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The characteristic polynomial of the Laplacian of graphs in (a, b)-linear classes

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Abstract

In this work we deal with the characteristic polynomial of the Laplacian of a graph. We present some general results about the coefficients of this polynomial. We present families of graphs, for which the number of edges m is given by a linear function of the number of vertices n. In some of these graphs we can find certain coefficients of the above-named polynomial as functions just of n.

Keywords: Laplacian of a graph; Spectral graph theory

1. Introduction

A comprehensive treatment of spectral graph theory is given in [1–4]. In Section 2, we describe how the coefficients of the characteristic polynomial of the Laplacian of a graph G, $p_{L(G)}(\lambda)$, are related to spanning forests. In Section 3 we provide an algebraic expression for q_2 and q_3, the third and fourth coefficients of $p_{L(G)}(\lambda)$, respectively. In the last section we calculate q_2 and q_3 for graphs in certain (a, b)-linear classes as functions of n, a and b.

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2. The Laplacian of a graph

Let G be a graph with n vertices. The Laplacian of G is defined as the $n \times n$ matrix

$$L(G) = \varDelta - A, \tag{2.1}$$

where A is the (0, 1)-adjacency matrix of G and $\varDelta$ is the diagonal matrix whose elements are the degrees of the vertices of G. We call $\varDelta$ the *matrix of degrees of* G or simply the *matrix of degrees*. The matrix $L(G)$ can be associated with a positive semidefinite quadratic form. We can see it in the following proposition for which the proof is in [7].

Proposition 2.1. *If* G *is a graph and the quadratic form related to* $L(G)$ *is given by*

$$q(x) = xL(G)x^{\mathrm{t}}, \quad x \in \mathbb{R}^n,$$

then q *is a positive semidefinite quadratic form.*

Consider $\omega(G)$, the number of connected components of G. The next result (for which the proof can also be found in [7]) shows a relation between $\omega(G)$, the number of vertices in G, and the rank of $L(G)$.

Proposition 2.2. *The rank of the Laplacian matrix is*

$$\operatorname{rank}(L(G)) = n - \omega(G).$$

The polynomial

$$p_{L(G)}(\lambda) = \det(\lambda I - L(G)) = \lambda^n + q_1\lambda^{n-1} + \cdots + q_{n-1}\lambda + q_n \tag{2.2}$$

is called **the characteristic polynomial of** $L(G)$. Its spectrum is

$$\zeta(G) = (\lambda_1, \ldots, \lambda_n), \tag{2.3}$$

where $\forall i$, $1 \leqslant i \leqslant n$, λ_i is an eigenvalue of $L(G)$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.

According to Propositions 2.1 and 2.2, $\forall i,\ 1 \leqslant i \leqslant n,\ \lambda_i$ is a non-negative real number; if G is connected then $\lambda_{n-1} = 0$ and $\lambda_n = 0$ whether or not G is connected.

Before introducing the first theorem, we have to consider the following definitions: For each $i \in \{1, \ldots, n\}$, let s_i be the number of spanning forests in G with i edges. Let these spanning forests be $\Theta_{if}\ \ (1 \leqslant f \leqslant s_i)$, and let $p(\Theta_{if})$ be the product of the numbers of vertices of the trees in Θ_{if}. Theorem 2.1 links the coefficients of the $p_{L(G)}(\lambda)$ to the spanning forests in G and its proof can be found in [1, Theorem 7.5].

Theorem 2.1. *The coefficients of the characteristic polynomial of* $L(G)$ *are given by*

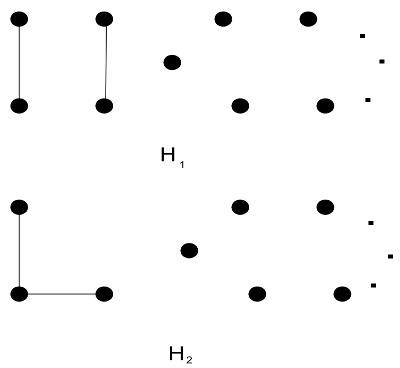

Fig. 1. H_1 and H_2 are spanning forests in G.

$$(-1)^i q_i = \sum_{f=1}^{s_i} p(\Theta_{if}), \quad 1 \leqslant i \leqslant n.$$

It follows from Theorem 2.1 that $q_0 = 1$; $q_n = 0$ and $q_1 = -2m$, where m is the number of edges of G. Furthermore, $q_{n-1} = (-1)^{n-1} nS(G)$, where $S(G)$ is the number of spanning trees in G. In the next section we calculate algebraic expressions for q_2 and q_3. These expressions provide us with q_2 and q_3 as functions of m, n, the degree sequence of vertices and the adjacency matrix of G.

3. The third and fourth coefficients of $p_{L(G)}(\lambda)$

From Theorem 2.1 we have

$$(-1)^2 q_2 = \sum_{f=1}^{s_2} p(\Theta_{2f}), \tag{3.1}$$

where each Θ_{2f} is a spanning forest in G with only two edges. We define an H_1-spanning forest in G as a spanning graph with two connected components isomorphic to P_2 and $(n-4)$ components isomorphic to K_1. We also define an H_2-spanning forest in G as a spanning graph with only one component isomorphic to P_3 and $(n-3)$ components isomorphic to K_1. Each Θ_{2f} is isomorphic to either an H_1- or an H_2-spanning forest in G, as displayed in Fig. 1.

The next theorem identifies the third coefficient q_2 of $p_{L(G)}(\lambda)$.

Theorem 3.1. *Let G be a graph with m edges and let $d = (d_1, \ldots, d_n)$ be its non-increasing degree sequence. The third coefficient in $p_{L(G)}(\lambda)$ is*

$$q_2 = 2m^2 - m - \tfrac{1}{2} \sum_{i=1}^{n} d_i^2.$$

Proof. From the hypotheses above and expression (3.1) we have

$$q_2 = 4\,\xi_{H_1}(G) + 3\,\xi_{H_2}(G), \tag{3.2}$$

where for $j = 1, 2$, $\xi_{H_j}(G)$ is the number of H_j-spanning forests in G.

When we calculate $\xi_{H_2}(G)$, we observe that each vertex i with $d_i \geqslant 2$ contributes $\binom{d_i}{2}$ towards the number of H_2-spanning forests, while the remaining vertices do not contribute at all. So,

$$\xi_{H_2}(G) = \sum_{i=1}^{n} \binom{d_i}{2}.$$

After some algebraic manipulations and, considering $\sum_{i=1}^{n} d_i = 2m$ we find

$$\xi_{H_2}(G) = \tfrac{1}{2}\sum_{i=1}^{n} d_i^2 - m. \tag{3.3}$$

To calculate $\xi_{H_1}(G)$ it is enough to use the number of all two-edge combinations. It follows that

$$\xi_{H_1}(G) = \frac{m(m-1)}{2} - \xi_{H_2}(G).$$

Consequently,

$$\xi_{H_1}(G) = \frac{m^2 + m - \sum_{i=1}^{n} d_i^2}{2}. \tag{3.4}$$

By substituting (3.3) and (3.4) into (3.2) we obtain

$$q_2 = 2m^2 - m - \tfrac{1}{2}\sum_{i=1}^{n} d_i^2. \qquad \square$$

Corollary 3.1. *If Δ is the matrix of degrees in G then*

$$q_2 = \tfrac{1}{2}\left[(\mathrm{tr}\Delta)^2 - \mathrm{tr}\Delta - \mathrm{tr}(\Delta^2)\right].$$

Proof. We obtain this result straight from Theorem 3.1, if we consider $\sum_{i=1}^{n} d_i = \mathrm{tr}[\Delta]$ and $\sum_{i=1}^{n} d_i^2 = \mathrm{tr}[\Delta^2]$. $\square$

In order to obtain an algebraic expression for q_3, the fourth coefficient in $p_{L(G)}(\lambda)$, we need to count all spanning forests Θ_{3f} with exactly three edges in G. Each Θ_{3f} is isomorphic to one of the four graphs displayed in Fig. 2, which we call H_j-spanning forests in G, $3 \leqslant j \leqslant 6$. Let $\xi_{H_j}(G)$ be the number of such forests ($3 \leqslant j \leqslant 6$).

Theorem 3.2. *Let G be a graph with m edges and let A be its adjacency matrix. Consider $d = (d_1, \ldots, d_n)$ its non-increasing degree sequence. Then,*

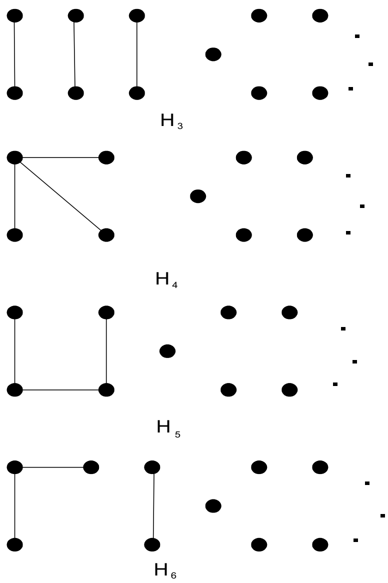

Fig. 2. H_3, H_4, H_5 and H_6 are spanning forests of G.

$$q_3 = \frac{1}{3}\left\{ -4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + \operatorname{tr}(A^3)\right\}.$$

Proof. From Theorem 2.1 and considering the forests displayed in Fig. 2, we have

$$(-1)^3 q_3 = \sum_{f=1}^{s_3} p(\Theta_{3f}), \tag{3.5}$$

and

$$(-1)^3 q_3 = 8\xi_{H_3}(G) + 6\xi_{H_6}(G) + 4(\xi_{H_4}(G) + \xi_{H_5}(G)). \tag{3.6}$$

In order to determine q_3, we need to find $\xi_{H_j}(G)$ for $j = 3, 4, 5$ and 6.

The procedure to calculate $\xi_{H_4}(G)$ is analogous to the one used in calculating $\xi_{H_2}(G)$, if we consider $d_i \geqslant 3$. Then, we have

$$\xi_{H_4}(G) = \frac{1}{6}\left(\sum_{i=1}^{n} (d_i^3 - 3d_i^2 + 2d_i)\right). \tag{3.7}$$

In order to evaluate $\xi_{H_6}(G)$ it is necessary to find the number of all spanning graphs with three edges that contain the path P_3. This value is

$$\xi_{H_2}(G)(m-2), \tag{3.8}$$

when $\xi_{H_2}(G)$ is given by (3.3). After that, we need to subtract all the spanning graphs with triangles of which there are $1/6\text{tr}(A^3)$ [5], and all the spanning forests $H_4(G)$, given by (3.7), and $H_5(G)$. We do not need to find the number of $H_5(G)$, as it will be eliminated in the calculations afterwards. So,

$$\xi_{H_6}(G) = \xi_{H_2}(G)(m-2) - \tfrac{1}{2}\text{tr}(A^3) - 3\xi_{H_4}(G) - 2\xi_{H_5}(G). \tag{3.9}$$

Finally to determine $\xi_{H_3}(G)$ we need to obtain the number of all spanning graphs in G with three edges, discarding the ones that contain a triangle, or an H_4, H_5 or H_6. Thus we arrive at

$$\begin{aligned}\xi_{H_3}(G) = {}& \tfrac{1}{6}[m(m-1)(m-2)] - \xi_{H_4}(G) - \xi_{H_5}(G)\\ & - \xi_{H_6}(G) - \tfrac{1}{6}\text{tr}(A^3).\end{aligned} \tag{3.10}$$

By substituting (3.7), (3.9) and (3.10) into (3.6) and after some manipulations, we obtain an expression for q_3:

$$q_3 = \frac{1}{3}\left\{ -4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + \text{tr}(A^3)\right\}. \qquad \square$$

Corollary 3.2. *If Δ is the matrix of degrees in G, then*

$$q_3 = -\tfrac{1}{6}(\text{tr}\Delta)^3 + \tfrac{1}{2}(\text{tr}\Delta)^2 + \tfrac{1}{2}(\text{tr}\Delta - 2)\text{tr}(\Delta^2) - \tfrac{1}{3}\text{tr}(\Delta^3) + \tfrac{1}{3}\text{tr}(A^3).$$

Proof. Straightfowardly from Theorem 3.2 and considering $\sum_{i=1}^{n} d_i^k = \text{tr}(\Delta^k)$, for $k \geqslant 1$. $\square$

4. The coefficients q_2 and q_3 of $p_{L(G)}(\lambda)$ for graphs in (a, b)-linear classes

Given a and $b \in Q^+$, we define the (a, b)-linear class, denoted by $L(a, b)$, to be the set of all connected graphs such that $m = an - b$. The $(1, 1)$-linear class coincides with the set of all trees and $L(1, 0)$ characterizes the set of connected graphs with only one cycle. Since the maximal outerplanar graphs (mops) have $m = 2n - 3$, they all belong to the $(2, 3)$-linear class. Nevertheless, this class does not only contain mops. Fig. 3 displays three mops and Fig. 4 shows a graph that is not a mop, but belongs to $L(2, 3)$.

In [6] we can find other well-known families of graphs in different (a, b)-linear classes, for specific pairs of rational numbers (a, b). For example, if $a = k/2, k \in N$, $n \geqslant k + 1$ and $b = 0$, the class $L(k/2, 0)$ contains the set of all k-regular graphs, while $L(3, 6)$ contains all maximal planar graphs.

For some graphs in the classes above, we can obtain q_2 and q_3 in terms of the number of vertices, directly from Theorems 3.1 and 3.2. For example:

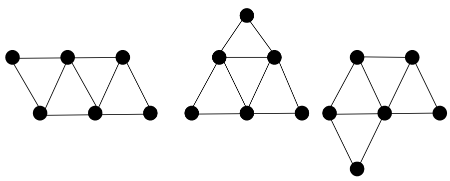

Fig. 3. Mops in $L(2, 3)$.

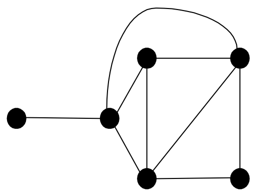

Fig. 4. A graph which is not a mop.

1. Consider $P_n \in L(1, 1)$. Then,

$$q_2 = 2n^2 - 7n + 6 \quad \text{and} \quad q_3 = \tfrac{1}{3}(-4n^3 + 30n^2 - 74n + 60).$$

2. Consider $C_n \in L(1, 0)$. Then for $n \geqslant 4$,

$$q_2 = 2n^2 - 3n \quad \text{and} \quad q_3 = \tfrac{1}{3}(-4n^3 + 18n^2 - 20n).$$

3. For $K_n \in L((n-1)/2, 0)$, then

$$q_2 = \tfrac{1}{2}(n^4 - 3n^3 + 2n^2) \quad \text{and} \quad q_3 = -\tfrac{n^6}{6} + n^5 - \tfrac{11n^4}{6} + n^3.$$

4. Let G be k-regular. Then $G \in L(k/2, 0)$, and

$$q_2 = \tfrac{1}{2}[k^2n^2 - n(k^2 + k)]$$

and

$$q_3 = \tfrac{1}{3}\big[-\tfrac{k^3}{2}n^3 + \tfrac{3}{2}n^2(k^2 + k^3) - n(k^3 + 3k^2) + \mathrm{tr}(A^3)\big].$$

5. For Fan graphs $F_n \in L(2, 3)$, displayed in Fig. 5, we calculate q_2 and q_3 as functions of n. Although it is easy to find these expressions a few other considerations might come in handy, such as: F_n has $m = 2n - 3$ and its number of triangles is $n - 2$. Morever, its non-increasing degree sequence is $(n - 1, 3, \ldots, 3, 2, 2)$ and consequently we have $\sum_{i=1}^{n} d_i^2 = n^2 + 7n - 18$ and $\sum_{i=1}^{n} d_i^3 = n^3 - 3n^2 + 30n - 66$. Applying all these results into Theorems 3.1 and 3.2 we find

$$q_2 = \tfrac{15}{2}n^2 - \tfrac{59}{2}n + 30$$

and

$$q_3 = -9n^3 + 67n^2 - 168n + 144.$$

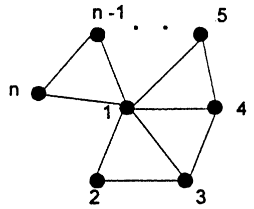

Fig. 5. Fan graph F_n.

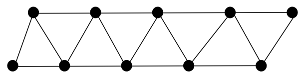

Fig. 6. Streamer graph S_{10}.

6. If n is even and $n \geqslant 6$, Streamer graphs, S_n, are mops whose non-increasing degree sequence is $d = (4, 4, \dots, 4, 3, 3, 2, 2)$. Fig. 6 shows an S_{10}.
   We can find q_2 and q_3 for the $p_{L(S_n)}(\lambda)$. It is enough to substitute the specific values for S_n: $1/6\operatorname{tr}(A^3) = n - 2$, $\sum_{i=1}^{n} d_i^2 = 16n - 38$; $\sum_{i=1}^{n} d_i^3 = 64n - 186$ and $m = 2n - 3$, into the general expressions for q_2 and q_3 given by Theorems 3.1 and 3.2, respectively. Then, we obtain

   $$q_2 = 8n^2 - 34n + 40$$

   and

   $$q_3 = 1/3[-32n^3 + 264n^2 - 766n + 792].$$

Proposition 4.1 provides upper bounds for q_2 and q_3 when $G \in L(a, b)$.

Proposition 4.1. *If $G \in L(a, b)$, then*

$$q_2 \leqslant 2[a^2n^2 - (2ab + a)n + b^2 + b]$$

and

$$q_3 \leqslant \tfrac{1}{3}\big\{8a^3n^3 + (10a^2 - 24a^2b)n^2 + (24ab^2 - 20ab - 10a)n + (-8b^3 + 10b^2 + 10b)\big\}.$$

Proof. For $G \in L(a, b)$, we have that

$$m = an - b;\ \sum_{i=1}^{n} d_i \leqslant \sum_{i=1}^{n} d_i^2 \leqslant \sum_{i=1}^{n} d_i^3,\ \sum_{i=1}^{n} d_i^2 \leqslant \sum_{i=1}^{n} d_i \sum_{i=1}^{n} d_i$$

and

$$\operatorname{tr}(A^3) \leqslant \sum_{i=1}^{n} d_i^2 - 2m.$$

After some manipulations, we obtain the result. □

5. Conclusions

In order to find the third coefficient q_2 in $p_{L(G)}(\lambda)$, all one needs to do is to count the number of H_2-spanning forests in G. On the order hand, determining q_3 is not that easy. It is necessary to calculate the number of several kinds of spanning forests in G, as shown in the proof of Theorem 3.2. This result, although dependent on adjacency matrix, allows a direct evaluation of q_3. If we only deal with graphs with no triangles, such as bipartite graphs, the expression of q_3 becomes quite handy. For certain graphs in (a, b)-linear classes we found expressions of q_2 and q_3 simply in terms of the number of vertices of G. Finally, for graphs in $L(a, b)$, we found upper bounds for q_2 and q_3.

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