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A Tensor Product Theorem for Quantum Linear Groups at Even Roots of Unity*

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The main result of this paper is a tensor product theorem for irreducible representations of the quantum general linear group $GL_q(n)$ where q is a primitive lth root of unity, l being an *even* integer. For odd l, an analogous result was proved by Parshall and Wang [PW]. Lusztig [L] proved a tensor product theorem for representations of quantized enveloping algebras, again where l is odd.

We prove that if l is divisible by 4, there is an isomorphism of $GL_q(n)$-modules

$$L^q(\lambda_0 + (l/2)\lambda_1) \cong L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}, \qquad (0)$$

where $L^q(\lambda)$ is the irreducible $GL_q(n)$-module of highest weight λ, $L(\lambda_1)^{(1)}$ is the $GL_q(n)$-module given by the Frobenius $l/2$ twist of the irreducible module $L(\lambda_1)$ for the (non-quantized) general linear group $GL(n)$ of highest weight λ, if λ_0 is $l/2$-restricted. (All these terms are defined below.) The Frobenius map is defined by sending elements x_{ij} of the coordinate ring of $GL(n)$ to the l'th power $X_{ij}^{l'}$ in the coordinate ring $K[GL_q(n)]$, where $l' = l/2$.

This is in analogy with the tensor product Theorem 9.2.2 of [PW], except that we are using $l' = l/2$ instead of l. The elements $X_{ij}^{l'}$ are *not* central in $K[GL_q(n)]$, unlike the lth powers used in [PW], and so our proof is different than that in [PW]. It is easy to see that our proof can be used to prove Steinberg's tensor product theorem for reductive (non-quantized) algebraic groups at prime characteristic.

If l is even but not divisible by 4, then (0) still holds, except that $L(\lambda_1)^{(1)}$ should be replaced by the Frobenius $l/2$ twist $L^{-1}(\lambda_1)^{(1)}$ of the irreducible module $L^{-1}(\lambda_1)$ for the quantum group $GL_{-1}(n)$ with parameter $q = -1$.

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We are concerned with the quantum linear group $GL_q(n)$, as defined in [RTF, M, PW]. (See also [TT].) There is really no such group, but there is a coordinate ring $K[GL_q(n)]$ where K is a field and q is a non-zero element of K. One first defines the coordinate ring of quantum $n \times n$ matrices over K, denoted $K[M_q(n)]$, to be the associative K-algebra generated by the n^2 variables X_{ij}, where i and j are between 1 and n, subject to the relations

$$X_{ri}X_{rj} = q^{-1}X_{rj}X_{ri}, \qquad i<j \tag{1}$$

$$X_{ri}X_{si} = q^{-1}X_{si}X_{ri}, \qquad r<s \tag{2}$$

$$X_{ri}X_{sj} = X_{sj}X_{ri}, \qquad \text{if} \quad r<s \quad \text{and} \quad i>j \tag{3}$$

$$X_{ri}X_{sj} - X_{sj}X_{ri} = (q^{-1}-q)\, X_{si}X_{rj}, \qquad \text{if} \quad r<s \quad \text{and} \quad i<j. \tag{4}$$

The quantum determinant D is defined by

$$D = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)} \cdots X_{n\sigma(n)} \tag{5}$$

where l is the standard length function on the symmetric group S_n. This element D is a central element of $K[M_q(n)]$, cf. [PW, 4.6.1]. Then $K[GL_q(n)]$ is defined to be the localization of $M_q(n)$ at D, and is a Hopf algebra, with co-multiplication defined by

$$\Delta\colon K[GL_n(q)] \to K[GL_n(q)] \otimes K[GL_n(q)], \qquad \Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$$

and co-unit ε and antipode γ defined by

$$\varepsilon(X_{ij}) = \delta_{ij}, \qquad \gamma(X_{ij}) = (-q)^{j-i} A(ji)\, D^{-1},$$

where $A(ji)$ is the quantum determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting row j and column i of the matrix (X_{kl}) (see [PW, Chap. 5]).

A closed subgroup H of $GL_q(n)$ has, by definition, a coordinate ring $K[GL_q(n)]/\mathfrak{a}$, where $\mathfrak{a}$ is a Hopf ideal of $K[GL_q(n)]$. We have the following Hopf ideals, and closed subgroups:

$\mathfrak{b}'$ generated by all X_{ij} with $i>j$, $\qquad K[B'_q] = K[GL_q(n)]/\mathfrak{b}'$

$\mathfrak{t}$ generated by all X_{ij} with $i \neq j$, $\qquad K[T_q] = K[GL_q(n)]/\mathfrak{t}$.

(We are using the notation of [PW] for the quantum Borel group B'_q of upper triangular matrices; in this paper we do not need to use the lower quantum Borel group B_q of [PW].)

One of the main tools in [PW] is the Frobenius morphism $F\colon GL_q(n)\to GL(n)$, defined in terms of the lth powers of elements of $K[GL_q(n)]$. Two of its most important properties are implied by the following to facts, if l is odd,

$$X^l_{ij} \text{ is a central element of } K[GL_q(n)], \tag{6}$$

and

$$\Delta(X^l_{ij})=\sum_{k=1}^{n} X^l_{ik}\otimes X^l_{kj} \tag{7}$$

[PW, 7.2.2 and 7.2.3]. When l is even, then (6) still holds, but (7) is false. To repair this we replace l in (6) and (7) by $l'=l/2$. Then the analogue of (7) is now true, but the analogue of (6) is not.

To define q-binomial coefficients (see, for example, [L, PW, Sect. 7]), we define, for positive integers $m\leqslant n$,

$$[n]_q=\frac{q^n-1}{q-1},\qquad \begin{bmatrix} n\\ m\end{bmatrix}_q=\frac{[n]_q\,[n-1]_q\cdots[n-m+1]_q}{[m]_q\,[m-1]_q\cdots[1]_q},\qquad \begin{bmatrix} n\\ 0\end{bmatrix}_q=1.$$

Proposition 1. *Let l be an even integer, and set $l'=l/2$. Let q be a primitive lth root of unity in K. We have in $K[M_q(n)]$*

$$X^{l'}_{ri}X_{rj}=-X_{rj}X^{l'}_{ri}\qquad \text{if } i\neq j \tag{8}$$

$$X^{l'}_{ri}X_{si}=-X_{si}X^{l'}_{ri}\qquad \text{if } r\neq s \tag{9}$$

$$X^{l'}_{ij}X_{rs}=X_{rs}X^{l'}_{ij}\qquad \text{if } i\neq r \text{ and } r\neq s. \tag{10}$$

Proof. Since q is a primitive $2l'$-root of unity, then $q^{l'}=q^{-l'}=-1$. To prove (8), suppose that $i<j$; then (1) implies

$$X^{l'}_{ri}X_{rj}=q^{-l'}X_{rj}X^{l'}_{ri}=-X_{rj}X^{l'}_{ri}.$$

If $i>j$ we get the same equation, but with $q^{l'}$ instead of $q^{-l'}$. It is clear that (9) follows in the same way, from (2). If $r<s$ and $i>j$, or if $r>s$ and $i<j$, then (10) follows from (3). If $r<s$ and $i<j$, we use the following result from [PW, 7.2b]:

$$X^k_{ij}X_{rs}-X_{rs}X^k_{ij}=(q^{-1}-q)\begin{bmatrix} k\\ 1\end{bmatrix}_{q^2}X^{k-1}_{ij}X_{is}X_{rj}.$$

Take $k=l'$; then $[l']_{q^2}=(q^{2l'}-1)/(q-1)=0$, so $X^{l'}_{ij}$ commutes with X_{rs}. This is also true if $r>s$ and $i>j$. This completes the proof.

At this point, in order to simplify the exposition, we make the following assumption, which holds until the last paragraph.

Assumption. The integer l is divisible by 4, $l' = l/2$, and q is a primitive lth root of unity in K.

PROPOSITION 2. *The subalgebra $K[M_q(n)]^{l'}$ of $K[M_q(n)]$ generated by $X_{ij}^{l'}$, where i and j are between 1 and n, is a commutative algebra isomorphic to the (non-quantized) coordinate ring $K[M(n)]$ of $n \times n$ matrices over K.*

Proof. From (8), we have

$$X_{ri}^{l'} X_{rj}^{l'} = (-1)^{l'} X_{rj}^{l'} X_{ri}^{l'} \qquad \text{if} \quad i \neq j$$

and $(-1)^{l'} = 1$ since we are assuming that l' is even. Similarly, $X_{ri}^{l'}$ commutes with $X_{si}^{l'}$ by (9), and $X_{ij}^{l'}$ commutes with $X_{rs}^{l'}$ by (10). This completes the proof.

PROPOSITION 3. *We have*

$$\Delta(X_{ij}^{l'}) = \sum_{k=1}^{n} X_{ik}^{l'} \otimes X_{kj}^{l'}.$$

Proof. As in the proof of [PW, 7.2.2], we have for any integer r between 1 and n

$$\left(\sum_{s \geq r} X_{is} \otimes X_{sj}\right)^{t} = \sum_{m=0}^{t} \begin{bmatrix} t \\ m \end{bmatrix}_{q^2} (X_{ir} \otimes X_{rj})^{m} \left(\sum_{s > r} X_{is} \otimes X_{sj}\right)^{t-m}.$$

If $t = l'$ then $[{}^{t}_{m}]_{q^2} = 0$ for $0 < m < l'$, which gives the result.

We now have a Frobenius map $F^{\#}: K[GL(n)] \to K[GL_q(n)]$, a Hopf algebra homomorphism defined by

$$F^{\#}(x_{ij}) = X_{ij}^{l'}.$$

PROPOSITION 4. *Let d be the ordinary determinant in $K[M(n)]$. Then $F^{\#}(d) = D^{l'}$.*

Proof. The proof is similar to [PW, 7.2.3]. As shown there,

$$D^{l'} = \sum_{j=1}^{n} (-q)^{l'(i-j)} X_{ij}^{l'} A(ij)^{l'} = \sum_{j=1}^{n} (-1)^{l'(i-j)} X_{ij}^{l'} A(ij)^{l'},$$

where the last equation holds because $q^{l'} = -1$. If $n = 2$ the result is proved; for bigger n use induction as indicated in [PW, 7.3.2].

We do not get a covering in the sense of [PW, 1.8], since the image of $F^\#$ is not central. We may define the Frobenius kernel as in [PW], whose coordinate ring $K[GL_q(n)_1]$ is by definition the factor ring of $K[GL_q(n)]$ by the two-sided ideal I generated by all $X_{ij}^{l'}$, for $i \neq j$ and by all $X_{ii}^{l'} - 1$, where i and j are between 1 and n. If $i \neq j$, then X_{ij} commutes with 1 but anti-commutes with $X_{ii}^{l'}$, so it follows that the image of X_{ij} in $K[GL_q(n)_1]$ is 0; indeed

$$X_{ij}(X_{ii}^{l'} - 1) + (X_{ii}^{l'} - 1)X_{ij} = -2X_{ij}$$

and since the characteristic of the field K is not 2 (since K has a primitive even root of unity) then $X_{ij} \in I$ if $i \neq j$. Thus $K[GL_q(n)_1]$ is much too small to be useful. We define $K[GL_q(n)_1 \cdot T]$ to be the factor ring $K[GL_q(n)]$ modulo the ideal generated by the $X_{ij}^{l'}$ for $i \neq j$. Then $K[GL_q(n)_1 \cdot T]$ behaves much as in [PW], since every X_{kl} either commutes or anti-commutes with $X_{ij}^{l'}$. Similarly we have the factor ring $K[(B'_q)_1 \cdot T]$ of $K[B_q]$ by the ideal generated by $X_{ij}^{l'}$ for $i < j$.

A $GL_q(n)$-module M is by definition a $K[GL_q(n)]$-comodule; it has a structure map $\tau_M : M \to M \otimes K[GL_q(n)]$ which satisfies $(\tau_M \otimes 1) \circ \tau_M = (1 \otimes \Delta) \circ \tau_M$. The restriction of M to any closed subgroup H of $GL_q(n)$ is the $K[H]$-comodule whose structure map is given by $(1 \otimes \zeta^\#) \circ \tau_M$, where $\zeta^\#$ is the natural epimorphism from $K[GL_q(n)]$ to its factor ring $K[H]$. If M is a T_q-module and if an element m of M satisfies $\tau_M(m) = m \otimes \lambda \in M \otimes K[T_q]$, where $\lambda = \prod_i X_{ii}^{n_i}$ for some integers n_i, then m is called a weight vector and λ is called a weight. The set of weights, denoted by $X(T_q)$, is an abelian group, and will as usual be written additively. A $GL_q(n)$-module M which has a unique B'_q-stable line is called a highest weight module; the weight of a generator m of this line is called the highest weight of M, and x is called a maximal vector. From [PW, 8.2.2], any irreducible $GL_q(n)$-module is a highest weight module, and any two such irreducibles with the same highest weight are isomorphic. An irreducible $GL_n(q)$-module with highest weight λ is denoted $L^q(\lambda)$.

We say that a weight λ is *l'-restricted* if $\lambda = \sum_{i=1}^n r_i \omega_i$ with $0 \leqslant r_i < l'$ for each i, where ω_i is the fundamental dominant weight $X_{11} X_{22} \cdots X_{ii}$.

For a rational $GL(n)$-module M, we can form a $GL_q(n)$-module by using the composition $(1 \otimes F^\#) \circ \tau_M$ from M to $M \otimes K[GL_q(n)]$, giving a $G_q(n)$-module $M^{(1)}$, called the Frobenius twist of the module M. We let B' denote the Borel subgroup of $GL(n)$ of upper triangular matrices; we have a Frobenius map $F^\# : K[B'] \to K[B'_q]$, and we have Frobenius twists of rational B'-modules.

If H is a closed subgroup of $GL_q(n)$ and M is an H-module, an element m of M is said to be fixed by H if $\tau_M(m) = m \otimes 1 \in M \otimes K[H]$. The H-fixed elements of M are denoted by M^H. In [PW] it is shown that if l is odd, the $(B')_1$-fixed points of a B'_q-module are the Frobenius twist of some

B'-module, using an elegant argument [PW, 2.10.2] (similar to one in [T]) which seems to depend on the centrality of $F^{\#}K[B']$ in $K[B'_q]$. Our next two results serve as replacements.

LEMMA 5. *Let x be a $(B'_q)_1 \cdot T$-fixed element of $K[B'_q]$. Then x is in the image of the Frobenius map from $K[B']$ to $K[B'_q]$.*

Proof. It follows from [PW, 3.5.1] that $K[B'_q]$ has a K-basis of the form

$$\mathscr{B} = \left\{ \prod_{i \leqslant j} X_{ij}^{t_{ij}} \mid t_{ij} \in \mathbf{Z},\ t_{ij} \geqslant 0 \text{ if } i < j \right\},$$

where the products are formed with respect to some fixed order of the X_{ij}'s. Let us fix an order: say that $X_{ij} < X_{rs}$ if $j < s$ or $j = s$ and $i < r$. We will write $m = m(t_{ij})$ if we need to emphasize the values of the exponents t_{ij} for some $m \in \mathscr{B}$.

Write x as a linear combination

$$x = \sum_{m \in \mathscr{B}} c_m m, \qquad c_m \in K.$$

Let supp x (the support of x) be the subset of those $m \in \mathscr{B}$ for which $c_m \neq 0$. Since x is $(B'_q) \cdot T$-fixed, then $(1 \otimes \zeta^{\#}) \circ \Delta(x) = x \otimes 1$, where $\zeta^{\#}: K[B'_q] \to K[(B'_q)_1 \cdot T]$ is the natural map, and $\Delta: K[B'_q] \to K[B'_q] \otimes K[B'_q]$ is the comultiplication. To compute $(1 \otimes \zeta^{\#}) \circ \Delta(m)$ for $m \in \operatorname{supp} x$, first apply Δ, giving

$$\prod_{i \leqslant j} \left(\sum_{k=i}^{j} X_{ik} \otimes X_{kj} \right)^{t_{ij}}, \tag{11}$$

expand this as a linear combination of elements of the form $m_1 \otimes m_2$, with m_1, m_2 in $\mathscr{B}$, then apply $\zeta^{\#}$ to each m_2, which amounts to deleting $m_1 \otimes m_2$ if $m_2 = m_2(r_{ij})$ has some exponent $r_{ij} \geqslant l'$ if $i \neq j$, and then collect terms. When expanding (11), because of the order we have chosen on the X_{ij}'s, to get elements m_2 in $\mathscr{B}$ we only use relations (1), (2), and (3), and never use relation (4). When expanding (11) for some $m \in \mathscr{B}$, if we take from each factor $(\sum_k X_{ik} \otimes X_{kj})^{t_{ij}}$ the summand $X_{ij} \otimes X_{jj}$ and raise this to the power t_{ij}, then multiply over all i, j, we get a term of the form

$$\prod_{i,j} X_{ij}^{t_{ij}} \otimes \prod_{i,j} X_{jj}^{t_{ij}} \tag{12}$$

and this is equal to $m \otimes 1$, since x is $(B'_q)_1 \cdot T$-fixed. Therefore for each j we have

$$t_{1j} + t_{2j} + \cdots + t_{jj} = 0. \tag{13}$$

Suppose that for some $m \in \operatorname{supp} x$, there is a pair (r, s) with $r < s$ for which t_{rs} is not divisible by l'. Expand (11) for this m as in (12), except that instead of taking $X_{rs}^{t_{rs}} \otimes X_{ss}^{t_{rs}}$ from the expansion of the factor $(\sum_k X_{rk} \otimes X_{ks})^{t_{rs}}$, consider for some integer h with $r \leqslant h < s$,

$$\begin{aligned}&(X_{rh} \otimes X_{hs})(X_{rs} \otimes X_{ss})^{t_{rs}-1}\\&\quad + (X_{rs} \otimes X_{ss})(X_{rh} \otimes X_{hs})(X_{rs} \otimes X_{ss})^{t_{rs}-2}\\&\quad + \cdots + (X_{rs} \otimes X_{ss})^{t_{rs}-1}(X_{rh} \otimes X_{hs}).\end{aligned}$$

Since $h < s$, this gives us

$$(1 + q^2 + q^4 + \cdots + q^{2(t_{rs}-1)})(X_{rh} \otimes X_{hs})(X_{rs} \otimes X_{ss})^{t_{rs}-1}.$$

We have $(1 + q^2 + q^4 + \cdots + q^{2(t_{rs}-1)}) = [t_{rs}]_{q^2}$ which is not 0, since t_{rs} is not divisible by l'. It follows that in the expansion of $\Delta(m)$ we get a non-zero multiple of

$$\prod_{(i,j)<(r,s)} X_{ij}^{t_{ij}} \otimes X_{ij}^{t_{ij}} (X_{rh} \otimes X_{hs})(X_{rs} \otimes X_{ss})^{t_{rs}-1} \prod_{(i,j)>(r,s)} X_{ij}^{t_{ij}} \otimes X_{ij}^{t_{ij}}.$$

This gives us a non-zero multiple of $m_1 \otimes m_2$ where $m_2 = X_{hs} X_{ss}^{-1}$, and m_1 looks exactly like m, except that the power of X_{rs} in m is precisely one more than in m_1, and the power of X_{rh} in m is one less than in m_1; write this symbolically as $m_1 \sim m X_{rs}^{-1} X_{hr}$. The term $m_1 \otimes m_2$ is a function f of m, r, s, and h.

For all $m(t_{rs}) \in \operatorname{supp} x$ for which l' does not divide t_{rs}, pick r to be minimal. Let $m(t_{ij})$ be an element of $\operatorname{supp} x$ for which t_{rs} is not divisible by l', for some s, for this smallest r. We must have $r < s$, since by (13) if l' does not divide t_{ss}, it also does not divide t_{rs} with $r < s$, and r is minimal.

Now for this m, pick any h with $r \leqslant h < s$, and let $m_1 \otimes m_2 = f(m, r, s, h)$. This term $m_1 \otimes m_2$ does not get deleted when we apply $\zeta^\#$ to m_2. So a non-zero multiple of $m_1 \otimes m_2$ must occur when we apply Δ to some other monomial $m'(t'_{ij})$ in $\operatorname{supp} x$, in order for these occurrences of $m_1 \otimes m_2$ to cancel out. The only way this can happen is that $m_1 \otimes m_2 = f(m', r', s', h')$ for some integers r', h', and s' with $r' \leqslant h' < s'$ and with $t'_{r's'}$ not divisible by l'. Then

$$m_2 = X_{hs} X_{ss}^{-1} = X_{h's'} X_{s's'}^{-1}$$

so $s = s'$ and $h = h'$. Further,

$$m_1 \sim m' X_{r's}^{-1} X_{r'h}, \qquad m_1 \sim m' X_{rs}^{-1} X_{rh}.$$

Since $m \neq m'$, then $r' \neq r$. By minimality of r, we have $r < r'$. Then $r < r' \leqslant h < s$. We picked h to be any number such that $r \leqslant h < s$. If we pick $h = r$, we obtain $r < r' \leqslant h = r$, which is impossible.

Therefore t_{ij} is divisible by l' for all $m \in \operatorname{supp} x$ and all i, j. This means that each $m \in \operatorname{supp} x$ is in the image of the Frobenius map, and so is x. This completes the proof.

LEMMA 6. *Let M be a B'_q-module, and suppose that M is generated by weight 0 vectors $x \in M$ which are $(B'_q)_1 \cdot T$-fixed. Then $M \cong N^{(1)}$ for some B'-module N.*

Proof. From [PW, 2.8.1 and 2.8.2], M embeds in a direct sum $\bigoplus_i K[B'_q]$ of copies of $K[B'_q]$. For one of the weight 0 generators x of M, x embeds as (x_i) where each x_i is a $(B'_q) \cdot T$-fixed element of $K[B'_q]$. Then Lemma 5 tells us that each x_i is the Frobenius applied to an element y_i of $K[B']$. Let N be the submodule of $\bigoplus_i K[B']$ generated by all the elements $y = (y_i)$. Then $M \cong N^{(1)}$.

PROPOSITION 7. *If $\lambda \in X(T_q)$ is l'-restricted, then the irreducible $GL_q(n)$-module $L^q(\lambda)$ is still irreducible as a $GL_q(n)_1 \cdot T$-module.*

Proof. In 9.3.4 of [PW], it is shown (for odd l) that the restriction of $L^q(\lambda)$ to $GL_q(n)_1$ is irreducible, if λ is l-restricted (where $GL_q(n)_1$ is defined as the lth power Frobenius kernel). This certainly fails in our situation, but the proof of 9.3.4 in [PW] shows that in our case, the restriction of $L^q(\lambda)$ to $GL_q(n)_1 \cdot T$ is irreducible, provided that we can show that $L^q(\lambda)$ has a unique $(B'_q)_1 \cdot T$-stable line. The proof of 9.3.2 in [PW], that $L^q(\lambda)$ has a unique $(B'_q)_1$-stable line in the odd case, goes through for our situation, except that we cannot use equation [PW, (9.3a)] about the fixed point dimension of $M \otimes (-\mu)^{(B'_q)_1}$, where μ is a weight of $M = L^q(\lambda)$, since $K[(B'_q)_1]$ is too small, as we are using the $l/2$-Frobenius. Consider all the vectors x which generate $(B'_q)_1 \cdot T$-stable lines of M of weight μ, and define $M(\mu)$ to be the B'_q-submodule of $M \otimes (-\mu)$ generated by the vectors $x \otimes (-\mu)$. It follows from the previous lemma that $M(\mu)$ is a B'-module. Then the proof of 9.3.2 of [PW] goes through, to show that $L^q(\lambda)$ indeed has a unique $(B'_q)_1 \cdot T$-stable line. This completes the proof of Proposition 7.

LEMMA 8. *Let $M^{(1)}$ be the Frobenius twist of a rational $GL(n)$-module. Then the restriction of $M^{(1)}$ to $GL_q(n)_1 \cdot T$ is a direct sum of one-dimensional submodules.*

Proof. The module $M^{(1)}$ can be embedded in a submodule of a direct sum of copies of $F^{\#}K[GL(n)]$. We have $\Delta(X_{ij}^{l'}) = \sum_k X_{ik}^{l'} \otimes X_{kj}^{l'}$ by

Proposition 3. Letting $\zeta^{\#}: K[GL_q(n)] \to K[GL_q(n)_1 \cdot T]$ be the natural map, we have $(1 \otimes \zeta^{\#}) \circ \Delta(X_{ij}^{l'}) = X_{ij}^{l'} \otimes X_{jj}^{l'}$. It follows that the restriction to $GL_q(n)_1 \cdot T$ of the module $F^{\#}K[GL_q(n)]$ is a direct sum of one-dimensional submodules. This is also true for $M^{(1)}$.

THEOREM. *Let q be a primitive lth root of 1, where l is divisible by 4, and let $l' = l/2$. Let $\lambda = \lambda_0 + l'\lambda_1$ be a dominant weight in $X(T_q)$ such that λ_0 is l'-restricted. Then there is an isomorphism of $GL_q(n)$-modules*

$$L^q(\lambda) \cong L^q(\lambda_0) \otimes L(\lambda_1)^{(1)},$$

where $L(\lambda_1)$ is the irreducible rational $GL(n)$-module of highest weight λ_1.

Proof. We show that $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is irreducible. As $GL_q(n)_1 \cdot T$-module, $L(\lambda_1)^{(1)}$ is a direct sum of one-dimensional submodules, and $L^q(\lambda_0)$ is irreducible. Then the restriction of $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ to $GL_q(n)_1 \cdot T$ is completely reducible, with isotypic components $N(\mu)$, where

$$N(\mu) = L^q(\lambda_0) \otimes L(\lambda_1)_{\mu}^{(1)},$$

where $L(\lambda_1)_\mu$ is the μ weight space of $L(\lambda_1)$. An irreducible $GL_q(n)_1 \cdot T$-submodule Y of $N(\mu)$ has highest weight $\lambda_0 + l'\mu$ and has a maximal vector of the form $x \otimes y$ where x is a maximal vector of $L^q(\lambda_0)$ and y is a vector in $L(\lambda_1)_{\mu}^{(1)}$. Then Y is equal to $L^q(\lambda_0) \otimes y$, and any $GL_q(n)_1 \cdot T$-submodule of $N(\mu)$ has the form $L^q(\lambda_0) \otimes U$ for some K-subspace U of $L(\lambda_1)_{\mu}^{(1)}$. Suppose that M is a non-zero $GL_q(n)$-submodule of $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$. Then as $GL_q(n)_1 \cdot T$-module, M is the direct sum of its intersections with the isotypic components $N(\mu)$, and thus M equals $L^q(\lambda_0) \otimes V$ for some K-subspace V of $L(\lambda_1)^{(1)}$. Since M is a $GL_q(n)$-submodule of $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ we claim that this forces V to be a $GL_q(n)$-submodule of $L(\lambda_1)^{(1)}$. To see this, pick K-bases $\{w_k\}$ of $L^q(\lambda_0)$ and $\{v_1, v_2, ..., v_m\}$ of $L(\lambda_1)^{(1)}$ where the first r vectors $\{v_1, ..., v_r\}$ are a K-basis of V. Take elements $w \in L^q(\lambda_0)$, $w \neq 0$, $v \in V$. Suppose that

$$\tau_{L^q(\lambda_0)}(w) = \sum_k w_k \otimes a_k, \qquad \tau_V(v) = \sum_i v_i \otimes b_i,$$

where each $a_k \in K[GL_q(n)]$ and each $b_i \in F^{\#}K[GL(n)] \subset K[GL_q(n)]$. Then we have

$$\tau_{L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}}(w \otimes v) = \sum_{k,i} w_k \otimes v_i \otimes a_k b_i.$$

Since $w \otimes v$ is an element of the $GL_q(n)$ submodule $L^q(\lambda_0) \otimes V$, then $a_k b_i = 0$ if $i > r$. Since a_k cannot be 0 for all k, it follows that $b_i = 0$ if $i > r$, which proves that V is indeed a $GL_q(n)$-submodule of $L(\lambda_1)^{(1)}$.

Since $L(\lambda_1)^{(1)}$ is irreducible, then $V = L(\lambda_1)^{(1)}$, hence $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is irreducible.

Since $L^q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ has a B'_q-stable line of highest weight $\lambda_0 + l'\lambda_1$, it follows that

$$L^q(\lambda_0) \otimes L(\lambda_1)^{(1)} \cong L^q(\lambda_0 + l'\lambda_1).$$

This completes the proof.

Now assume that l is even but not divisible by 4. Proposition 2 must be changed, since $(-1)^{l'} = -1$, so the subalgebra $K[M_q(n)]^{l'}$ of $K[M_q(n)]$ generated by the l'th powers of the X_{ij}'s is no longer commutative: $X_{ri}^{l'}$ anti-commutes with $X_{rj}^{l'}$ if $i \neq j$, and $X_{ri}^{l'}$ anti-commutes with $X_{si}^{l'}$ if $r \neq s$. It follows that $K[M_q(n)]^{l'}$ is isomorphic to the coordinate ring $K[M_{-1}(n)]$ of quantum matrices at the parameter $q = -1$. Then the Frobenius map is defined by taking l'th powers of elements X_{ij} of the coordinate ring $K[GL_{-1}(n)]$ of the quantum general linear group at $q = -1$. Our tensor product theorem still holds, except that the Frobenius twist is applied not to $GL(n)$-modules but to $GL_{-1}(n)$-modules. We omit the details. The tensor product theorem for l twice an odd number is mentioned in [PW, 10.5.6], using the co-algebra isomorphism [PW, 10.5.4] between $K[GL_q(n)]$ and $K[GL_{-q}(n)]$.

References

[L] G. Lusztig, Modular representations and quantum groups, *Contemp. Math.* **82** (1989), 59–78.

[M] Yu. I. Manin, "Quantum Groups and Non-commutative Geometry," Université de Montréal, 1988.

[PW] B. Parshall and J.-P. Wang, Quantum linear groups, *Mem. Amer. Math. Soc.* **439** (1991).

[RTF] N. Yu. Reshetikhin, L. A. Takhtajan, and L. D. Fadeev, Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1990), 193–225.

[TT] E. Taft and J. Towber, Quantum deformation of flag schemes and Grassmann schemes. I. A q-deformation of the shape-algebra for $GL(n)$, *J. Algebra* **142** (1991), 1–36.

[T] M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras, *Manuscripta Math.* **7** (1972), 251–270.