Duality for Minmax $B$-vex Programming
Involving $n$-Set Functions

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A minmax programming problem involving several $B$-vex $n$-set functions is considered. Necessary and sufficient optimality theorems and Wolfe type duality results are established. Duality for the $n$-set generalized (minmax) fractional programming problem is derived as a special case of the main problem.

1. INTRODUCTION

Bector [2] introduced the concept of a strong pseudoconvex function and used it to establish the nature of quotients, products, rational powers, and compositions of convex-like functions [3]. Later, Bector and Singh in [9] introduced $B$-vex functions as a generalization of strong pseudo convex functions and discussed their various properties. Further properties of such functions were discussed in [35]. Hanson [29] introduced the concept of invex functions and showed that an appropriately defined optimization problem containing invex functions satisfies the Karush–Kuhn–Tucker optimality conditions.

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In the recent past some duality results have been obtained for minmax programming problems involving several ratios, called minmax fractional or generalized fractional programming problems [5–7, 15, 17, 23–25, 30, 37, 38, 40]. Of particular interest are those by Crouzeix [23], Crouzeix et al. [24, 25], Jagannathan and Schaible [30], Chandra et al. [19], and Bector and Suneja [6]. Crouzeix [23] and Crouzeix et al. [24, 25] used the quasiconvex duality theory developed by Crouzeix [23] while Jagannathan and Schaible [30] obtained the duality results via Farkas' lemma. For such problems, Chandra et al. [19] studied duality through the ratio game approach, and Bector and Suneja [6] presented Lagrange duality. Other duality results for this class of problems have been obtained by Bector et al. [7], and Xu [37, 38]. Minmax optimization problems occur frequently in many important areas like game theory, Chebychev approximation, economics, and financial planning. Some of the basic results of such problems are found in books such as those by Danskin [26] and Demyanov and Molozemov [27]. A variety of applications of minmax fractional programming problems are given in [1, 24, 28].

Recently, Bector et al. [10, 11] unified the concept of B-vex functions and invex functions, naming such functions as B-invex functions. Independently, Jeyakumar and Mond [31] introduced the idea of V-invex functions which are similar to B-invex functions [10, 11]. Both B-invex functions [10, 11] and V-invex [31] functions unify the duality of vector valued fractional programs [12–14, 17, 20, 31]. A useful consequence of B-vexity is that pseudolinear multiobjective and minmax programming problems and certain nonlinear multiobjective fractional and minmax (generalized) fractional programming problems do not require a separate treatment for duality, and all results on optimally conditions and duality for them can be derived by using the general concept of B-vexity.

General theory for optimizing n-set functions was first developed by Morris [34] who, for fractions of a single set, obtained results that are similar to the standard mathematical programming problem. Corley [22] developed an optimization theory for programming problems with n-set functions, established optimality conditions, and obtained Lagrangian duality. Zalmai [39] considered several practical applications for a class of nonlinear programming problems involving a single objective and differentiable n-set functions, and established several sufficient and duality results under generalized p-convexity conditions.

Bector et al. [16] established sufficient optimally conditions and proved duality results for multiobjective programming problems with differentiable n-set functions. In [17] Bector et al. considered a class of multiobjective fractional programming problems in which the objectives are ratios of appropriately restricted differentiable n-set functions, introduced Wolfe's dual [36] along the lines of Bector [4], and established duality results in
terms of properly efficient solutions. A relationship with a certain vector-valued saddle point of a Lagrangian was also established.

In the present paper we consider a minmax programming problem involving several $B$-vex $n$-set functions and for it establish necessary and sufficient optimality theorems and Wolfe type duality [36, 4] results. Furthermore, we consider the $n$-set generalized (minmax) fractional programming problem and derive the Wolfe type duality results for it as a special case of the main problem.

2. NOTATION, DEFINITIONS, AND PRELIMINARIES

Throughout the paper we assume that $(X, A, \mu)$ is a finite atomless measure space with $L_1(X, A, \mu)$ separable. We also assume that $S$ is a subset of $A^n = A \times A \times \cdots \times A$, the $n$-fold product of the $\sigma$-algebra $A$ of subsets of a given set $X$. Let $d$ be the pseudometric on $A^n$ defined by

$$d((R_1, R_2, \ldots, R_n), (S_1, S_2, \ldots, S_n))$$

$$= \left[ \sum_{i=1}^{n} \mu^2(R_i \Delta S_i) \right]^{1/2}, \quad R_i, S_i \in A \quad \forall i = 1, 2, \ldots, n,$$

where $R_i \Delta S_i$ denotes the symmetric difference for $R_i$ and $S_i$. Thus $(A^n, d)$ is a pseudosymmetric space which will serve as the domain for most of the functions used in the present paper. Thus $h \in L_1(X, A, \mu)$ and $Z \in A$ with indicator (characteristic) function $I_Z \in L_\infty(X, A, \mu)$; the general integral $\int Z h d \mu$ will be denoted by $\langle h, I_Z \rangle$.

We now give the following definitions along the lines of Zalmai [39].

**Definition 2.1.** A set function $H : A \to R^2$ is said to be differentiable at $S^* \in A$ if there exists $DH_{S^*} \in L_1(X, A, \mu)$, called the derivative of $H$ at $S^*$, such that

$$H(S) = H(S^*) + \langle DH_{S^*}, I_{S^*} - I_S \rangle + V_H(S^*, S),$$

where $V_H(S^*, S)$ is $o(d(S^*, S))$, i.e., $\lim_{d(S^*, S) \to 0} V_H(S^*, S)/d(S^*, S) = 0$.

We now define the differentiation for an $n$-set function.

**Definition 2.2.** Let $F : A^n \to R^1$ and $(S^*_1, S^*_2, \ldots, S^*_n) \in A^n$. Then $F$ is said to have a partial derivative at $(S^*_1, S^*_2, \ldots, S^*_n)$ with respect to its $i$th argument $S_i$ if the set function

$$H(S_i) = F(S^*_1, S^*_2, \ldots, S^*_{i-1}, S_i, S^*_{i+1}, \ldots, S^*_n)$$
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has derivative $DF_{S^*_i}$ at $S^*_i$. In that case we define the $i$th partial derivative of $F$ at $(S^*_1, S^*_2, \ldots, S^*_n)$ to be $D_iF_{S^*_1, \ldots, S^*_n} = DF_{S^*_i}$, $i = 1, 2, \ldots, n$.

**Definition 2.3.** Let $F: A^n \to R^3$ and $(S^*_1, S^*_2, \ldots, S^*_n) \in A^n$. Then $F$ is said to be differentiable at $(S^*_1, S^*_2, \ldots, S^*_n)$ if all the partial derivatives $DF_{S^*_i}$, $i = 1, 2, \ldots, n$, exist and satisfy

$$F(S_1, S_2, \ldots, S_n) = F(S^*_1, S^*_2, \ldots, S^*_n) + \sum_{i=1}^{n} \left( D_iF_{S^*_1, \ldots, S^*_n} I_{S_i} - I_{S^*_i} \right)$$

$$+ W_F[(S^*_1, S^*_2, \ldots, S^*_n), (S_1, S_2, \ldots, S_n)],$$

where $W_F[(S^*_1, S^*_2, \ldots, S^*_n), (S_1, S_2, \ldots, S_n)]$ is

$$o\{d[(S^*_1, S^*_2, \ldots, S^*_n), (S_1, S_2, \ldots, S_n)]\}
$$

for all $(S_1, S_2, \ldots, S_n) \in A$.

**Definition 2.4.** Let $F: A^n \to R^1$ be differentiable. Then $F$ is said to be convex (strictly convex) if for

$$(R_1, R_2, \ldots, R_n), (S_1, S_2, \ldots, S_n) \in A^n
$$

$$[F(R_1, \ldots, R_n) - F(S_1, \ldots, S_n)] \geq (>) \sum_{i=1}^{n} \left( D_iF_{S_1, S_2, \ldots, S_n} I_{R_i} - I_{S_i} \right).$$

**Definition 2.5.** Differentiable functions $F_i: A^n \to R^1$, $i = 1, 2, \ldots, p$ are said to be additively convex (additively strictly convex) on $A^n$ if for $R = (R_1, \ldots, R_n)$, and $S = (S_1, \ldots, S_n) \in A^n$,

$$\sum_{i=1}^{p} \left[ F_i(R_1, \ldots, R_n) - F_i(S_1, \ldots, S_n) \right]$$

$$\geq (>) \sum_{i=1}^{p} \sum_{j=1}^{n} \left( D_jF_{iS_1, S_2, \ldots, S_n} I_{R_j} - I_{S_j} \right).$$

Next we introduce the following definition of $n$-set $B$-vex (strictly $B$-vex) functions.

**Definition 2.6.** A differentiable function $F: A^n \to R^1$ is said to be properly $B_1$-vex (properly strictly $B_1$-vex) on $A^n$ if there exists $B_i: A^n \times
$A^n \to R^1_+ \setminus \{0\}$, such that for $R = (R_1, \ldots, R_n)$ and $S = (S_1, \ldots, S_n) \in A^n$, 

$$B_i(R, S)[F(R_1, \ldots, R_n) - F(S_1, \ldots, S_n)]$$

$$\geq (>) \sum_{i=1}^{n} \left\langle D_iF_{S_1, S_2, \ldots, S_n}, I_{R_i} - I_{S_i} \right\rangle .$$

**Definition 2.7.** Differentiable functions $F_i: A^n \to R^1$, $i = 1, 2, \ldots, p$ are said to be additively properly $B_i$-vex (additively properly strictly $B_i$-vex) on $A^n$ if there exist $B_i: A^n \times A^n \to R^1_+ \setminus \{0\}$, $i = 1, 2, \ldots, p$ such that for $R = (R_1, \ldots, R_n)$, and $S = (S_1, \ldots, S_n) \in A^n$, 

$$\sum_{i=1}^{p} B_i(R, S)[F_i(R_1, \ldots, R_n) - F_i(S_1, \ldots, S_n)]$$

$$\geq (>) \sum_{i=1}^{p} \sum_{i=1}^{n} \left\langle D_iF_{S_1, S_2, \ldots, S_n}, I_{R_i} - I_{S_i} \right\rangle .$$

We say that,

(i) $F$ is concave, or properly $B_i$-cave (properly strictly $B_i$-cave) on $A^n$ if and only if $-F$ is convex, or properly $B_i$-vex (properly strictly $B_i$-vex), respectively, on $A^n$.

(ii) $F_i$, $i = 1, 2, \ldots, p$ are additively concave (additively strictly concave), or additively properly $B_i$-cave (additively properly strictly $B_i$-cave) on $A^n$ if and only if $-F_i$, $i = 1, 2, \ldots, p$, are additively convex (additively strictly convex), properly $B_i$-vex (additively properly strictly $B_i$-vex), respectively, on $A^n$.

**Remarks 2.1.** (i) If we set $B_i(R, S) = 1$, Definition 2.6 reduces to Definition 2.4, the definition of a convex (strictly convex) function, and Definition 2.7 reduces to Definition 2.5, the definition of additively convex (additively strictly convex) functions.

(ii) In the above definitions of strictly convex, additively strictly convex, properly strictly $B_i$-vex, and additively properly strictly $B_i$-vex functions, we take $R \neq S$, $R, S \in A^n$.

In the sequel we shall use

Minimize $F(S_1, S_2, \ldots, S_n)$

subject to

$H_j(S_1, S_2, \ldots, S_n) \leq 0, \quad j = 1, 2, \ldots, m$ (NP)

$(S_1, S_2, \ldots, S_n) \in A^n \in A^n$. 
Definition 2.8. A point \((S_1^*, S_2^*, \ldots, S_n^*) \in A^n\) is said to be a regular feasible solution for (NP) if there exists \((\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_n) \in A^n\) such that

\[
H_j(S_1^*, S_2^*, \ldots, S_n^*) + \sum_{i=1}^{n} \left( D_i H_{jS_i^*, i} S_1^*, \ldots, S_n^* \right) I_{s_i} - I_{s_i^*}) < 0,
\]

\(j = 1, 2, \ldots, m.\)

Theorem 2.1 [22, 34]. Let \((S_1^*, S_2^*, \ldots, S_n^*)\) be a regular optimal solution of (NP). Then there exists \(u^* = (u_1^*, u_2^*, \ldots, u_m^*) \in R^m_+\) (nonnegative orthant of \(R^m\)) such that

\[
\left\{ D_i F_{S_i^*, i} S_1^*, \ldots, S_n^* + \sum_{j=1}^{m} u_j^* D_i H_{jS_i^*, i} S_1^*, \ldots, S_n^* I_{s_i} - I_{s_i^*} \right\} \geq 0, \quad \forall S_i \in A, i = 1, 2, \ldots, n
\]

\[u_j^* H_j(S_1^*, S_2^*, \ldots, S_n^*) = 0 \quad (k = 1, 2, \ldots, m)\]

\[H_j(S_1^*, S_2^*, \ldots, S_n^*) \leq 0 \quad (j = 1, 2, \ldots, m)\]

\[u^* = (u_1^*, u_2^*, \ldots, u_m^*) \geq 0.\]

3. Main Problem and Optimality Conditions

We now consider the following generalized minmax programming problem \((P)\) involving differentiable \(n\)-set functions:

\[q^* = \min_{(S_1, S_2, \ldots, S_n) \in A^n} \max_{1 \leq i \leq p} [Q_i(S_1, S_2, \ldots, S_n)] \quad \text{(P)}\]

subject to

\[Q_{ij}(S_1, S_2, \ldots, S_n) \leq 0, \quad i = 1, 2, \ldots, p \text{ and } j = 1, 2, \ldots, m \quad (3.1)\]

\[S = (S_1, S_2, \ldots, S_n) \in A^n, \quad (3.2)\]

where

(A-1) \(A^n\) is the \(n\)-fold product of a \(\sigma\)-algebra \(A\) of subsets of a given set \(X,\)

(A-2) Each \(Q_i\) for \(i = 1, 2, \ldots, p,\) and each \(Q_{ij}\) for \(i = 1, 2, \ldots, p\) and \(j = 1, 2, \ldots, m,\) is a real valued differentiable properly \(B_1\)-vex function defined on \(A^n.\)
We now consider the following programming problem (EP) which is equivalent to (P) in the sense of Lemmas 3.1 and 3.2 given below,

\[
\begin{align*}
\text{minimize} & \quad q \\
\text{subject to} & \quad Q_i(S_1, S_2, \ldots, S_n) \leq q, \quad i = 1, 2, \ldots, p \tag{3.3} \\
& \quad Q_{ij}(S_1, S_2, \ldots, S_n) \leq 0 \quad (i = 1, 2, \ldots, p; \; j = 1, 2, \ldots, m) \tag{3.4} \\
& \quad (S_1, S_2, \ldots, S_n) \in A^n. \tag{3.5}
\end{align*}
\]

**Lemma 3.1.** Let \((S_1, S_2, \ldots, S_n) \in A^n\) be (P)-feasible. Then there exists a \(q\) such that \((S_1, S_2, \ldots, S_n, q) \in A^{n+1}\) is (EP)-feasible, and if \((S_1, S_2, \ldots, S_n, q) \in A^{n+1}\) is (EP)-feasible then \((S_1, S_2, \ldots, S_n, q) \in A^n\) is (P)-feasible.

**Lemma 3.2.** Let \((S_1^*, S_2^*, \ldots, S_n^*) \in A^n\) be (P)-optimal. Then there exists a \(q\) such that \((S_1^*, S_2^*, \ldots, S_n^*, q) \in A^{n+1}\) is (EP)-optimal, and if \((S_1^*, S_2^*, \ldots, S_n^*, q) \in A^{n+1}\) is (EP)-optimal then \((S_1^*, S_2^*, \ldots, S_n^*) \in A^n\) is (P)-optimal.

If (EP) is a convex programming problem, we can easily derive optimality conditions and Wolfe type [22, 39] duality. However, as in (A-2), if we take \(Q\) to be a properly \(B\)-vex function on \(A^n\), then \(Q(x) - q\) in (3.3) is not a \(B\)-vex function on \(A^{n+1}\). This phenomenon necessitates a separate study of minmax fractional programming problems. In the present paper it is seen that the notion of proper \(B\)-vexity and additive \(B\)-vexity facilitates the study of minmax fractional programming with \(n\)-set functions in a unified manner and provides a Wolfe type dual [36].

**Lemma 3.3.** Let \(Q_i\) and each \(Q_{ij}, j = 1, 2, \ldots, m\) be properly \(B\)-vex on \(A^n\).

(i) If \(\lambda_i \geq 0\), and \(y_{ij} \geq 0, j = 1, 2, \ldots, m\), then the function \(\lambda_i Q_i + \sum_{j=1}^m y_{ij} Q_{ij}\) is properly \(B_i\)-vex on \(A^n\).

(ii) Additionally, if \(Q_i\) for which \(\lambda_i > 0\) is properly strictly \(B_i\)-vex on \(A^n\), and/or at least one of \(Q_{ij}, j = 1, 2, \ldots, m\), for which the corresponding \(y_{ij} > 0\) is properly strictly \(B_i\)-vex on \(A^n\), then \(\lambda_i Q_i + \sum_{j=1}^m y_{ij} Q_{ij}\) is properly strictly \(B_i\)-vex on \(A^n\).

**Theorem 3.1 (Necessary Condition).** Let \((S_1^*, S_2^*, \ldots, S_n^*)\) be a regular (P)-optimal solution. Then there exist \(q^* \in R^1, \lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)\), and
\[ y^* = (y^*_{1j}, y^*_{2j}, \ldots, y^*_{nj}), \text{ } j = 1, 2, \ldots, m, \text{ } \text{such that} \]

\[
\begin{aligned}
&\left( \sum_{i=1}^{p} \lambda^*_i D_i Q_{i(S^*_1, S^*_2, \ldots, S^*_n)} + \sum_{i=1}^{p} \sum_{j=1}^{m} y^*_{ij} D_j Q_{ij(S^*_1, S^*_2, \ldots, S^*_n), I_{S_i}, I_{S_j}} \right) \\
&\geq 0 \quad \forall S_i \in A, \text{ } r = 1, 2, \ldots, n \tag{3.5}
\end{aligned}
\]

\[
\lambda^*_i [Q_i(S^*_1, S^*_2, \ldots, S^*_n) - q^*] = 0 \quad (i = 1, 2, \ldots, p) \tag{3.6}
\]

\[
y^*_{ij} Q_{ij}(S^*_1, S^*_2, \ldots, S^*_n) = 0 \quad (i = 1, 2, \ldots, p; \text{ } j = 1, 2, \ldots, m) \tag{3.7}
\]

\[
Q_i(S^*_1, S^*_2, \ldots, S^*_n) \leq q^* \quad (i = 1, 2, \ldots, p) \tag{3.8}
\]

\[
Q_{ij}(S^*_1, S^*_2, \ldots, S^*_n) \leq 0 \quad (i = 1, 2, \ldots, p; \text{ } j = 1, 2, \ldots, m) \tag{3.9}
\]

\[
\sum_{i=1}^{n} \lambda^*_i = 1 \tag{3.10}
\]

\[
(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n) \geq 0, \quad y^*_ij \geq 0 \quad (i = 1, 2, \ldots, p; \text{ } j = 1, 2, \ldots, m), \tag{3.11}
\]

**Proof.** Since \((S^*_1, S^*_2, \ldots, S^*_n)\) is a regular \((P)\)-optimal solution, therefore, by Lemmas 3.1 and 3.2, \((S^*_1, S^*_2, \ldots, S^*_n)\) is a regular \((EP)\)-optimal solution. Using Theorem 2.1 for \((EP)\), there exist \((\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n) \geq 0, \quad y^*_ij \geq 0 \quad (i = 1, 2, \ldots, p; \text{ } j = 1, 2, \ldots, m), \) that satisfy (3.5)–(3.11).

**Theorem 3.2 (Sufficient Condition).** Assume that there exist \((\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n) \geq 0, \quad y^*_ij \geq 0 \quad (i = 1, 2, \ldots, p; \text{ } j = 1, 2, \ldots, m), \) and \((S^*_1, S^*_2, \ldots, S^*_n, q^*) \in A^{n+1} \) that satisfy (3.5)–(3.11). Then \((S^*_1, S^*_2, \ldots, S^*_n)\) is \((P)\)-optimal.

**Proof.** Using Assumption (A-2) and Definition 2.6, for all \((EP)\)-feasible \((S_1, S_2, \ldots, S_n, q)\) we have

\[
B_i(S, S^*)[\lambda^*_i Q_i(S_1, S_2, \ldots, S_n) - \lambda^* Q_i(S^*_1, S^*_2, \ldots, S^*_n)] \\
\geq \sum_{r=1}^{n} \left( \lambda^*_i D_i Q_{i(S_1, S_2, \ldots, S_n), I_{S_i}, I_{S_r}} - I_{S_r} \right) \quad \text{for } i = 1, 2, \ldots, p \tag{3.12}
\]
and

\[
B_i(S, S^*) \left[ \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S_1, \ldots, S_n) - \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S^*_1, \ldots, S^*_n) \right]
\geq \sum_{r=1}^{n} \left( \sum_{j=1}^{m} y^R_{ij} D_r Q_{ij}(s^*_1, s^*_2, \ldots, s^*_n), I_{S_r} - I_{S^*_r} \right) \quad \text{for } i = 1, 2, \ldots, p.
\]

Adding (3.12) and (3.13) we have

\[
B_i(S, S^*) \left[ \lambda^R_i Q_i(S_1, \ldots, S_n) + \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S_2, \ldots, S_n) \right] - \left( \lambda^R_i Q_i(S^*_1, \ldots, S^*_n) + \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S^*_2, \ldots, S^*_n) \right)
\geq \sum_{r=1}^{n} \left( D_r \left( \lambda^R_i Q_i + \sum_{j=1}^{m} y^R_{ij} Q_{ij} \right), s^*_1, s^*_2, \ldots, s^*_n, I_{S_r} - I_{S^*_r} \right) \quad \text{for } i = 1, 2, \ldots, p.
\]

Summing both sides of (3.14) over \( i = 1, 2, \ldots, p \), we obtain

\[
\sum_{i=1}^{p} B_i(S, S^*) \left[ \lambda^R_i Q_i(S_1, \ldots, S_n) + \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S_2, \ldots, S_n) \right] - \left( \lambda^R_i Q_i(S^*_1, \ldots, S^*_n) + \sum_{j=1}^{m} y^R_{ij} Q_{ij}(S^*_2, \ldots, S^*_n) \right)
\geq \sum_{r=1}^{n} \left( \sum_{i=1}^{p} \lambda^R_i D_r Q_{is^*_1, s^*_2, \ldots, s^*_n} + \sum_{i=1}^{p} \sum_{j=1}^{m} y^R_{ij} D_r Q_{is^*_1, s^*_2, \ldots, s^*_n}, I_{S_r} - I_{S^*_r} \right)
\]

Summing both sides of (3.5) over \( r = 1, 2, \ldots, n \) we obtain

\[
\sum_{r=1}^{n} \left( \sum_{i=1}^{p} \lambda^R_i D_r Q_{is^*_1, s^*_2, \ldots, s^*_n} + \sum_{i=1}^{p} \sum_{j=1}^{m} y^R_{ij} D_r Q_{is^*_1, s^*_2, \ldots, s^*_n}, I_{S_r} - I_{S^*_r} \right) \geq 0
\]
and (3.15) and (3.16) yield

\[
\sum_{i=1}^{p} B_i(S,S^*) \left[ \lambda_i^e Q_i(S_{1}, \ldots, S_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S_{1}, \ldots, S_n) 
\right.
\]

\[
- \left( \lambda_i^e Q_i(S_{1}^*, \ldots, S_n^*) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S_{1}^*, \ldots, S_n^*) \right) \right] \geq 0. \quad (3.17)
\]

Using (3.3), (3.4), (3.11), (3.6), and (3.7), we obtain from (3.17),

\[
\sum_{i=1}^{p} B_i(S,S^*) \left[ \lambda_i^e (q - q^*) \right] \geq 0
\]

or

\[
(q - q^*) \left[ \sum_{i=1}^{p} B_i(S,S^*) \lambda_i^e \right] \geq 0. \quad (3.18)
\]

\(B_i(S,S^*) > 0\) for \(i = 1, 2, \ldots, p\), along with (3.10) and (3.11) yields \(\sum_{i=1}^{p} B_i(S,S^*) \lambda_i^e > 0\), which in view of (3.18) gives \(q \geq q^*\) for all \((EP)\)-feasible solutions \((S_1, S_2, \ldots, S_n, q)\). This implies that

\[
(S_{1}^*, S_{2}^*, \ldots, S_n^*, q^*) \in A^{n+1}
\]

is \((EP)\)-optimal. Hence, by Lemma 3.2, \((S_{1}^*, S_{2}^*, \ldots, S_n^*)\) is \((P)\)-optimal.

**Remarks 3.1.**

(i) From (3.14) we observe that Theorem 3.2 can still be proved if Assumption (A-2) in \((P)\) is replaced by the following. Each \(\lambda_i Q_i(S_{1}, \ldots, S_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S_{1}, \ldots, S_n), \lambda_i \geq 0, y_{ij} \geq 0, i = 1, 2, \ldots, p\), is a differentiable properly \(B\)-vex function on \(A^n\).

(ii) From (3.12) and (3.13) we obtain that Theorem 3.2 can be proved if Assumption (A-2) in \((P)\) is replaced by the following. \(Q_i, i = 1, 2, \ldots, p\) are additively properly \(B\)-vex functions and \(Q_{ij}, i = 1, 2, \ldots, p\) are additively properly \(B\)-vex functions, on \(A^n\).

(iii) Inequality (3.15) yields that Theorem 3.2 can be proved if Assumption (A-2) in \((P)\) is replaced by the following. Functions \(\lambda_i Q_i(S_{1}, \ldots, S_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S_{1}, \ldots, S_n), \lambda_i \geq 0, y_{ij} \geq 0, i = 1, 2, \ldots, p\) are additively properly \(B\)-vex functions on \(A^n\).
4. DUAL PROBLEM AND DUALITY THEOREMS

Hence after we shall use, for notational convenience, the following notations for (EP) and (ED),

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \geq 0, \quad T = (T_1, T_2, \ldots, T_n) \in A^n, \]

\[ S = (S_1, S_2, \ldots, S_n) \in A^n \]

\[ Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p1} & y_{p2} & \cdots & y_{pm} \end{bmatrix} \in \mathbb{R}^{p \times m} \]

is the matrix of Lagrange multipliers for the constraints of (EP),

\[ y^i = (y_{i1}, y_{i2}, \ldots, y_{im}), \quad i = 1, 2, \ldots, m, \]

\[ Q_i(S) = Q_i(S_1, S_2, \ldots, S_n), \quad i = 1, 2, \ldots, p \]

\[ Q_{ij}(S) = Q_{ij}(S_1, S_2, \ldots, S_n), \quad i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, m \]

\[ L_i(T, \lambda_i, y^i) = \lambda_i Q_i(T_1, \ldots, T_n) + \sum_{j=1}^{m} y_{ij} Q_{ij}(T_1, \ldots, T_n). \]

In view of the above notation, (EP) becomes

\[
\begin{align*}
\min & \quad q \\
\text{subject to} & \quad F_i(S) \leq q \quad (i = 1, 2, \ldots, p) \quad (4.1) \\
& \quad Q_{ij}(S) \leq 0 \quad (i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, m) \quad (4.2) \\
& \quad S \in A^n.
\end{align*}
\]

Theorems 3.1 and 3.2 provide us the motivation for introducing the following maximization problem as the dual problem (ED) for the minimization problem (EP),

\[
\begin{align*}
\max & \quad v \\
\text{subject to} & \quad \left( \sum_{t=1}^{p} D_t(L_i(T, \lambda_i, y^i))T_1, \ldots, T_n, I_s - I_{t_s} \right) \geq 0 \quad \text{for all } S \in A^n \quad (4.3) \\
& \quad L_i(T, \lambda_i, y^i) \geq \lambda_i v \quad (i = 1, 2, \ldots, p) \quad (4.4) \\
& \quad \sum_{i=1}^{n} \lambda_i = 1 \quad (4.5) \\
& \quad \lambda \geq 0, \quad Y \geq 0, \quad \lambda \in \mathbb{R}^p, \quad Y \in \mathbb{R}^{p \times m}, \quad v \in \mathbb{R}^{1 \times m}, \quad T \in A^n. \quad (4.6)
\end{align*}
\]
**Theorem 4.1 (Weak Duality).** Let $S \in A^n$ be $(P)$-feasible and $(\lambda, \nu, T, Y)$ be $(ED)$-feasible.

(H-1) For $i = 1, 2, \ldots, p$, let $Q_i$ and $Q_{ij}$ be additively properly $B_i$-vex functions on all feasible solutions of $(P)$ and $(ED)$.

Then, $\nu \leq q$.

**Proof.** Since $(\lambda, \nu, T, Y)$ is $(ED)$-feasible, therefore, it satisfies (4.7)–(4.10). As in Theorem 3.2, using Assumption (A-2), Definition 2.6, and (4.3), we have for all $(EP)$-feasible solutions $(S, q)$ and $(ED)$-feasible solutions $(\lambda, \nu, T, Y)$,

$$\sum_{i=1}^{p} B_i(S, T) \left[ L_i(S, \lambda_i, y^i) - L_i(T, \lambda_i, y^i) \right] \geq 0. \quad (4.7)$$

Using (3.3), (3.4), (4.6), and (4.4) in (4.7) we obtain

$$\sum_{i=1}^{p} B_i(S, T) \left[ \lambda_i(q - \nu) \right] \geq 0, \quad \text{or} \quad (q - \nu) \left[ \sum_{i=1}^{p} B_i(S, T) \lambda_i \right] \geq 0. \quad (4.8)$$

Now $B_i(S, T) > 0$ for $i = 1, 2, \ldots, p$. This along with (4.5) and (4.6) yields $\sum_{i=1}^{p} B_i(S, T) \lambda_i > 0$, which in conjunction with (4.8) gives $q \geq \nu$ for all $(EP)$-feasible solutions $(S, q)$ and all $(ED)$-feasible solutions $(\lambda, \nu, T, Y)$. This proves the theorem.

**Remarks 4.1.** As in Remarks 3.1 (i)–(iii), we observe that Theorem 4.1 holds if we replace (H-1) by any one of the following hypotheses.

(i) Each $L_i(T, \lambda_i, y_i)$, $\lambda_i \geq 0$, $y_i \geq 0$, $i = 1, 2, \ldots, p$, is a differentiable properly $B_i$-vex function on all feasible solutions of $(P)$ and $(ED)$.

(ii) $Q_i$, $i = 1, 2, \ldots, p$ are additively properly $B_i$-vex functions and $Q_{ij}$, $i = 1, 2, \ldots, p$ are additively properly $B_i$-vex functions, on all feasible solutions of $(P)$ and $(ED)$.

(iii) Functions $L_i(T, \lambda_i, y^i)$, $\lambda_i \geq 0$, $y^i \geq 0$, $i = 1, 2, \ldots, p$, are additively properly $B_i$-vex functions on all feasible solutions of $(P)$ and $(ED)$.

**Corollary 4.1.** Let $(S^*, q^*)$ be $(EP)$-feasible and let $(\lambda^*, \nu^*, T^*, Y^*)$ be $(ED)$-feasible with $q^* = \nu^*$. Then $S^*$ is $(P)$-optimal, and $(\lambda^*, \nu^*, T^*, Y^*)$ is $(ED)$-optimal.

**Theorem 4.2 (Strong Duality).** Suppose $(S^*, q^*) \in A^{n+1}$ is $(EP)$-optimal. Then there exist $\lambda^* \in R^p$, $Y^* \in R^{p \times m}$, $\lambda^* \geq 0$, $Y^* \geq 0$ such that $(\lambda^*, q^*, S^*, Y^*)$ is $(ED)$-optimal, and the objective value of $(EP)$ at $(S^*, q^*)$ is equal to the objective value of $(ED)$ at $(\lambda^*, q^*, S^*, Y^*)$. 

Proof. Since \((S^*, q^*) \in \mathbb{A}^{n+1}\) is (EP)-optimal, therefore, there exist \(\lambda^* \in \mathbb{R}^p, Y^* \in \mathbb{R}^{p \times m}, \lambda^* \geq 0, Y^* \geq 0\), such that (3.5)–(3.11) hold. Condition (3.5) yields that \((\lambda^*, q^*, S^*, Y^*)\) satisfies (4.3); (3.6) and (3.7) imply that \((\lambda^*, q^*, S^*, Y^*)\) satisfies (4.4); whereas (3.10) and (3.11) imply that \((\lambda^*, q^*, S^*, Y^*)\) satisfies (4.5) and (4.6), respectively. Hence, \((\lambda^*, q^*, S^*, Y^*)\) is (ED)-feasible. Also, we see that the value of the (ED)-objective is \(q^*\), which is the same as the (EP)-objective. Hence, using Corollary 4.1, \((\lambda^*, q^*, S^*, Y^*)\) is (ED)-optimal.

**Theorem 4.3 (Strict Converse Theorem).** Suppose \((S^*, q^*) \in \mathbb{A}^{n+1}\) is an optimal solution of (ED).

\((\mathbb{H}-2)\) If on all feasible solutions of \((\mathbb{P})\) and \((\mathbb{ED})\), \(Q_i, i = 1, 2, \ldots, p\) are properly \(B_i\)-vex functions and \(Q_{ij}, i = 1, 2, \ldots, p, j = 1, 2, \ldots, m\), are properly \(B_i\)-vex functions, and additionally, at least one of \(Q_i, i = 1, 2, \ldots, p\), is properly strictly \(B_i\)-vex, and/or at least one of \(Q_{ij},\) with the corresponding \(y_{ij} > 0, i = 1, 2, \ldots, p, j = 1, 2, \ldots, m\), is a properly strictly \(B_i\)-vex function.

Then \((\lambda^*, v^*, T^*, Y^*) = (S^*, q^*)\).

Proof. We assume that \((\hat{\lambda}, \hat{\nu}, \hat{T}, \hat{Y}) \neq (S^*, q^*)\) and exhibit a contradiction. Since \((S^*, q^*)\) is (EP)-optimal, therefore, there exist \(\lambda^* \in \mathbb{R}^p, Y^* \in \mathbb{R}^{p \times m}\), such that \((\lambda^*, q^*, S^*, Y^*)\) is (ED)-optimal, and (3.5)–(3.11) hold at \((\lambda^*, q^*, S^*, Y^*)\).

Since \((\lambda, \nu, T, Y)\) is (ED)-optimal, therefore,

\[
q^* = \hat{\nu}. \tag{4.9}
\]

Using the facts \((\lambda^*, q^*, S^*, Y^*)\) satisfies (3.5)–(3.11) and both \((\lambda^*, q^*, S^*, Y^*)\), \((\hat{\lambda}, \hat{\nu}, \hat{T}, \hat{Y})\) are (ED)-feasible, and using the hypothesis of the theorem, Lemma 3.3, Definition 2.6, we have

\[
\sum_{i=1}^{p} B_i(S^*, \hat{T}) \left[ L_i(S^*, \lambda^i, y^{*i}) - L_i(\hat{T}, \hat{\lambda}^i, \hat{y}^{*i}) \right] \geq 0. \tag{4.10}
\]

Using (3.3), (3.4), (4.6), and (4.4) in (4.7) we obtain

\[
\sum_{i=1}^{p} B_i(S^*, \hat{T}) \left[ \lambda_i(q^* - \hat{\nu}) \right] \geq 0, \quad \text{or} \quad (q^* - \hat{\nu}) \left[ \sum_{i=1}^{p} B_i(S^*, \hat{T}) \lambda_i \right] > 0. \tag{4.11}
\]

Now \(B_i(S^*, \hat{T}) > 0\) for \(i = 1, 2, \ldots, p\). This along with (4.5) and (4.6) yields \(\sum_{i=1}^{p} B_i(S^*, \hat{T}) \lambda_i > 0\), which in conjunction with (4.11) gives \(q^* > \hat{\nu}\) for all (EP)-feasible solutions \((S, q)\) and all (ED)-feasible solutions \((\lambda, \nu, T, Y)\). This contradicts (4.9). Hence, \((\hat{\lambda}, \hat{\nu}, \hat{T}, \hat{Y}) \neq (S^*, q^*)\).
Remark 4.2. As in Remark 4.1, we observe that Theorem 4.3 holds if we replace (H-2) by any one of the following hypotheses.

(i) On all feasible solutions of (P) and (ED), each \( L_i(T, \lambda_i, y^i) \), \( \lambda_i \geq 0, y^i \geq 0, i = 1, 2, \ldots, p \), is a differentiable properly \( B \)-vex function and additionally at least one \( L_i(T, \lambda_i, y^i) \) is properly strictly \( B \)-vex function.

(ii) On all feasible solutions of (P) and (ED), \( Q_i, i = 1, 2, \ldots, p \) are additively properly \( B \)-vex functions and \( Q_{ij}, i = 1, 2, \ldots, p \), are additively properly \( B \)-vex functions, and additionally either \( Q_i, i = 1, 2, \ldots, p \), are additively strictly properly \( B \)-vex functions and/or \( Q_{ij}, i = 1, 2, \ldots, p \), are additively strictly properly \( B \)-vex functions.

(iii) Functions \( L_i(T, \lambda_i, y^i) \), \( \lambda_i \geq 0, y^i \geq 0, i = 1, 2, \ldots, p \), are additively strictly properly \( B \)-vex function on all feasible solutions of (P) and (ED).

Below we give (D-1), and (D) as alternative formulations of (ED). Both (D-1) and (D) can be obtained easily from (ED).

\[
\begin{align*}
\text{max} & \sum_{i=1}^{p} (L_i(T, \lambda_i, y^i)) \\
& \text{subject to} \\
& \left( \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i)) \right)_{T_1, \ldots, T_p, I_S, I_T} \geq 0 \quad \text{for all } S \in A^n \quad \text{(D-1)} \\
& \sum_{i=1}^{n} \lambda_i = 1 \\
& \lambda \geq 0, Y \geq 0, \lambda \in R^p, Y \in R^{p \times m}, v \in R^l, y^i \in R^m, T \in A^n
\end{align*}
\]

\[
\begin{align*}
\text{min} & \max_{T \in A^n} \left[ \frac{L_1(T, \lambda_1, y^1)}{\lambda_1}, \ldots, \frac{L_p(T, \lambda_p, y^p)}{\lambda_p} \right] \\
& \text{subject to} \\
& \left( \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i)) \right)_{T_1, \ldots, T_p, I_S, I_T} \geq 0 \quad \text{for all } S \in A^n \quad \text{(D)} \\
& \sum_{i=1}^{n} \lambda_i = 1 \\
& \lambda \geq 0, Y \geq 0, \lambda \in R^p, Y \in R^{p \times m}, v \in R^l, y^i \in R^m, T \in A^n.
\end{align*}
\]
5. APPLICATION

Generalized Fractional Programming. In the present section we consider
the following minmax (generalized fractional programming) problem in-
volving \( n \)-set functions as an application of the results proved in the earlier
sections and relate it to a special case of (VP),

\[
q^* = \min_{(s_1, \ldots, s_n) \in A^n} \max_{1 \leq i \leq p} \left[ \frac{F_i(s_1, s_2, \ldots, s_n)}{G_i(s_1, s_2, \ldots, s_n)} \right]
\]

(GFP)

subject to \( H_j(s_1, s_2, \ldots, s_n) \leq 0, \quad j = 1, 2, \ldots, m \)
\[
(s_1, s_2, \ldots, s_n) \in A^n
\]

(B1) \( A^n \) is the \( n \)-fold product of a \( \sigma \)-algebra \( A \) of subsets of a given
set \( X \).

(B2) \( F_i, G_i \) for \( i = 1, 2, \ldots, p \) and \( H_j \) for \( j = 1, 2, \ldots, m \) are real
valued differentiable functions defined on \( A^n \).

(B3) For \( i = 1, 2, \ldots, p \), \( F_i \) is a convex and nonnegative function, \( G_i \)
is a concave and positive function, and whenever a \( G_i \) is both convex and
concave the corresponding \( F_i, i = 1, 2, \ldots, p \), is not necessarily restricted
to be nonnegative.

(B4) For \( j = 1, 2, \ldots, m \), \( H_j \) is a convex function.

(B5) \( \lambda_i F_i(S) + y_{ij} H_j(S) \geq 0 \) on \( A^n \) for all \( \lambda_i \geq 0, y_{ij} \geq 0, i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, m \). \( \lambda_i F_i(S) + y_{ij} H_j(S) \) need not be nonnegative
on \( A^n \) when \( G_i(S) \) is both convex and concave on \( A^n \).

From (GFP) we now obtain, along the lines of Bector [2], Bector et al.
[4], and Chandra et al. [6] the following transformed generalized fractional
programming problem (TGFP),

\[
q^* = \min_{(s_1, \ldots, s_n) \in A^n} \max_{1 \leq i \leq p} \left[ \frac{F_i(s_1, s_2, \ldots, s_n)}{G_i(s_1, s_2, \ldots, s_n)} \right]
\]

(TGFP)

subject to \( H_i(s_1, s_2, \ldots, s_n) \leq 0, \quad i = 1, 2, \ldots, p \), and \( j = 1, 2, \ldots, m \)
\[
(s_1, s_2, \ldots, s_n) \in A^n
\]
Setting
\[ Q_i = \frac{F_i(S_1, S_2, \ldots, S_n)}{G_i(S_1, S_2, \ldots, S_n)}, \quad i = 1, 2, \ldots, p, \]
and
\[ Q_{ij} = \frac{H_j(S_1, S_2, \ldots, S_n)}{G_i(S_1, S_2, \ldots, S_n)}, \quad i = 1, 2, \ldots, p \text{ and } j = 1, 2, \ldots, m \]
we observe that (TGFP) is of the same form as (P).

The following lemma relates (GFP) and (TGFP).

**Lemma 5.1.** (i) \((S_1, S_2, \ldots, S_n) \in A^n\) is (GFP)-feasible if and only if it is (TGFP) feasible.

(ii) \((S_1^*, S_2^*, \ldots, S_n^*) \in A^n\) is (GFP)-optimal if and only if it is (TGFP) optimal.

We now state the following lemmas that we shall use in the (GFP) duality.

**Lemma 5.2.** Let \(F, G : A^n \to R^2\) be differentiable functions and let \(Q = F/G\). If

(i) \(G\) is concave and strictly positive, and

(ii) \(F\) is convex and nonnegative (\(F\) need not be nonnegative if \(G\) be both convex and concave),

then \(Q\) is a properly \(B\)-\(\text{vec}\) function on \(A^n\) with \(B(R, S) = G(R)/G(S) > 0\) for all \(R\) and \(S\) in \(A^n\).

If the function in the numerator is strictly convex, and/or the function in the denominator is strictly concave on \(A^n\), then \(Q = F/G\) is a properly strictly \(B\)-\(\text{vec}\) function on \(A^n\).

**Lemma 5.3.** Let \(Q_i\) and \(Q_{ij}\) be as in (P). Then the function \(\lambda_i Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S) = [(\lambda_i F_i(S) + \sum_{j=1}^{m} y_{ij} H_j(S))/G_i(S)]\) is a properly \(B\)-\(\text{vec}\) function on \(A^n\) with \(B_i(R, S) = G_i(R)/G_i(S) > 0\) for all \(R\) and \(S\) in \(A^n\). If at least one function in the numerator is strictly convex, and/or the function in the denominator is strictly concave on \(A^n\), then \(\lambda_i Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S) = [(\lambda_i F_i(S) + \sum_{j=1}^{m} y_{ij} H_j(S))/G_i(S)]\) is properly strictly \(B\)-\(\text{vec}\) on \(A^n\) with \(B_i(R, S) = G_i(R)/G_i(S) > 0\) for all \(R\) and \(S\) in \(A^n\).

**Lemma 5.4.** The functions \(\lambda_i Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S) = [(\lambda_i F_i(S) + \sum_{j=1}^{m} y_{ij} H_j(S))/G_i(S)]\), \(i = 1, 2, \ldots, p\) are additively properly \(B\)-\(\text{vec}\) functions on \(A^n\) with \(B_i(R, S) = G_i(R)/G_i(S) > 0\), \(i = 1, 2, \ldots, p\) for all \(R\) and \(S\) in \(A^n\).

For \(i = 1, 2, \ldots, p\) if at least one function in the numerator is strictly convex, and/or at least one function in the denominator is strictly concave on \(A^n\), then the functions \(\lambda_i Q_i(S) + \sum_{j=1}^{m} y_{ij} Q_{ij}(S) = [(\lambda_i F_i(S) + \sum_{j=1}^{m} y_{ij} H_j(S))/G_i(S)]\), \(i = 1, 2, \ldots, p\) are additively properly \(B\)-\(\text{vec}\) functions on \(A^n\) with \(B_i(R, S) = G_i(R)/G_i(S) > 0\), \(i = 1, 2, \ldots, p\) for all \(R\) and \(S\) in \(A^n\).
The sum \( \sum_{j=1}^{m} y_{ij} H_j(S)/G_i(S) \), \( i = 1, 2, \ldots, p \) are additively properly strictly B-convex on \( A^n \).

In view of assumptions (B1)–(B5), Lemmas 5.1–5.4, and Remarks 5.1(i)–(iii), we see that the results of Sections 2, 3, and 4 become applicable to (GFP). Taking

(i) \( L_i(T, \lambda_i, y_i) = [(\lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T))/G_i(T)], \ i = 1, 2, \ldots, p \),

(ii) \( F_i \geq 0 \) and convex on \( A^n \), \( G_i > 0 \) and concave on \( A^n \), and if \( G_i \) is both convex and concave on \( A^n \) then \( F_i \) need not be nonnegative on \( A^n \),

(iii) \( \lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T) \geq 0 \) for all \( i = 1, 2, \ldots, p, T \in A^n \), and if \( G_i \) is both convex and concave on \( A^n \) then \( \lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T) \) need not be nonnegative on \( A^n \), we can easily have, using (ED), the following (GFD-1), (GFD-2), (GFD-3), and (GFD) as duals to (GFP).

\[
\begin{align*}
\max \quad & v \\
\text{subject to} & \\
\left\langle \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i))_{T_1, \ldots, T_n}, I_S - I_T \right\rangle & \geq 0 \quad \text{for all } S \in A^n \\
\left[ \frac{\lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T)}{G_i(T)} \right] & \geq \lambda_i v, \quad i = 1, 2, \ldots, p \\
\sum_{i=1}^{n} \lambda_i & = 1 \\
\lambda \geq 0, \ Y \geq 0, \ \lambda \in \mathbb{R}^p, \ Y \in \mathbb{R}^{p \times m}, \ v \in \mathbb{R}^q, \ y^i \in \mathbb{R}^m, \ T \in A^n.
\end{align*}
\]

\[
\begin{align*}
\max \sum_{i=1}^{p} (L_i(T, \lambda_i, y^i)) = \sum_{i=1}^{p} \left[ \frac{\lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T)}{G_i(T)} \right] \quad \text{(GFD-2)}
\end{align*}
\]

subject to

\[
\left\langle \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i))_{T_1, \ldots, T_n}, I_S - I_T \right\rangle \geq 0 \quad \text{for all } S \in A^n \\
\sum_{i=1}^{n} \lambda_i & = 1 \\
\lambda \geq 0, \ Y \geq 0, \ \lambda \in \mathbb{R}^p, \ Y \in \mathbb{R}^{p \times m}, \ v \in \mathbb{R}^q, \ y^i \in \mathbb{R}^m, \ T \in A^n.
\]
\[
\max \left( \sum_{i=1}^{p} \left( \lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T) \right) \right) \\
\sum_{i=1}^{p} \lambda_i G_i(T)
\]

subject to
\[
\left( \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i)) \right)_{\tau_1, \ldots, \tau_n, I_{S_i}} - I_{T_i} \geq 0 \quad \text{for all } S \in A^n
\]
\[
\sum_{i=1}^{n} \lambda_i = 1
\]
\[
\lambda \geq 0, \ Y \geq 0, \ \lambda \in \mathbb{R}^p, \ Y \in \mathbb{R}^{p \times m}, \ \nu \in \mathbb{R}^1, \ y^i \in \mathbb{R}^m, \ T \in A^n.
\]

\[
\min \max_{T \in A^n} \left( \sum_{i=1}^{p} \left( \lambda_i F_i(T) + \sum_{j=1}^{m} y_{ij} H_j(T) \right) / \lambda_i G_i(T) \right)
\]

subject to
\[
\left( \sum_{i=1}^{p} D_i(L_i(T, \lambda_i, y^i)) \right)_{\tau_1, \ldots, \tau_n, I_{S_i}} - I_{T_i} \geq 0 \quad \text{for all } S \in A^n
\]
\[
\sum_{i=1}^{n} \lambda_i = 1
\]
\[
\lambda \geq 0, \ Y \geq 0, \ \lambda \in \mathbb{R}^p, \ Y \in \mathbb{R}^{p \times m}, \ \nu \in \mathbb{R}^1, \ y^i \in \mathbb{R}^m, \ T \in A^n.
\]

CONCLUSION

In the present paper we present necessary and sufficient optimality conditions, and a Wolfe type [36] dual for a minmax primal problem (P) in which each of the objective functions and the constraint functions is an appropriately restricted properly \(B\)-vex \(n\)-set functions. Weak, strong, and strictly converse duality theorems are proved. Results for the duality of a certain generalized fractional programming problem (GFP) are shown to follow as a special case. Alternative duals both for (P) and (GFP) are also presented. The results presented in this paper can be easily extended under appropriate restrictions for \((F, B)\)-vex functions, \(B\)-invex functions, \((F, B)\)-invex functions, \((F, \rho, B)\)-vex functions, \((F, \rho, B)\)-invex functions, continuous programming, and for variational programming problems.
REFERENCES


