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Suboptimal Control of Fixed-End-Point Minimum Energy Problem via Singular Perturbation Theory

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1. INTRODUCTION

In optimization problems, it is urged from practical points of view to develop effective synthesis methods based on an approximate model obtainable by reducing the system dimension. For instance, the numerical calculation of optimal control needs enormous endeavor with increase of dimension of the control system, because the generated canonical system equation is "$2n$-dimensional" when the original system is "n-dimensional". Moreover, the essential difficulty lies in that one must solve it under two-point boundary conditions. Various methods have been discussed on solving the boundary value problem, e.g., sweep method [1], shooting method [2], quasilinearization [3], invariant imbedding [4], etc.

This paper gives a powerful tool for the analysis of the fixed-terminal minimum energy problems with the aid of the Singular Perturbation Theory [5–7] and the Riccati transformation [8]. The Riccati transformation is efficient in computation of the so-called ill-conditioned two-point boundary value problem. The Singular Perturbation Theory has two aspects: one is to consider the convergence of the solution of the full system to that of the reduced system as a small parameter, whose existence makes the system order higher, tends to zero; the other is to construct an asymptotic expansion of the solution, which can offer a desired approximate solution by truncation. In both the aspects, the stability of the boundary layer system plays a crucial role [5—7].

Kokotović *et al.* first used the Singular Perturbation Theory in developing a method to reduce the system order in optimization problems [9]. In their studies of the linear regulator problem, it is needed that the boundary layer system should be asymptotically stable [9], or that the state matrix of the boundary layer system should be stable and the full system should satisfy a peculiar condition [10], or that the boundary layer system should be controllable and observable [11].

The method in this paper assumes that the state matrix of the boundary layer system is either positively or negatively stable and that it satisfies a certain condition similar to observability. The method is based upon Vasil'eva's theory [6] and enables us to calculate the terminal values, under which higher order (outer) correction system is to be solved, by utilizing the inner system, which is equivalent to the boundary layer system, without solving it throughout the interval (see Section 6).

2. Problem Statement

The state equation is

$$\frac{d}{dt}x(t,\epsilon) = A(t,\epsilon)\,x(t,\epsilon) + B(t,\epsilon)\,u(t,\epsilon),$$
$$A = \begin{pmatrix} A_1 & A_2 \\ \epsilon^{-1}A_3 & \epsilon^{-1}A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ \epsilon^{-1}B_2 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1}$$

where x_1 is an n-dimensional vector, x_2 m-dimensional, control vector u r-dimensional, and ϵ is a positive small parameter. The boundary conditions are

$$x(t_0) = \xi, \qquad x(t_f) = \eta, \tag{2}$$

or, in the partitioned form,

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(t_f) \\ x_2(t_f) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

The problem is to minimize the performance index

$$J = \tfrac{1}{2}\int_{t_0}^{t_f} (x'Qx + u'Ru)\,dt. \tag{3}$$

As well-known, the optimal control u^* is given as follows:

$$u^* = -R^{-1}B'p. \tag{4}$$

The canonical equation is

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{5}$$

where p is a costate vector of x. The canonical system (5) under the boundary condition (2) is ill-conditioned in the sense of Mufti *et al.* [8], i.e., in our case the boundary values of costate p are not specified.

3. Riccati Transformation

Now we define a Riccati transformation [8]

$$x(t) = K(t)\,p(t) + g(t), \tag{6}$$

where $K(t)$ is an $(n+m) \times (n+m)$-matrix and $g(t)$ is an $(n+m)$-dimensional vector. Notice that (6) is different from the conventional Riccati transformation [13] in control theory, which is given as

$$p(t) = \hat{K}(t)\,x(t) + \hat{g}(t). \tag{7}$$

By simple manipulation, we obtain the differential equation of Riccati type for $K(t)$ and the associated differential equation for $g(t)$ as follows

$$\dot{K} = AK + KA' + KQK - BR^{-1}B', \tag{8}$$

$$\dot{g} = (A + KQ)\,g. \tag{9}$$

Equations (8) and (9) are solved under the initial conditions

$$K(t_0) = 0 \tag{10}$$

and

$$g(t_0) = \xi \tag{11}$$

or under the final conditions

$$K(t_f) = 0, \tag{10'}$$

$$g(t_f) = \eta. \tag{11'}$$

Partitioning K and g into the forms

$$K = \begin{pmatrix} K_1 & K_2 \\ K_2' & \epsilon^{-1}K_3 \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{12}$$

we get the equations for K_i, $i = 1, 2, 3$;

$$\begin{aligned}\dot{K}_1 = {} & A_1K_1 + K_1A_1' + A_2K_2' + K_2A_2' + K_1Q_1K_1 \\ & + K_2Q_2K_1 + K_1Q_2K_2' + K_2Q_2K_2' - B_1R^{-1}B_1',\end{aligned} \tag{13a}$$

$$\begin{aligned}\epsilon\dot{K}_2 = {} & K_2A_4' + K_2Q_3K_3 + K_1A_3 + K_1Q_2K_3 - B_1R^{-1}B_2' \\ & + \epsilon(A_1K_2 + A_2K_3 + K_1Q_1K_2 + K_2Q_2K_2),\end{aligned} \tag{13b}$$

$$\begin{aligned}\epsilon\dot{K}_3 = {} & A_4K_3 + K_3A_4' + K_3Q_3K_3 - B_2R^{-1}B_2' \\ & + \epsilon(A_3K_2 + K_2'A_3' + K_3Q_2'K_2 + K_2'Q_2K_3) + \epsilon^2K_2'Q_1K_2,\end{aligned} \tag{13c}$$

and for g_i, $i = 1, 2$,

$$\dot{g}_1 = (A_1 + K_1Q_1 + K_2Q_2) g_1 + (A_2 + K_1Q_2 + K_2Q_2) g_2 , \tag{14a}$$

$$\epsilon\dot{g}_2 = (A_3 + K_3Q_2' + \epsilon K_2'Q_1) g_1 + (A_4 + K_3Q_3 + \epsilon K_2'Q_2) g_2 , \tag{14b}$$

where Q_i's are the elements of $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{pmatrix}$. We call the system equations (13) and (14) with (10) and (11) the "forward full system" and Eqs. (13) and (14) with (10′) and (11′) the "backward full system".

After K's and g's are determined under the terminal conditions, we get the optimal control u^* by substituting

$$p(t) = K(t)^{-1} (x(t) - g(t)) \tag{15}$$

into Eq. (4), and the optimal trajectory is given by

$$\dot{x} = (A - BR^{-1}B'K^{-1}) x + BR^{-1}B'K^{-1}g, \tag{16}$$

where it is assumed that K is invertible, so that Eq. (16) holds for $t \in (t_0, t_f]$ for the forward system or for $t \in [t_0, t_f)$ for the backward system.

4. Asymptotic Expansion and Approximate Solution

In this section is given a systematic method to construct the set of the recursive equations to give the asymptotic expansion of solutions.

The reduced equations for K_i's and g_i's are obtained by letting $\epsilon = 0$ in Eqs. (13) and (14),

$$\begin{aligned} \dot{K}_1^0 = {} & A_1K_1^0 + K_1^0A_1' + A_2K_2^{0\prime} + K_2^0A_2' + K_1^0Q_1K_1^0 \\ & + K_2^0Q_2K_1^0 + K_1^0Q_2K_2^{0\prime} + K_2^0Q_2K_2^{0\prime} - B_1R^{-1}B_1', \end{aligned} \tag{17a}$$

$$0 = K_2^0A_4' + K_2^0Q_3K_3^0 + K_1^0A_3 + K_1^0Q_2K_3^0 - B_1R^{-1}B_2', \tag{17b}$$

$$0 = A_4K_3^0 + K_3^0A_4' + K_3^0Q_3K_3^0 - B_2R^{-1}B_2', \tag{17c}$$

$$\dot{g}_1^0 = (A_1 + K_1^0Q_1 + K_2^0Q_2) g_1^0 + (A_2 + K_1^0Q_2 + K_2^0Q_3) g_2^0, \tag{18a}$$

$$0 = (A_3 + K_3^0Q_2') g_1^0 + (A_4 + K_3^0Q_3) g_2^0, \tag{18b}$$

with the initial conditions for the forward reduced system

$$K_1^0(t_0) = 0 \tag{19}$$

and

$$g_1^0(t_0) = \xi_1 , \tag{20}$$

or with the final conditions for the backward reduced system

$$K_1^0(t_f) = 0 \tag{19'}$$

and

$$g_1^0(t_f) = \eta_1 \,. \tag{20'}$$

The superscript zero means that the corresponding variables are of the reduced system.

Such a reduction is intuitively carried out by a designer when he deals with a real physical system by neglecting a small parameter, as in the example of the Prompt Jump Approximation in nuclear reactor dynamics [12].

Now expanding K_i's and g_i's into Taylor series, we get

$$K_i = K_i^0 + \epsilon K_i^1 + O(\epsilon^2), \tag{21}$$

$$g_i = g_i^0 + \epsilon g_i^1 + O(\epsilon^2). \tag{22}$$

The reduced equations, as simply seen, correspond to the first terms in Eqs. (21) and (22). In order to obtain the first correction equations for the second terms in these equations, we differentiate the full equations with respect to ϵ, then ϵ is set to be zero,

$$\begin{aligned}\dot{K}_1^1 = A_1K_1^1 + K_1^1A_1 + A_2K_2^{1\prime} + K_2^1A_2 + K_1^1Q_1K_1^0 \\ + K_1^0Q_1K_1^1 + K_2^1Q_2K_1^0 + K_2^0Q_2K_1^1 + K_1^1Q_2K_2^{0\prime} \\ + K_1^0Q_2K_2^{1\prime} + K_2^1Q_2K_2^{0\prime} + K_2^0Q_2K_2^{1\prime},\end{aligned} \tag{23a}$$

$$\begin{aligned}0 = K_2^1A_4' + K_2^1Q_3K_3^0 + K_2^0Q_3K_2^{1\prime} + K_1^1A_3 + K_1^1Q_2K_3^0 \\ + K_1^0Q_2K_3^1 + A_1K_2^0 + A_2K_3^0 + K_1^0Q_1K_1^0 + K_2^0Q_2'K_2^0 - \dot{K}_2^0,\end{aligned} \tag{23b}$$

$$\begin{aligned}0 = A_4K_3^1 + K_3^1A_4' + K_3^1Q_3K_3^0 + K_3^0Q_3K_3^1 + A_3K_2^0 \\ + K_2^{0\prime}A_3' + K_3^0Q_2'K_2^0 + K_2^{0\prime}Q_2K_3^0 - \dot{K}_3^0,\end{aligned} \tag{23c}$$

$$\begin{aligned}\dot{g}_1^1 = (A_1 + K_1^0Q_1 + K_2^0Q_2)\,g_1^1 + (K_1^1Q_1 + K_2^1Q_2)\,g_1^0 \\ + (A_2 + K_1^0Q_2 + K_2^0Q_3)\,g_2^1 + (K_1^1Q_2 + K_2^1Q_3)\,g_2^0,\end{aligned} \tag{24a}$$

$$\begin{aligned}0 = (A_3 + K_3^0Q_2')\,g_1^1 + (K_3^1Q_2' + K_2^{1\prime}Q_1)\,g_1^0 \\ + (A_4 + K_3^0Q_3)\,g_2^1 + (K_3^1Q_3 + K_2^{0\prime}Q_2)\,g_2^0 - \dot{g}_2^0.\end{aligned} \tag{24b}$$

In deriving the above equations, A_i's, B_i's, R, and Q are assumed to be independent of ϵ in order to avoid introducing superfluous notations, but this assumption is not essential in our argument. The initial conditions (the final conditions) under which Eqs. (23) and (24) are solved are not generally set to be zero. This situation is peculiar to the singular perturbation theory. Following Vasil'eva's integral formula [6], we obtain the relations between the initial conditions (the final conditions) and the boundary layer system.

The boundary layer system plays a principal role in the singular perturbation theory, and, in this case, for the forward system it results in

$$\frac{d\bar{K}_2(\tau)}{d\tau} = \bar{K}_2(\tau)\, A_4' + \bar{K}_2(\tau)\, Q_3 \bar{K}_3(\tau) + K_1^0(t)\, A_3 + K_1^0(t)\, Q_2 \bar{K}_3(\tau), \tag{25}$$

$$\frac{d\bar{K}_3(\tau)}{d\tau} = A_4 \bar{K}_3(\tau) + \bar{K}_3(\tau)\, A_4' + \bar{K}_3(\tau)\, Q_3 \bar{K}_3(\tau) - B_2 R^{-1} B_2', \tag{26}$$

$$\frac{d\bar{g}_2(\tau)}{d\tau} = (A_3 + \bar{K}_3(\tau)\, Q_2')\, g_1^0(t) + (A_4 + \bar{K}_3(\tau)\, Q_3)\, \bar{g}_2(\tau), \tag{27}$$

where the independent variable is τ, and t is regarded as a fixed parameter. The above system is considered as a kind of stretched system derived by the "left stretching transformation" $\tau = [(t - t_0)/\epsilon]$, $\tau \in [0, \infty)$. The terminal conditions for these equations are

$$\bar{K}_2(\tau = 0) = 0, \tag{28}$$

$$\bar{K}_3(\tau = 0) = 0, \tag{29}$$

$$\bar{g}_2(\tau = 0) = \xi_2 . \tag{30}$$

For the backward system, we can derive easily the backward boundary layer system in place of Eqs. (25)–(27) by introducing the "right stretching transformation" $\tau' = [(t_f - t)/\epsilon]$, $\tau' \in [0, \infty)$

$$\bar{K}_2(\tau' = 0) = 0, \tag{28'}$$

$$\bar{K}_3(\tau' = 0) = 0, \tag{29'}$$

$$\bar{g}_2(\tau' = 0) = \eta_2 . \tag{30'}$$

Then the terminal conditions for the first correction equations are given as follows for the forward system:

$$\begin{aligned} K_1^1(t_0) = \int_0^\infty & [(A_2 \bar{K}_2'(\tau) + \bar{K}_2(\tau)\, A_2' + \bar{K}_2(\tau)\, Q_2 K_1^0 + K_1^0 Q_2 \bar{K}_2'(\tau)) \\ & - (A_2 K_2^0 + K_2^0 A_2' + K_2^0 Q_2 K_1^0 + K_1^0 Q_2 K_2^{0\prime})]\, d\tau, \end{aligned} \tag{31}$$

$$\begin{aligned} g_1^1(t_0) = \int_0^\infty & [(\bar{K}_2(\tau) - K_2^0)\, Q_2 g_1^0 + (A_2 + K_1^0 Q_2)\, (\bar{g}_2(\tau) - g_2^0) \\ & + \bar{K}_2(\tau)\, Q_3 \bar{g}_2(\tau) - K_2^0 Q_3 g_2^0]\, d\tau, \end{aligned} \tag{32}$$

where $\bar{K}_2$, $\bar{K}_3$, and $\bar{g}_2$ are the solutions of the boundary layer system (25), (26), (27) with the boundary conditions (28), (29), (30). For the backward system, the variable τ in Eqs. (31) and (32) should be replaced by τ', and instead of the initial conditions (28)–(30), we adopt the final conditions (28′)–(30′) for the backward boundary layer system, then obtaining $K_i^1(t_f)$ and $g_i^1(t_f)$.

A similar procedure is applied to solve the recursive equation for the higher order, and it is possible to construct an approximate solution whose accuracy is a desired one. As discussed later, whether the forward or backward system should be selected depends on the property of the boundary layer system.

5. Suboptimum Trajectory

Between the variables K, g in Eq. (6) and $\hat{K}, \hat{g}$ in Eq. (7), there are following relations:

$$\hat{K}(t) = K(t)^{-1} \tag{33}$$

and

$$\hat{g}(t) = -K(t)^{-1} g(t). \tag{34}$$

Expanding $K(t)^{-1}$ into Taylor series in ϵ, we have (see Appendix)

$$\begin{aligned}\hat{K}(t) &= K(t)^{-1}\\ &= \begin{pmatrix} K_1^{0-1} & 0 \\ 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} K_1^{1-1} + K_1^{-2} K_2 K_3^{-1} K_2' & -K_1 K_2 K_3^{-1} \\ -K_3^{-1} K_2' K_1^{-1} & K_3^{-1} \end{pmatrix} + O(\epsilon^2)\\ &= \hat{K}^0 + \epsilon \hat{K}^1 + O(\epsilon^2)\\ &= \begin{pmatrix} \hat{K}_0 & 0 \\ 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \hat{K}_1 & \hat{K}_2 \\ \hat{K}_2' & \hat{K}_3 \end{pmatrix} + O(\epsilon^2),\end{aligned} \tag{35}$$

where the superscript zero is omitted in the coefficient matrix of ϵ for simplicity.

The optimum trajectory is given by Eq. (16). Expanding each variable into Taylor series in ϵ and comparing coefficients of like powers of ϵ thereof, we have for the reduced system

$$\dot{x}_1^0 = (A_1 - E_1 \hat{K}_0)\, x_1^0 + A_2 x_2^0 + (E_1 \hat{K}_0 + E_2 \hat{K}_2')\, g_1^0 + E_2 \hat{K}_3' g_2^0, \tag{36a}$$

$$0 = (A_3 - E_2 \hat{K}_0)\, x_1^0 + (A_4 - E_3 \hat{K}_3)\, x_2^0 + E_2 \hat{K}_0 g_1^0 + E_3 \hat{K}_3' g_2^0, \tag{36b}$$

with

$$x_1^0(t_0) = \xi_1 \qquad \text{or} \qquad x_1^0(t_f) = \eta_1,$$

where

$$E_1 = B_1 R^{-1} B_1', \qquad E_2 = B_1 R^{-1} B_2', \qquad E_3 = B_2 R^{-1} B_2'.$$

For the first correction system

$$\begin{aligned}\dot{x}_1^1 = (A_1 - E_1\hat{K}_0)\, x_1^1 + A_2 x_2^1 + (E_1\hat{K}_0 + E_2\hat{K}_2')\, g_1^1 + E_2\hat{K}_3' g_2^1 \\ - E_1\hat{K}_2 x_2^0 + (E_1\hat{K}_2' + E_1\hat{K}_1)\, g_1^0,\end{aligned} \tag{37a}$$

$$\begin{aligned}0 = (A_3 - E_2\hat{K}_0)\, x_1^1 + (A_4 - E_3\hat{K}_3)\, x_2^1 + E_2\hat{K}_0 g_1^1 + E_3\hat{K}_3' g_2^1 \\ - E_3\hat{K}_2' x_1^0 - E_2'\hat{K}_2 x_2^0 + (E_3\hat{K}_2' + E_2'\hat{K}_1)\, g_1^0 + E_2'\hat{K}_2 g_2^0,\end{aligned} \tag{37b}$$

with terminal conditions for the forward system

$$x_1^1(t_0) = \int_0^\infty [A_2(\bar{x}_2(\tau) - x_2^0) + E_2\hat{K}_2'(\bar{g}_2(\tau) - g_2^0)]\, d\tau, \tag{38}$$

where $\bar{x}_2(\tau)$ in the integrand is a solution of the boundary layer equation

$$\frac{d\bar{x}_2(\tau)}{d\tau} = (A_3 - E_2\hat{K}_0)\, x_1^0 + (A_4 - E_3\hat{K}_3)\, \bar{x}_2(\tau) + E_2'\hat{K}_0 g_1^0 + E_3\hat{K}_2'\bar{g}_2(\tau), \tag{39}$$

with

$$\bar{x}_2(\tau = 0) = \xi_2\,. \tag{40}$$

For the backward system, similar conditions are easily obtained by the same procedure as in the preceding section. It is remarked that we need to solve the forward system for x_i's when we solve the forward system for K_i's and g_i's and vice versa. This situation is different from the case using the conventional Riccati transformation.

After determining K_i's, g_i's, and x_i's, the suboptimal control u_{sub} is given as follows:

$$u_{\text{sub}} = u^0 + \epsilon u^1, \tag{41}$$

where

$$\begin{aligned}u^0 &= -R^{-1}[(B_1'\hat{K}_0 + B_2'\hat{K}_2')\,(x_1^0 - g_1^0) + B_2'\hat{K}_3(x_2^0 - g_2^0)],\\ u^1 &= -R^{-1}[(B_1'\hat{K}_0 + B_2'\hat{K}_2')\,(x_1^1 - g_1^1) + B_2'\hat{K}_3(x_2^1 - g_2^1)\\ &\quad + B_1'\hat{K}_1(x_1^0 - g_1^0) + B_1'\hat{K}_2(x_2^0 - g_2^0)].\end{aligned}$$

6. Simple Example

We shall show the outline of constructing the terminal values of higher order system for

$$\frac{dx_1}{dt} = x_1 - u, \qquad \epsilon\frac{dx_2}{dt} = x_1 - x_2 + u, \tag{42}$$

where x_1, x_2, and u are scalars with $x_i(t_0) = \xi_i$ and $x_i(t_f) = 0$ prescribed. The performance index to be minimized is given as follows:

$$J = \tfrac{1}{2} \int_{t_0}^{t_f} (x'x + ru^2)\, dt. \tag{43}$$

The Riccati transformation $x(t) = K(t)\, p(t) + g(t)$ yields

$$\frac{dk_1}{dt} = k_1{}^2 + 2k_2 + k_2{}^2 - \frac{1}{r}, \tag{44a}$$

$$\epsilon \frac{dk_2}{dt} = k_1 - k_2 + k_3 + k_2 k_3 - \frac{1}{r} + \epsilon k_1 k_2, \tag{44b}$$

$$\epsilon \frac{dk_3}{dt} = -2k_3 + k_3{}^2 - \frac{1}{r} + 2\epsilon k_2 + \epsilon^2 k_2{}^2, \tag{44c}$$

for each element of $K = \begin{pmatrix} k_1 & k_2 \\ k_2 & \epsilon^{-1}k_3 \end{pmatrix}$, where k_1, k_2, and k_3 are scalars.

The boundary layer system of Eq. (44) is

$$\frac{d\bar{k}_2}{d\tau} = k_1{}^0(t_0) - \bar{k}_2 + \bar{k}_3 + \bar{k}_2\bar{k}_3 - \frac{1}{r} \tag{45a}$$

and

$$\frac{d\bar{k}_3}{d\tau} = -2\bar{k}_3 + \bar{k}_3{}^2 - \frac{1}{r}. \tag{45b}$$

Now we have an asymptotically stable point as $\tau \to \infty$ for $\bar{k}_3$,

$$\bar{k}_3(\infty) = 1 - (1 + r^{-1})^{1/2} < 0, \tag{46}$$

by equating the right-hand side of Eq. (45b) to zero, where

$$\bar{k}_3 = 1 + (1 + r^{-1})^{1/2} > 0$$

is excluded because of Lemma 1 in Section 7. For $\bar{k}_2$, we get from Eqs. (45a) and (46)

$$\bar{k}_2(\infty) = -1 - (1 + r^{-1})^{1/2}. \tag{47}$$

Initial value of $k_1{}^1(t_0)$ is given from Eq. (31)

$$k_1{}^1(t_0) = \int_0^\infty [2\bar{k}_2(\tau) + \bar{k}_2^2(\tau) - 2k_2{}^0(t_0) - k_2^{02}(t_0)]\, d\tau. \tag{48}$$

Eliminating $\bar{k}_2(\tau)$ in Eq. (48) by using Eqs. (45a), (45b), (46), and (47), we get

$$\begin{aligned} k_1^1(t_0) &= \left[-\frac{1}{2(1+r^{-1})^{1/2}}\bar{k}_2^2(\tau) - \frac{1+(1+r^{-1})^{1/2}}{1+r^{-1}}\bar{k}_2(\tau)\right]_0^\infty \\ &= \frac{-2+(r^{-1}-2)(1+r^{-1})^{1/2}}{2(1+r^{-1})^{1/2}}, \end{aligned} \tag{49}$$

which can be obtained by simple manipulation under the asymptotic stability of Eqs. (45a) and (45b), considering $\bar{k}_2(0) = 0$. Now we can compute the reduced system for $k_i^0(t)$ and the first correction system for $k_i^1(t)$, and similar procedures can be applied to higher order terms $k_i^j(t)$, $j > 1$ and to the recursive equations for $g_i^j(t)$, $x_i^j(t)$.

7. Basic Theorems

We have derived a representation of the approximate solution in the form of asymptotic expansion. Here the basic theorems are given upon which the validity of the approximate series depends.

Some conditions are needed to establish main theorems and lemmas. We first state the prerequisite conditions, and then the basic theorems are offered.

In the regular optimization problem, the following conditions are usually assumed to hold for $t \in [t_0, t_f]$ and $\epsilon \in [0, \epsilon_0]$:

C1. A_i's and B_i's are holomorphic with respect to t and ϵ.

C2. R and Q are positive definite and holomorphic with respect to t and ϵ.

C3. ξ and η are continuous with respect to ϵ.

We introduce an important condition in an analogous form to the observability referring to Reid [15], as follows:

C4. rank $\| Q_3, A_4Q_3, A_4^2Q_3, \ldots, A_4^{m-1}Q_3 \| = m$.

We add the following two conditions which are essential to the singular perturbation theory:

C5a. A_4 is stable.

C5b. $-A_4$ is stable.

These conditions (C5a and C5b) are exclusive, and main theorems need either one.

LEMMA 1. *If conditions* C1–C5a (C1–C4 *and* C5b) *hold, then the forward (backward) boundary layer equation* (26) *has an isolated and asymptotically stable solution as* $\tau \to \infty$ ($\tau' \to \infty$), *which is negative definite (positive definite) and is given by equating the right-hand side of Eq.* (26) *to zero. Moreover,* $A_4 + \bar{K}_3 Q_3(-A_4 - \bar{K}_3 Q_3)$ *is a stable matrix.*

Lemma 1 plays a decisive role and directly leads to Lemma 2.

LEMMA 2. *If conditions* C1–C5a (C1–C4 *and* C5b) *hold, then the forward (backward) boundary layer equations* (25) *and* (27) *have, respectively, an isolated and asymptotically stable solution, which is given as in Lemma* 1.

Lemmas 1 and 2 lead to the main theorems.

THEOREM 1. *If conditions* C1–C5a (C1–C4 *and* C5b) *hold for the forward system (backward system), then the following convergence relations between the reduced solutions and the full solutions are satisfied:*

$$\lim_{\epsilon\to 0} K_1(t) = K_1^0(t) \qquad \text{for } t \in [t_0, t_f] \quad (t \in [t_0, t_f]),$$

$$\lim_{\epsilon\to 0} K_i(t) = K_i^0(t) \qquad \text{for } t \in (t_0, t_f] \quad (t \in [t_0, t_f)), \quad i = 2, 3,$$

$$\lim_{\epsilon\to 0} g_1(t) = g_1^0(t) \qquad \text{for } t \in [t_0, t_f] \quad (t \in [t_0, t_f]),$$

$$\lim_{\epsilon\to 0} g_2(t) = g_2^0(t) \qquad \text{for } t \in (t_0, t_f] \quad (t \in [t_0, t_f)).$$

Following Vasil'eva [6], we have Theorem 2 which gives error estimations.

THEOREM 2. *If conditions* C1–C5a (C1–C4 *and* C5b) *hold, then there exist bounded functions* $U_i(t, \epsilon)$ *and* $V_i(t, \epsilon)$ *such that*

$$K_1(t) = \sum_{j=0}^{m} K_1^j(t)\,\epsilon^j + U_1(t, \epsilon)\,\epsilon^{m+1}, \qquad \text{for } t \in [t_0, t_f],$$

$$K_i(t) = \sum_{j=0}^{m} K_i^j(t)\,\epsilon^j + U_i(t, \epsilon)\,\epsilon^{m+1}, \qquad \text{for } t \in [t_0 + \delta, t_f] \quad (t \in [t_0, t_f - \delta]), \quad i = 2, 3,$$

$$g_1(t) = \sum_{j=0}^{m} g_1^j(t)\,\epsilon^j + V_1(t, \epsilon)\,\epsilon^{m+1}, \qquad \text{for } t \in [t_0, t_f],$$

$$g_2(t) = \sum_{j=0}^{m} g_2^j(t)\,\epsilon^j + V_2(t, \epsilon)\,\epsilon^{m+1}, \qquad \text{for } t \in [t_0 + \delta, t_f] \quad (t \in [t_0, t_f - \delta]),$$

where

$$\delta = -C\epsilon \log \epsilon, \qquad C \text{ is independent of } \epsilon.$$

Now we have similar theorems for x and u, and these are obtained by simple manipulation directly from Theorems 1 and 2, so that the detail is omitted.

The proofs of these theorems and lemmas are shown directly or by simple modification from relevant theorems in [5, 6, 15, 16]. Lemma 1 is proved from Theorem A in [15]. Note that in the boundary layer system t is regarded as a parameter, and, therefore, the matrices in the system are constant, and all the assumptions of Theorem A in Reid are satisfied. Each Jacobian matrix of the right-hand sides of the boundary layer systems (25)–(27) with respect to $\bar{K}_2$, $\bar{K}_3$, and $\bar{g}_2$, respectively, is $A_4 + \bar{K}_3 Q_3$, so the proof of Lemma 2 is deduced straight from Lemma 1.

Theorem 1 is a modified one of the popular theorem [16] in the two-point boundary value problem of singular perturbation type. In this regard, Tikhonov's convergence theorem [5] in the initial value problem should be referred to. Theorem 2 is an extended one of Theorem 40.1 in Wasow [5].

8. Concluding Remarks

We developed a systematic technique to construct an approximate solution of the ill-conditioned two-point boundary value problem of singular perturbation type. The transformation (6) is closely connected with the sweep method in the variational calculation [1]. It can also be established through invariant imbedding [17, Ch. 15].

We note that the full system expressed by Eqs. (8) and (9) can be solved consistently with the initial conditions (10) and (11) or with the final conditions (10′) and (11′). The recursive system, however, can be solved not forward but backward and vice versa, according to the properties of the boundary layer systems.

Without loss of generality, we can fix then end-value η to be zero, and then we have the advantage that we need not solve the equations for g_i's of the backward system because these have only trivial solutions.

Recently, O'Malley [12], etc., developed another technique to treat the singularly perturbed linear regulator problem from the different point of view via Turrittin's works and Harris's (see references of O'Malley [12] for example) on two-point boundary value problem of singular perturbation type. Their method may be applied to the fixed-terminal optimization problem and allows a more general property of boundary layer system, that its matrix of canonical system

$$G = \begin{pmatrix} A_4 & -B_2 R^{-1} B_2' \\ -Q_3 & -A_4 \end{pmatrix}$$

should have m-eigenvalues with negative real part and m-eigenvalues with positive real part, which includes obviously our conditions C5's as a special case. However, the generated recursive system must be solved under two-point boundary value conditions, and since the boundary layer occurs at both ends of the interval considered, it must treat both of them in order to make "boundary layer corrections".

The asymptotic solution we derived corresponds to the "outer expansion," and the "boundary layer corrections" in the sense of O'Malley [12], which enable the convergence to be uniform in $t \in [t_0, t_f]$, can be made in our case in the same way as given in Wasow [5] or Vasil'eva [6] by solving the recursive "stretched (inner) system".

Note that the stretched system of degenerated order m can be solved by regular perturbation theory, and the singularity of the original system is removed away. We add that on the property of the asymptotic expansion of the performance index, the reader is referred to Sannuti and Kokotović [10] and O'Malley [12].

Appendix

The well-known inversion formula for a partitioned matrix leads to

$$\begin{bmatrix} K_1 & K_2 \\ K_2' & \epsilon^{-1}K_3 \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{A.1}$$

where

$$\begin{aligned}
A &= [K_1 - \epsilon K_2 K_3^{-1} K_2']^{-1} = K_1^{-1} + \epsilon K_1^{-2} K_2 K_3^{-1} K_2' + O(\epsilon^2), \\
B &= -[K_1 - \epsilon K_2 K_3^{-1} K_2']^{-1} [\epsilon K_2 K_3^{-1}] = -\epsilon K_1^{-1} K_2 K_3^{-1} + O(\epsilon^2), \\
C &= -\epsilon K_3^{-1} K_2' [K_1 - \epsilon K_2 K_3^{-1} K_2']^{-1} = -\epsilon K_3^{-1} K_2' K_1^{-1} + O(\epsilon^2), \\
D &= [\epsilon^{-1} K_3 - K_2' K_1^{-1} K_2]^{-1} = \epsilon K_3^{-1} + O(\epsilon^2).
\end{aligned}$$

Then we have

$$K^{-1} = \begin{bmatrix} K_1^{-1} + \epsilon K_1^{-2} K_3^{-1} K_2' & -\epsilon K_1^{-1} K_2 K_3^{-1} \\ -\epsilon K_3^{-1} K_2' K_1^{-1} & \epsilon K_3^{-1} \end{bmatrix} + O(\epsilon^2). \tag{A.2}$$

Substituting Eq. (21) into Eq. (A.2), we have Eq. (35).

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