiency. Let $S$ be an integer-pair sequence which is graphic and satisfies the conditions C1 to C10. Let $H$ be a realization of $S$ and let $G$ be the canonical realization of $S$ obtained from Lemma 2.4. Analogous to the notations developed for $G$, we have the following for $H$: $H_{ii}$ denotes $H[V_i, V_j]$, $d_{ij}(x)$ denotes the degree of $x$ in $H_{ij}$.

We will prove the sufficiency by showing that $H$ is isomorphic to $G$. To this end we state and prove the following claims.

**Claim 1.** For fixed $i$ and all $j$ the degree sequence of the vertices of $V_i$ in the graph $H_{ij}$ is same as in $G_{ij}$.

For $l_i = 1$ it is clearly true. So let $l_i \geq 2$. Also then we need only check for $j \neq i$, such that $l_j \geq 2$ and $0 < k(d_i, d_j) < l_i l_j$.

Suppose for all $j \neq i$, $k(d_i, d_j) = 0$ or $l_i l_j$, then $H_{ii}$ (respectively $G_{ii}$) has to be regular as the degree in $H$ ($G$) of all vertices of $V_i$ is same. Thus $\Pi(H_{ii}) = \Pi(G_{ii})$ and the claim holds.

So now we suppose there exists $j \neq i$ such that $0 < k(d_i, d_j) < l_i l_j$. Then we have the following five exhaustive cases.

**Case (a).** There exists $j \neq i$ such that $l_j \geq 2$ and $l_j \leq k(d_i, d_j) \leq l_i l_j - l_i$. Now if $l_i | k(d_i, d_j)$, then from C2 we see that all vertices of $V_i$ have same degree in $H_{ij}$ ($G_{ij}$). Hence the claim holds.

So let $l_j \nmid k(d_i, d_j)$. So $X_i \neq 0$, $Y_i \neq 0$. Hence $X_i = Y_i = l_i$ from C2. This implies that there are exactly two values of $r$ such that $\Delta_r \neq \delta_r$. One of them is $j$, let the other be $m$. Then, from C2, we get that for all $r \neq j, m$, $k(d_i, d_r) = 0$ or $l_i l_r$ and so for all these $r$ claim holds. Also, from C2, either

$$0 < k(d_i, d_i) \leq 1 \quad \text{or} \quad \frac{1}{2} l_i(l_i - 1) - 1 \leq k(d_i, d_i) \leq \frac{1}{2} l_i(l_i - 1)$$
and so the claim is true for i too. If $m \neq i$, then, from C2, $H_i$ ($G_i$) has exactly one edge or no edge, if $\Delta_i \geq 2$; and if $\Delta_i = 1$, then $H_i$ and $G_i$ are isomorphic. Hence in any case claim is true for $m$. As $\Delta_i \neq \delta_i$ only for $r = m, j$ and claim is true for $m$ so it is true too for $j$ (as all vertices of $V_i$ have same degree in $H$). Hence the claim holds.

Case (b). Not in Case (a) and there exists $j \neq i$ such that $l_i \geq 2$ and $1 < k(d_i, d_j) < l_i$.

From C3 we get that $X_i = l_i$. If $k(d_i, d_j) < \frac{1}{2} l_i$, then, from C5, we get that $X_i = \sum_{j \in J} k(d_i, d_j) + 2k(d_i, d_j)$ where for all $j \in J$ we have $l_j \neq k(d_i, d_j)$ and $i \notin J$. Also, from C5, we get that for all $m \neq j, m \neq i$ $k(d_i, d_m) = 0$ or $l_i l_m$. Hence each vertex of $V_i$ has exactly one of the edges counted in the expression for $X_i$ above as $X_i = l_i$. Thus the claim holds.

If $k(d_i, d_j) = \frac{1}{2} l_i (l_i - 1) - 1$. Then $\Pi(H_i) = H_i$. Then as $X_i = l_i$ and $2 \leq k(d_i, d_j) < l_i$ so for all $m \neq j$, $\Delta_m = \delta_m$. As we are not in Case (a) so $k(d_i, d_m) = 0$ or $l_i l_m$. Moreover $k(d_i, d_j) = 2$. Hence the claim holds.

If $k(d_i, d_j) = \frac{1}{2} l_i (l_i - 1)$, then, from C5, we get that $X_i = \sum_{j \in J} k(d_i, d_j)$ with $J$ defined as before. Similarly here, too, the claim holds.

Case (c). Not in Case (a) and there exists $j \neq i$ such that $l_i \geq 2$ and $l_i l_j - l_i < k(d_i, d_j) < l_i l_j - 1$.

C6 ensures that this follows by considering $H$ and Case (b).

Case (d). Not in any of the previous cases and there exist $j, m \neq i$ such that $0 < k(d_i, d_j) < l_i l_j$ and $0 < k(d_i, d_m) < l_i l_m$.

If $X_i = Y_i = 0$, then we are in Case (a). Hence by C1 we have either $X_i = l_i$ or $Y_i = l_i$. Suppose $X_i = l_i$. Then we use C7 and proof is similar to that of Case (b). If $Y_i = l_i$, then C8 shows that we are in a case similar to Case (c).

Case (e). Not in any of the previous cases. Hence there exists $j \neq i$ such that either $k(d_i, d_j) = 1$ or $l_i l_j - 1$; or $l_j = 1$ with $0 < k(d_i, d_j) < l_i l_j$; and for all $m \neq i, j$ $k(d_i, d_m) = 0$ or $l_i l_m$. Then clearly claim is true for $j$ and hence also $\Pi(H_i) = H_i$.

Thus the claim is proved.

From Claim 1 we immediately have the following.

Claim 2. $\Pi(H_{ij}) = H_{ij}$ for $1 \leq i, j \leq n$.

So now we may denote $\Delta_i(H)$ ($\delta_i(H)$) by $\Delta_{ij}$ ($\delta_{ij}$) also. We now require the following definitions for the next two claims.

We say that $i$ is paired if there is a $j$ such that

$$l_i, l_j \geq 2 \text{ and } \max\{l_i, l_j\} \leq k(d_i, d_j) \leq l_i l_j - \max\{l_i, l_j\};$$

further, we say that $i, j$ are a pair and $i$ is paired with $j$. Otherwise we say that $i$ is not paired.

For $x \in V_i$ we define $f(x) = (d_{i1}(x), d_{i2}(x), \ldots, d_{in}(x))$, $f'(x) = (d'_{i1}(x), d'_{i2}(x), \ldots, d'_{in}(x))$. 

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A map \( \phi : V \to V \) is said to be permissible if \( \phi \) is 1-1 and for \( 1 \leq i \leq n, \ x \in V_i \) implies \( \phi(x) \in V_i \) and \( f'(x) = f(\phi(x)) \).

Observe that Claim 2 and C1 shows that a permissible map exists.

If \( uv \) is an edge (nonedge) in \( H \), then we say \( \phi \) maps it onto an edge (nonedge) if \( \phi(u)\phi(v) \) is an edge (nonedge) in \( G \). Hence a permissible map \( \phi \) which maps all edges of \( H \) onto edges will be an isomorphism of \( H \) onto \( G \). Note that if \( l_i = 1 \), then any permissible map \( \phi \) is an isomorphism from \( H[V_i, V] \) onto \( G[V_i, V] \).

Hence we only consider \( i \) such that \( l_i \geq 2 \) to see how we may obtain an isomorphism from a given permissible map \( \phi \). To this end we have the following two claims.

Claim 3. If \( i \) is paired with \( j \) and \( \phi \) is any permissible map, then we can get a permissible map \( \phi' \) such that \( \phi' = \phi \) on \( V - (V_i \cup V_j) \) and \( \phi' \) maps all edges in \( H \) with at least one point in \( V_i \) onto edges of \( G \).

By Claim 2 and C2 we get that \( \Pi(H_{ij}) = \Pi_{ij} \) has unique realization by bipartite graph. So there is an isomorphism \( \psi \) from \( H_{ij} \) onto \( G_{ij} \) with \( \psi(V_i) = V_i \). Define \( \phi' \) as follows: \( \phi' = \psi \) on \( V_i \cup V_j \) and \( \phi' = \phi \) on \( V - (V_i \cup V_j) \). To check that this \( \phi' \) serves. Clearly all edges of \( H_{ij} \) are mapped onto edges by \( \phi' \).

If \( l_i \mid k(d_i, d_j) \), then, from C2, we see that for all \( m \neq i, j \) all edges of \( H_{im} \) are mapped onto edges. Also for \( x \in V_i: f'(x) = f(\phi'(x)) \). Similarly if \( l_i \mid k(d_i, d_j) \).

If \( l_i \neq k(d_i, d_j) \), then \( X_i, Y_i \neq 0 \). So from C2 we get that \( X_i = Y_i = l_i \) and hence there is exactly one edge with all points in \( V_i \). Hence the vertex \( x \) in \( V_i \) which has the only edge or nonedge in \( H_{ij} \) has its degree in \( H_{ii} \) different from the degree in \( H_{ij} \) of all other vertices of \( V_i \). Hence \( x \) is mapped by \( \phi', \) and hence \( \phi' \), to \( y \) the vertex of same degree in \( G_{ij} \). As a permissible map exists so \( j'(x) = f(y) \). Hence in \( G \) also \( y \) has the only edge or nonedge in \( G_{im} \).

If \( m = i \), then there is exactly one edge or nonedge, say \( xy \), in \( H_{ii} \) by C2. Then \( d'_i(x) = d'_i(y) \) and for all \( z \in V_i, z \neq x, y \) we have \( d'_i(z) \neq d'_i(x) \). So \( xy \) is mapped by \( \phi' \) onto the two vertices of \( V_i \) which have their degree in \( G_{ij} \) same as \( d'_i(x) \). Hence \( \phi' \) maps \( xy \) onto the only edge or nonedge in \( G_{ii} \). Similarly if \( l_i \mid k(d_i, d_j) \).

Clearly \( \phi' \) is permissible in each case. Hence the claim is proved.

Claim 4. If \( i \) is not paired; there is no \( j \neq i \) such that \( l_i \leq k(d_i, d_j) \leq l_i - 1 \); and \( \phi \) is any permissible map, then we can get a permissible map \( \phi' \) such that \( \phi' = \phi \) on \( V - V_i \) and \( \phi' \) maps all edges in \( H \) with at least one point in \( V_i \) onto edges.

Suppose for all \( j \) \( H_{ij} \) is complete or empty, then clearly so is \( G_{ij} \) and hence we can take \( \phi' = \phi \). So let there exist a \( j \) such that \( H_{ij} \) is neither complete nor empty. If \( j = i \) and \( l_i \mid 2k(d_i, d_j) \), then, by C9, we know that for all \( m \neq i, H_{im} \) is either complete or empty. Also by C9 and Lemma 2.3 we see that \( \Pi_{ii} \) is unigraphic. Hence there is an isomorphism \( \psi \) from \( H_{ii} \) onto \( G_{ii} \). We then define \( \phi' \) as follows: \( \phi' = \psi \) on \( V_i \) and \( \phi' = \phi \) on \( V - V_i \). This \( \phi' \) serves.
If \( j = i \) and \( l < 2k(d_i, d_j) \), then there is \( m \neq i \) such that \( \Delta_m \neq \delta_m \). In particular \( G_m \) is neither complete nor empty. So the only case remaining is that there is a \( j \neq i \) such that \( 0 < k(d_i, d_j) < l_j \). We make the following subcases, which are exhaustive.

**Subcase (a).** There exists \( j \neq i \) such that \( l_j \gg 2 \) and \( l_i = l_j \). Then by C3 we know that \( X_i = l_j \). Also, by C5, either

\[
k(d_i, d_j) < \frac{1}{2}l_i \quad \text{or} \quad k(d_i, d_j) > \frac{1}{2}l_i(l_i - 1) - 1.
\]

Suppose \( k(d_i, d_j) = \frac{1}{2}l_i(l_i - 1) - 1 \) and \( k(d_i, d_j) > 0 \). Applying Lemma 2.2 to \( H_{ij} \) we can get \( x, y \in V_i, u, v \in V_i \) such that \( xu, yv \in E(H_i) \). Note \( u \neq v \) as \( d'(u) = d'(v) = 1 \). As \( \phi \) is permissible so we can get \( u', v' \in V_i \) such that \( \phi(u)u', \phi(v)v' \in E(G_i) \). Similarly \( u' \neq v' \). Define \( \phi' \) as follows: \( \phi'(u) = u'; \phi'(v) = v'; \phi' = \phi \) on \( V - V_i \); and \( \phi' \) on \( V_i, \{u, v\} \) is any 1-1 map onto \( V_i, \{u', v'\} \). To check that \( \phi' \) serves. As \( u, v, (u', v') \) have degree \( \Delta_i = 1 = \delta_i \) in \( H_{ij} \) and \( X_i = l_i \) so they have degree \( \delta_i = l_i - 2 \) in \( H_{i}(G_{ij}) \). Hence the only nonedge in \( H_{ij} \) is \( uv \). So all edges of \( H_{ij} \) are mapped onto edges. The vertices of \( V_i \) have degree \( \delta_i = l_i - 2 \) in \( H_{ij} \). Thus all edges of \( H_{ij}, H_{ij}, H_{ij} \) for \( m \neq j, i \) are mapped onto edges.

Suppose \( k(d_i, d_j) = \frac{1}{2}l_i \) or \( k(d_i, d_j) = 1 \). Then from C5 it can be seen that \( \Delta_m \neq \delta_m \) implies \( \Delta_m = 1 \); for \( m \neq i \), \( \Delta_m = \delta_m \) implies \( \delta_m = 0 \) or \( l_m \); and \( \Delta_i = \delta_i \) implies \( \delta_i = l_i - 1 \) or 0. Let \( m \neq i \), \( \Delta_m = 1 \neq \delta_m \) and \( x \in V_m \). Now as \( \phi \) is permissible \( \phi\) is permissible so \( \phi(m)(x) = \phi(m)(\phi(x)) = k \) (say). Let \( xu_1, \ldots, xu_k \in E(H_{im}) \) and \( \phi(x)v_1, \ldots, \phi(x)v_k \in E(G_{im}) \). Then we define \( \phi'(u_r) = v_r \) for \( 1 \leq r \leq k \). Further if \( 0 < k(d_i, d_j) < \frac{1}{2}l_i \) and \( \{u_1, v_1, \ldots, u_k v_k\} = E(H_{ij}) \) and \( \{u_1' v_1', \ldots, u_k' v_k'\} = E(G_{ij}) \) we define \( \phi'(u_s) = u_s' \) and \( \phi'(v_s) = v_s' \) for \( 1 \leq r \leq k \). Thus we can define \( \phi' \) on \( V_i \). \( \phi' \) is well defined as \( X_i = l_i \). Clearly this \( \phi' \) serves.

**Subcase (b).** There exists \( j \neq i \) such that \( l_j \gg 2 \) and \( l_i = l_j \). Then from C3 it can be seen that \( \Delta_m \neq \delta_m \) implies \( \Delta_m = 1 \); for \( m \neq i \), \( \Delta_m = \delta_m \) implies \( \delta_m = 0 \) or \( l_m \); and \( \Delta_i = \delta_i \) implies \( \delta_i = l_i - 1 \) or 0. Let \( m \neq i \), \( \Delta_m = 1 \neq \delta_m \) and \( x \in V_m \). Now as \( \phi \) is permissible \( \phi\) is permissible so \( \phi(m)(x) = \phi(m)(\phi(x)) = k \) (say). Let \( xu_1, \ldots, xu_k \in E(H_{im}) \) and \( \phi(x)v_1, \ldots, \phi(x)v_k \in E(G_{im}) \). Then we define \( \phi'(u_r) = v_r \) for \( 1 \leq r \leq k \). Further if \( 0 < k(d_i, d_j) < \frac{1}{2}l_i \) and \( \{u_1, v_1, \ldots, u_k v_k\} = E(H_{ij}) \) and \( \{u_1' v_1', \ldots, u_k' v_k'\} = E(G_{ij}) \) we define \( \phi'(u_s) = u_s' \) and \( \phi'(v_s) = v_s' \) for \( 1 \leq r \leq k \). Thus we can define \( \phi' \) on \( V_i \). \( \phi' \) is well defined as \( X_i = l_i \). Clearly this \( \phi' \) serves.

**Subcase (c).** Not in any of the previous subcases and there exist \( j, m \neq i \) such that \( 0 < k(d_i, d_j) < l_i \) and \( 0 < k(d_i, d_m) < l_m \).

If \( X_i = Y_i = 0 \), then we have \( l_i \leq k(d_i, d_j) \leq l_j \), a contradiction. Hence, by C1, either \( X_i = l_i \) or \( Y_i = l_i \). Suppose \( X_i = l_i \), but \( Y_i \neq l_i \). Then we have also for \( r = j, m \neq i \) and \( 0 < k(d_i, d_j) < l_i \) as we are not in any of the previous cases. Using this and C7 we proceed as in Subcase (a). Similarly we are done if \( Y_i = l_i \) and \( X_i \neq l_i \). If \( X_i = Y_i = l_i \), then \( j, m \) are the only two values \( r \) for which \( H_{ir} \) is not complete or empty. Then we can take \( \phi' = \phi \).

**Subcase (d).** Not in any of the previous cases.

Here (same case as Case (e) of Claim 1) we take \( \phi' = \phi \) if \( l_i < 2k(d_i, d_j) \). If \( l_i > 2k(d_i, d_j) \), then by C10 we know that \( P_{ii} \) is unigraphic. Let \( \psi \) be an isomorphism from \( H_{ii} \) onto \( G_{ii} \). Then we define \( \phi' = \phi \) on \( V - V_i \) and \( \phi' = \psi \) on \( V_i \).

This proves the claim.

Now note that if \( i \) is paired with \( j \), then \( i \) is not paired with any \( m \neq j \) by C2.
Hence distinct pairs are disjoint. So we begin with a permissible map, say \( \phi \), which we know exists, and modify it successively for each distinct pair, according to Claim 3, and then for each \( i \) satisfying the conditions of Claim 4, according to Claim 4. Let \( \phi_0 \) be the permissible map we get finally. Now, if any \( i \) is not covered in the above steps, then we know that it is not paired and there is a \( j \) such that \( l_i, l_j \geq 2 \) and \( l_i, l_j = k(d_i, d_j) = l_i - l_j \). Then as \( j \) is not paired with \( i \) and C2 is satisfied, there cannot be any \( m \neq j \) such that \( l_i, l_j \geq 2 \) and \( l_i, l_j = k(d_i, d_m) = l_i - l_j \). Hence \( i \) has been covered in the above steps and so all edges of \( H_i \) are mapped onto edges by \( \phi_0 \). All other edges in \( H \) with at least one point in \( V_i \) are mapped onto edges by any permissible map: this follows from C2 and fact that any edge with at least one point in \( V_i \), where \( l_i, l_j = 1 \), is mapped onto an edge by any permissible map. Hence we see that \( \phi_0 \) is an isomorphism of \( H \) onto \( G \).

This completes the proof of the theorem.

3. Unidigraphic integer-pair sequences

As there may be non-isolated vertices in a digraph with indegree 0 so we require the following analogous definitions in the case of digraphs.

Let \( S = ((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) \) be a sequence of ordered pairs of non-negative integers.

We then have (following [3]): \( A \) is the sequence \( (a_1, a_2, \ldots, a_n) \); \( B \) is the sequence \( (b_1, b_2, \ldots, b_n) \); \( A^* \) is the set of distinct members of \( A \); \( B^* = \{d_1, d_2, \ldots, d_n\} \) is the set of distinct members of \( B \); \( k'(r, s) \) is the number of times the ordered integer-pair \((r, s)\) occurs in \( S \); \( k'(r) \) is the number of times \( r \) occurs in \( A \) and \( B \).

It is shown in [3] that if \( S \) is digraphic, then \( 0 \notin B^* \) and any digraph realization of \( S \) has \( k'(d_i)/d_i \) vertices of indegree \( d_i \). Hence \( k'(d_i)/d_i \) is a positive integer. Also then \( k(0) \) is the number of times 0 occurs in \( A \).

Thus for digraphic \( S \) we have the following: For \( 1 \leq i \leq n \) we define:

\[
l_i = \frac{k'(d_i)}{d_i},
\]

\[
X_i = \sum_{j=1}^{n} ((k'(d_i, d_j))(\text{mod } l_j)) + (k'(0, d_i)(\text{mod } l_i)),
\]

\[
Y_i = \sum_{j=1}^{n}((- k'(d_i, d_j))(\text{mod } l_i)) + ((- k'(0, d_i))(\text{mod } l_i)).
\]

Also when \( S \) is digraphic any digraph realization \( G \) of \( S \) is considered on the vertex set \( V(G) = \bigcup_{i=0}^{n} V_i \) where for \( 1 \leq i \leq n \), \( |V_i| = l_i \) and \( d_G^-(x) = d_i \) for \( x \in V_i \); if \( 0 \notin A^* \), then \( V_0 = \emptyset \); and if \( 0 \in A^* \), then for \( x \in V_0 \), \( d_G^-(x) = 0 \).

As before for \( 0 \leq i, j \leq n \) we denote \( G[V_i, V_j] \) by \( G_i \) and \( G[V_i] \) by \( G_i \) or \( G_{ui} \). \( \Delta_i^+(G) \) and \( \delta_i^+(G) \) \( (\Delta_i^-(G) \) and \( \delta_i^-(G) \) denote respectively the maximum and minimum out-degree (indegree) in \( G_{ij} \) of a vertex in \( V_i \) \( (V_i) \).
For the bipartite digraph $G_{ij}$ ($i \neq j$) the bipartition is always taken to be $V_i \cup V_i$ and $G_{ij}$ is said to be semiregular on both sides if $G_{ij}$ considered as an undirected bipartite graph is. (Note the digraph $G_{ij}$ is antisymmetric.)

The maximum and minimum outdegree (indegree) of a vertex in a digraph $G$ is denoted respectively by $\Delta^+(G)$ and $\Delta^-(G)$ ($\Delta^-(G)$ and $\Delta^+(G)$). A digraph $G$ is said to be semiregular if $\Delta^+(G) - \Delta^-(G) \leq 1$ and $\Delta^-(G) - \Delta^+(G) \leq 1$.

We now give a canonical realization of a digraphic integer-pair sequence.

**Lemma 3.1.** If $S$ is a digraphic integer-pair sequence, then there is a digraph realization $G$ of $S$ such that for every $i, j, 1 \leq i \neq j \leq n$, $G_i$ is semiregular and $G_{ij}$ is semiregular on both sides.

**Proof.** Let $H$ be a digraph realization of $S$. Let $M^+_{ij}$ ($M^-_{ij}$) be the number of vertices in $V_i$ ($V_j$) that have maximum outdegree (indegree) in $H[V_i, V_j]$.

Let

$$f(H) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta^+_{ij}(H) - \delta^+_{ij}(H) + \Delta^-_{ij}(H) - \delta^-_{ij}(H) + M^+_{ij} + M^-_{ij}).$$

Then of all the realizations of $S$ we choose one, $G$, such that $f(G)$ is minimum.

**Claim.** $G$ satisfies the conditions of Lemma 3.1.

Suppose there is $i$ such that $G_i$ is not semiregular. Then either $\Delta^+_i - \delta^+_i \geq 2$ or $\Delta^-_i - \delta^-_i \geq 2$. If $\Delta^+_i - \delta^+_i \geq 2$, then we can get distinct $u, v, w$ in $V_i$ such that in $G_i$ outdegrees of $u, v$ are $\Delta^+_i, \delta^+_i$ respectively and $uw \in E(G), vw \notin E(G)$. We then obtain $G'$ from $G$ by replacing $uw$ by $vw$ in $E(G)$. Note $S_{G'} = S_G$. Further $f(G') < f(G)$, a contradiction.

If $\Delta^-_i - \delta^-_i \geq 2$, then there exist distinct $u, v, w$ in $V_i$ such that in $G_i$ indegrees of $u, v$ are $\Delta^-_i, \delta^-_i$ respectively and $wu \in E(G), wv \notin E(G)$. Also there is $m \neq i$ such that $d^-_{G_m}(v) > d^-_{G_m}(u)$ and the cycle is $x \in V_m$ such that $xu \notin E(G), xv \in E(G)$. $G \rightarrow I(wuxv) \rightarrow G'$. Then $f(G') < f(G)$, a contradiction. Hence $G_i$ is semiregular.

Now, assuming that for $1 \leq i \leq n$, $G_i$ is semiregular, it can be similarly shown that $G_{ij}$ is semiregular on both sides for $1 \leq i \neq j \leq n$.

Hence the lemma is proved.

We now give the theorem characterizing unidigraphic integer-pair sequences.

**Theorem 3.1.** Let $S$ be an integer-pair sequence. Then $S$ is unidigraphic if and only if $S$ is digraphic and satisfies the following conditions:

**P1.** Either $X'_i = Y'_i = 0$ or $X'_i = l'_i$ or $Y'_i = l'_i$ for $1 \leq i \leq n$.

**P2.** $k'(d_i, d_i) \in \{0, 1, l'_i(l'_i - 1) - 1, l'_i(l'_i - 1)\}$ for $1 \leq i \leq n$.

**P3.** If $1 \leq i \neq j \leq n$ and $l'_i, l'_j \geq 2$, then $k'(d_i, d_i) \in \{0, 1, l'_i(l'_i - 1), l'_j l'_i\}$. 
P4. If \( k'(d_i, d_i) \not\in \{0, l_i(l'_i - 1)\} \), then \( X'_i = Y'_i = l'_i \) for \( 1 \leq i \leq n \).

P5. If \( 1 \leq i \neq j \leq n, l'_i \geq 2 \) and \( k'(d_i, d_i) \not\in \{0, l'_i l'_m\} \), then for all \( m \neq i, j, k'(d_m, d_i) = 0 \) or \( l'_m(l'_i - 1) \).

P6. If \( 0 \in A^* \), then either \( k(0) = 1 \) or there is \( j \) such that \( l'_i = 1 \) and \( k(0) = k'(0, d_j) \).

Proof. In the following we take \( G \) to be the canonical realization of Lemma 3.1. Further if \( 0 \in A^* \) we assume that \( V(G) = \bigcup_{i=0}^n V_i \) where \( |V_i| = k(0) \) and for all \( x \in V_0, d_G(x) = 0 \). As \( G \) has no isolated vertices so \( d_G(x) = 0 \) for all \( x \in V_0 \).

We also write \( \Delta^+_i, \Delta^-_i, \delta^+_i, \delta^-_i, d^+_i(x), d^-_i(x) \) for \( \Delta_i(G), \Delta_i(G), \delta_i(G), \delta_i(G), d_G(x), d_G(x) \) respectively.

Now to prove the necessity.

(1) The proof of that P1 holds is similar to that of Assertion 1.

(2) Suppose \( 2 \leq k'(d_i, d_i) \not\in l'_i(l'_i - 1) - 2 \). Now as \( G_E \) is semiregular so \( \Delta^+_i = \Delta^-_i = \delta^+_i = \delta^-_i \). So if \( \Delta^+_i = 1 \) or \( l'_i - 1 \), then we have distinct vertices \( x, y, u, v \) in \( V_i \) such that \( xu, yu \in E(G) \) and \( xu, yu \not\in E(G) \). Then we obtain \( G' \) from \( G \) by replacing \( xu \) by \( yu \) in \( E(G) \) where we assume, without loss of generality, that \( d^+_i(y) \geq d^+_i(x) \). Then \( G'[V_i] \) is not semiregular, contradicting the fact that \( S \) is unidigraphic. So we take \( 2 \leq \Delta^-_i \leq l'_i - 2 \). Let \( x \in V_i \) be such that \( d^-_i(x) = \Delta^-_i \). So there is \( y \in V_i \) such that \( xy \in V_i \). But \( d^-_i(y) \geq \Delta^-_i \). Hence there is \( w \in V_i \) such that \( w \in E(G) \). We obtain \( G' \) from \( G \) by replacing \( w \) by \( y \) in \( E(G) \). Then \( \Delta^-_i(G') > \Delta^-_i \). A contradiction. Hence P2 holds.

(3) Suppose we have \( i \neq j \), \( l'_i, l'_j \geq 2 \) and \( 2 \leq k'(d_i, d_j) \not\in l'_i l'_j - 2 \). Then \( G_E \) considered as an undirected bipartite graph satisfies the conditions of Lemma 2.2. Hence we get \( x, y, u, v \) as stated there. Let \( d^+_i(x) \geq d^+_i(y) \). We replace \( yu \) by \( xu \) to get a contradiction as above. Hence P3 holds.

(4) Clearly P4 holds if \( l'_i = 2 \). So let \( l'_i \geq 3 \). By P2, \( k'(d_i, d_i) = 1 \) or \( l'_i(l'_i - 1) - 1 \). Hence there is \( w \) in \( V_i \) which has unique indegree in \( G_{\Delta_i} \). If \( X'_i \neq l'_i \) or \( Y'_i \neq l'_i \), then there are \( j, m \neq i, 0 \leq j \neq m \leq n \) and vertices \( x, y \not\in w \) in \( V_i \), such that \( d^-_j(x) > d^-_j(y) \) and \( d^-_m(y) > d^-_m(x) \). So in \( G_{\Delta_i} \) the only arc or nonarc may be taken to be \( xw \) or \( yw \) to give two non-isomorphic realizations of \( S \). Hence P4 holds.

(5) Suppose there is \( m \neq i, j \) such that \( 0 < k'(d_m, d_i) < l'_i l'_m \). Then there are \( x, y \in V_i \) such that \( d^-_m(x) > d^-_m(y) \). Then we may have \( d^+_i(x) > d^+_i(y) \) or \( d^+_i(y) > d^+_i(x) \) to give two non-isomorphic realizations of \( S \). A contradiction. Similarly it may be shown that \( k'(d_i, d_i) = 0 \) or \( l'_i(l'_i - 1) \). Hence P5 holds.

(6) If P6 does not hold, then we can get a realization in which the number of vertices with indegree 0 is less than \( k(0) \). A contradiction. Hence P6 holds.

Now to prove the sufficiency.

Let \( S \) satisfy the conditions and \( H \) be any realization of \( S \). From P2 and P3 we see that \( \Delta^-_i(H) = \Delta^-_i, \Delta^-_i(H) = \Delta^-_i, \delta^-_i(H) = \delta^-_i \) and \( \delta^-_i(H) = \delta^-_i \). Also from P6 we see that there are \( k(0) \) vertices in \( V(H) \) of indegree 0. Hence we can take \( V(H) = V(G) = V \).

Let \( g^+_i(x) \) and \( g^-_i(x) \) denote the outdegree and indegree respectively of \( x \) in \( H[V_i, V_i], \) where \( x \in V \) and \( V \).
Let \( f(x) = ((d^+_1(x), d^+_2(x), \ldots, d^+_n(x)), (d^-_0(x), d^-_1(x), \ldots, d^-_n(x))) \) and \( f'(x) = ((g^+_1(x), g^+_2(x), \ldots, g^+_n(x)), (g^-_0(x), g^-_1(x), \ldots, g^-_n(x))) \) if \( x \in V_i \). If \( k(0) = 0 \), then \( d^+_0(x) = g^-_0(x) = 0 \) for all \( x \in V \).

A map \( \phi: V \rightarrow V \) is said to be permissible if \( \phi \) is 1-1 and \( x \in V_i \) \( 0 \leq i \leq n \), implies \( \phi(x) \in V_i \) and \( f'(x) = f(\phi(x)) \).

**Claim.** If \( l'_i \geq 2 \), then \( \{f(x): x \in V_i\} = \{f'(x): x \in V_i\} \).

Suppose there is a \( j \neq i \) such that \( \Delta_{ij}^+ \neq \delta_{ij}^+ \), then as \( S \) is digraphic so \( j \neq 0 \). Hence from P5 we get that \( \Delta_{ii}^+ = \delta_{ii}^+ \), \( \Delta_{ii}^- = \delta_{ii}^- \) and for all \( m \neq i, j, 0 \Delta_{im}^+ = \delta_{im}^+, \Delta_{mi}^- = \delta_{mi}^- \). Also by P6 \( \Delta_{ii}^- = \delta_{ii}^- = 0 \) as indegree in \( H \) of all \( l_i^1 \) vertices of \( V_i \) is same. Hence the claim holds in this case.

Suppose \( \Delta_{ii}^+ \neq \delta_{ii}^+ \) and there is no \( j \neq i, 1 \leq j \leq n \), such that \( \Delta_{ij}^+ \neq \delta_{ij}^+ \). Note there is a vertex of unique outdegree distinct from the vertex of unique indegree in both \( G_i \) and \( H_i \) by P2. From P4 we get that \( X_i = Y_i = l_i^1 \) and hence the claim holds.

So for all \( j, 1 \leq j \leq n \), \( \Delta_{ij}^+ = \delta_{ij}^+ \). If there is a \( p, 0 \leq p \leq n \) such that \( \Delta_{pi}^+ \neq \delta_{pi}^+ \), then also Claim holds as we know that either \( X_i = l_i^1 \) or \( Y_i \neq l_i^1 \) by P1. If there is no such \( p \), then, too, the claim holds.

Hence the claim is proved.

It follows from the above claim, as we need not check for \( V_i \) with \( l_i^1 = 1 \) and for \( V_o \) that there is a permissible map \( \phi \). It can easily be seen that \( \phi \) is an isomorphism from \( H \) onto \( G \).

This completes the proof of the theorem.

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**References**


