1. INTRODUCTION

Least squares finite element methods for boundary value problems associated with partial differential equations have become more and more popular lately, mainly because they are not subject to the Brezzi-Babuska conditions, but also because these methods have a series of other advantages over classical finite element methods, such as freedom in choosing the finite element spaces, easy implementation and programming, application to a wide range of problems, and the fact that the resulting discrete system can be solved by a variety of algebraic methods, being symmetric and positive definite. An extensive coverage of these methods can be found in [1]. Also, a good outline of the advances made so far in developing these methods can be found in [2] and the references therein. A large number of studies that numerically demonstrate the efficiency of the least squares finite element methods seem to be ahead of the development of the mathematical context and the error analysis attached to these experiments. Nevertheless, the mathematical foundation is a key step to the full success of these methods, as it indicates the extent of efficiency and the directions to be taken for further development.

In the present article, we consider the problem

\[-\text{div}(A\nabla u) + qu = f, \quad \text{in } \Omega,\]  \hspace{1cm} (1.1)

\[u = 0, \quad \text{on } \Gamma_D,\] \hspace{1cm} (1.2)

\[(A\nabla u) \cdot \nu = 0, \quad \text{on } \Gamma_N,\] \hspace{1cm} (1.3)
where $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $3$), $\Omega$ is a connected bounded convex polygonal domain with a Lipschitz boundary $\partial \Omega$, $\Gamma_D \cup \Gamma_N = \partial \Omega$, the measure of $\Gamma_D$ is strictly positive, and $\nu$ is the outward unit vector directed normal to the boundary. Also, $A : (L^2(\Omega))^n \to (L^2(\Omega))^n$ is a linear operator, and $q = q(x)$ is a function defined for $x \in \Omega$.

If $A$ is the identity operator, the analysis of a least squares finite element method applied to this problem has been made in [3], and extended in [4] for the case where $A$ is an operator and $q = 0$. Also, a number of basic error estimates have been obtained in [5] and [6] for the case where $A$ is a symmetric $n \times n$ matrix and $A\nu$ represents the multiplication of $A$ by the vector $\nabla u$. In all these articles, the method is based on first writing problem (1.1)-(1.3) as a boundary value problem associated with a first-order system of partial differential equations, and then reformulating the problem as the minimization of a least squares functional associated with the first-order system, over an appropriate space. In [3] and [4], this space is a subspace of $H^1(\Omega) \times (H^1(\Omega))^n$, while in [5] it is a subspace of $H^1(\Omega) \times H^{div}(\Omega)$.

We shall follow the same procedure for problem (1.1)-(1.3), by first writing it as the following equivalent problem:

$$A \nabla u - \phi = 0, \quad \text{in } \Omega, \quad (1.4)$$

$$-\text{div} \phi + qu = f, \quad \text{in } \Omega, \quad (1.5)$$

$$u = 0, \quad \text{on } \Gamma_D, \quad (1.6)$$

$$\phi \cdot \nu = 0, \quad \text{on } \Gamma_N, \quad (1.7)$$

then formulating it as the minimization of a least squares functional. The goal of this article is to make the same type of analysis as made in [3] and [4], but over a subspace of $H^1(\Omega) \times H^{div}(\Omega)$, since this is the natural context that arises from the very formulation of the least squares problem, without additional smoothness assumptions (see also [7] and [8]). At the same time, while working on $H^1(\Omega) \times H^{div}(\Omega)$, we extend the results in [3] by considering the case where $A$ is a linear operator (not necessarily the identity operator), and extend the results in [4] by considering cases where $q$ is bounded, $q \neq 0$. We also extend and improve the results in [5], by proving additional error estimates for $u$ and $\phi$ in the $L^2$-norms (some of which assume the grid decomposition property of the finite element spaces [3]), and by considering the case where $A$ is an operator (not necessarily a symmetric matrix).

We now introduce a series of notations.

Denote by $(\cdot, \cdot)_0$, $(\cdot, \cdot)_1$, and $(\cdot, \cdot)_{\text{div}}$ the inner products on the spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H^{div}(\Omega)$, respectively, given by $(u, v)_0 = \int_{\Omega} uv$, $(u, v)_1 = \int_{\Omega} (uv + \nabla u \cdot \nabla v)$, and $(\phi, \psi)_{\text{div}} = \int_{\Omega} (\phi \cdot \psi + \text{div} \phi \cdot \text{div} \psi)$. Let $\| \cdot \|_0$, $\| \cdot \|_1$, and $\| \cdot \|_{\text{div}}$ be the norms induced by these inner products, respectively. We shall also use the notation $\| \cdot \|_{\text{div}}$ for the norm induced by the inner product $(\cdot, \cdot)_0$ on $(L^2(\Omega))^n$, i.e., $(\phi, \psi)_0 = \int_{\Omega} \phi \cdot \psi$. Denote by $A^*$ the adjoint operator of $A$ with respect to the $(L^2(\Omega))^n$ inner product. Let $\| \cdot \|_{-1}$ denote the norm on $H^{-1}(\Omega)$, defined by $\| f \|_{-1} = \sup_{u \in H^1(\Omega), \| u \|_1 = 1} |(f, u)_0|$, where $H^1(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma \}$. Let

$$\mathcal{V}_0 = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D \} \quad (1.8)$$

and

$$\mathcal{S}_0 = \{ \psi \in H^{\text{div}}(\Omega) : \psi \cdot \nu = 0 \text{ on } \Gamma_N \}. \quad (1.9)$$

Notice that for $v \in \mathcal{V}_0$ and $\psi \in \mathcal{S}_0$, the Stokes theorem gives

$$(\psi, \nabla v)_0 + (\text{div} \psi, v)_0 = 0. \quad (1.10)$$

Also notice that the Poincaré-Friedrichs inequality holds true for functions in $\mathcal{V}_0$; i.e., there exists a positive constant $C_F$ such that

$$\| \xi \|_0 \leq C_F \| \nabla \xi \|_0, \quad \text{for all } \xi \in \mathcal{V}_0. \quad (1.11)$$
Section 2 of this paper is dedicated to formulating the least squares problem, showing the existence and uniqueness of a solution, and deriving other basic results.

In Section 3, we apply a finite element method to the least squares formulation and derive basic error estimates for the couple \((u, \phi)\) in the norm of \(H^1(\Omega) \times H_{div}(\Omega)\). In addition, we derive \(L^2\)-error estimates separately for \(u\), and finally, assuming the grid decomposition property, we derive \(L^2\)-error estimates of the same order for \(\phi\).

Section 4 presents a number of computational results that support the estimates obtained in Section 3, and complement the results provided in [3-5].

### 2. LEAST SQUARES FORMULATION

Assume that for \(f \in L^2(\Omega)\), problem (1.1)-(1.3) has a unique solution. Also assume the unique solvability of problem (1.1)-(1.3) when \(A\) is replaced with \(A^\ast\). Note that if \(f \in L^2(\Omega)\), then the solution \(u\) of problem (1.1)-(1.3) will satisfy \(A\nabla u \in \mathcal{S}_0\), and (1.10) has the form

\[
(A\nabla u, \nabla v)_0 + (\text{div} A\nabla u, v)_0 = 0.
\]  

(2.1)

The same comments apply to \(A^\ast\).

Assume that \(\mathcal{S}_0\) is an invariant subspace for \(A\) and \(A : \mathcal{S}_0 \to \mathcal{S}_0\) is invertible. Note that in this case \(A^\ast : \mathcal{S}_0 \to \mathcal{S}_0\) is also invertible.

In the analysis that follows, we shall use the following hypotheses.

(i) Assume that there exists an \(\alpha > 0\) such that

\[
\alpha \|\phi\|_0 \leq (A\phi, \phi)_0, \quad \text{for all } \phi \in \mathcal{S}_0,
\]

and the same inequality holds true when \(A\) is replaced by \(A^{-1}\). Notice that in this case inequality (2.2) also holds true when \(A\) is replaced by \(A^\ast\) or \(A^{-1}\ast\).

(ii) Assume that there exists a \(\beta > 0\) such that

\[
(A\phi, \psi)_0 \leq \beta \|\phi\|_0 \|\psi\|_0, \quad \text{for all } \phi, \psi \in \mathcal{S}_0,
\]

and the same inequality holds true when \(A\) is replaced by \(A^{-1}\). Notice that in this case inequality (2.3) also holds true when \(A\) is replaced by \(A^\ast\) or \(A^{-1}\ast\). Also notice that inequality (2.3) implies

\[
\|A\phi\|_0 \leq \beta \|\phi\|_0, \quad \text{for all } \phi \in \mathcal{S}_0.
\]

(2.4)

(iii) Assume that there exists a \(\gamma \geq 0\) such that

\[
\gamma \leq \frac{\alpha}{C_F^2},
\]

and

\[
-\gamma \leq q(x) \leq \beta, \quad \text{for all } x \in \bar{\Omega}.
\]

We may assume without loss of generality that \(\gamma \leq \beta\).

Notice that Hypothesis (iii) includes the case where \(q(x) = 0\) for all \(x \in \bar{\Omega}\), and the case where \(\alpha \leq q(x) \leq \beta\) for all \(x \in \bar{\Omega}\). It also includes the case where \(A = I\) (the identity operator) and \(q(x) = -k^2\) for all \(x \in \bar{\Omega}\), which corresponds to the Helmholtz equation.

Whenever necessary, we shall assume that \(A\nabla \xi \in (H^{s-1}(\Omega))^n\), provided that \(\xi \in H^s(\Omega)\), where \(s \geq 1\). We make the same assumption for \(A^\ast\).

A least squares functional attached to system (1.4)-(1.7) is the following:

\[
J(v, \psi) := \|A\nabla v - \psi\|_0^2 + \| - \text{div} \psi + qv - f\|_0^2, \quad \text{for } v \in V_0 \text{ and } \psi \in \mathcal{S}_0.
\]

(2.7)
Now the given problem can be reformulated as the following optimization problem [3]:

$$\text{minimize } J(v, \psi) \text{ over all } v \in V_0 \text{ and } \psi \in S_0. \quad (2.8)$$

Let $B(\ldots)$ be the bilinear form on $V_0 \times S_0$, defined by

$$B((u, \phi), (v, \psi)) := (A\nabla u - \phi, A\nabla v - \psi)_0 + (-\text{div}\phi + qu, -\text{div}\psi + qv)_0. \quad (2.9)$$

The minimization problem (2.8) leads to the following least squares variational formulation, obtained by setting the first variation of $J$ equal to zero, which means:

find $(u, \phi) \in V_0 \times S_0$ such that

$$B((u, \phi), (v, \psi)) = (f, -\text{div}\psi + qv)_0, \quad \text{for all } (v, \psi) \in V_0 \times S_0. \quad (2.10)$$

The first result concerning the form $B$ is a key step in showing the existence and uniqueness of a solution for problem (2.8), and also in deriving error estimates for the finite element approximation of equation (2.10) that will follow in the next section.

**Theorem 1.** Assume that (i)-(iii) hold. Then there exist positive constants $C_1$ and $C_2$, independent of $u, \phi, v,$ and $\psi$, such that

$$C_1 (\|u\|_0^2 + \|\phi\|_{W_0}^2) \leq B((u, \phi), (u, \phi)), \quad (2.11)$$

and

$$|B((u, \phi), (v, \psi))| \leq C_2 (\|u\|_0^2 + \|\phi\|_{W_0}^2)^{1/2} (\|v\|_0^2 + \|\psi\|_{W_0}^2)^{1/2}. \quad (2.12)$$

**Proof.** For inequality (2.11), we use some of the ideas in [5] and [6]. We show that there exist positive constants $C_3'$ and $C_4'$ such that

$$C_3' \|u\|_0^2 \leq B((u, \phi), (u, \phi)) \quad (2.13)$$

and

$$C_4' \|\phi\|_{W_0}^2 \leq B((u, \phi), (u, \phi)), \quad (2.14)$$

so that (2.11) will hold with $2C_1 = \min\{C_3', C_4'\}$. In fact, due to the Poincaré-Friedrichs inequality (1.11), for (2.13) it is sufficient to show that there exists a positive constant $C_3$ such that

$$C_3' \|\nabla u\|_0^2 \leq B((u, \phi), (u, \phi)). \quad (2.15)$$

For this, let $t > 0$ be such that

$$t < \frac{2(\alpha - \gamma C_3')}{\alpha + C_3'}. \quad (2.16)$$

Notice that (1.10) implies the following identity:

$$B((u, \phi), (u, \phi)) = \|(A - tI)\nabla u - \phi\|_0^2 + \| - \text{div}\phi + (q - t)u\|_0^2$$

$$+ t^2 \|u\|_0^2 + 2t(\nabla u, u)_0 + 2t(A\nabla u, \nabla u)_0. \quad (2.17)$$

(here, $I$ denotes the identity operator on $(L^2(\Omega))^n$). Now if (i) and (iii) are used, and taking into account that the first two terms on the right-hand side are positive, we obtain

$$B((u, \phi), (u, \phi)) \geq -t^2 \|u\|_0^2 - t^2 \|\nabla u\|_0^2 - 2t\gamma \|u\|_0^2 + 2t\alpha \|\nabla u\|_0^2. \quad (2.18)$$

Inequality (1.11) further implies

$$B((u, \phi), (u, \phi)) \geq \{2t\alpha - t^2 - (t^2 + 2t\gamma) C_3'\} \|\nabla u\|_0^2, \quad (2.19)$$
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and therefore, (2.15) holds true with

$$C'_3 = 2t\alpha - t^2 - (t^2 + 2t\gamma)C'_p = t\left(1 + C'_p^2\right)\left\{\frac{2(\alpha - \gamma C'_p^2)}{\alpha + C'_p^2} - t\right\},$$

(2.20)

which is strictly positive, because (2.5) and (2.16) hold.

For (2.14), notice first that from the definition of $B$ we have

$$\|A\nabla u - \phi\|_0 \leq B((u, \phi), (u, \phi))^{1/2}$$

(2.21)

and

$$\| - \text{div}\phi + qu\|_0 \leq B((u, \phi), (u, \phi))^{1/2}.$$  

(2.22)

Combining the triangle inequalities

$$\|\phi\|_0 \leq \|A\nabla u - \phi\|_0 + \|A\nabla u\|_0$$

(2.23)

and

$$\|\text{div}\phi\|_0 \leq \| - \text{div}\phi + qu\|_0 + \|qu\|_0,$$

(2.24)

with (2.21), (2.22), (2.4), and (2.6), we obtain

$$\|\phi\|_{\text{div}} \leq \sqrt{2} \left(B((u, \phi), (u, \phi))^{1/2} + \beta\|u\|_1\right).$$

(2.25)

Now (2.14) follows from (2.13) and (2.25), and the proof of (2.11) is complete.

To see that inequality (2.12) holds true, notice that the definition of $B$ implies that

$$|B((u, \phi), (v, \psi))| \leq \|A\nabla u - \phi\|_0 \|A\nabla u - \psi\|_0 + \| - \text{div}\phi + qu\|_0 - \| - \text{div}\psi + qv\|_0.$$  

(2.26)

Now triangle inequalities (2.4) and (2.6) imply

$$\|A\nabla u - \phi\|_0 \leq \|A\nabla u\|_0 + \|\phi\|_0 \leq \beta + (1)(\|u\|_1 + \|\phi\|_{\text{div}})$$

(2.27)

and

$$\| - \text{div}\phi + qu\|_0 \leq \|\text{div}\phi\|_0 + \|qu\|_0 \leq \beta + (1)(\|u\|_1 + \|\phi\|_{\text{div}}),$$

(2.28)

and therefore, (2.26) implies

$$|B((u, \phi), (v, \psi))| \leq 4(\beta + 1)^2 \left(\|u\|_1^2 + \|\psi\|_{\text{div}}^2\right)^{1/2} \left(\|v\|_1^2 + \|\psi\|_{\text{div}}^2\right)^{1/2},$$

(2.29)

so that (2.12) holds true with $C_2 = 4(\beta + 1)^2.$

This theorem shows that the application

$$(u, \phi) \mapsto B((u, \phi), (u, \phi))^{1/2}$$

(2.30)

defines a norm over the product space $V_0 \times S_0,$ and this norm is equivalent with the norm

$$(u, \phi) \mapsto \left(\|u\|_1^2 + \|\phi\|_{\text{div}}^2\right)^{1/2}.$$  

(2.31)

For $(u, \phi) \in V_0 \times S_0,$ let

$$||(u, \phi)|| := B((u, \phi), (u, \phi))^{1/2}.$$  

(2.32)

Theorem 1 also shows that it is natural to consider problem (2.10) posed on a subspace of $H^1(\Omega) \times H_{\text{div}}(\Omega),$ and use the norm $||(\cdot)||$ to estimate the errors for the finite element approximation that will follow in Section 3 (see also [5]).

Note that inequalities (2.11) and (2.12) can also be written as follows:

$$C_1|||(u, \phi)|||^2 \leq B((u, \phi), (u, \phi)),$$

(2.33)

and

$$|D((u, \phi), (v, \psi))| \leq C_2|||(u, \phi)|| |||(v, \psi)|||,$$

(2.34)

and that for $f \in L^2(\Omega),$ (i)–(iii) imply the following inequality:

$$||(f, -\text{div}\psi + qv)||_0 \leq 2(1 + \beta)||f||_0|||(v, \psi)|||.$$  

(2.35)

Now taking into account the bilinearity of $B(\cdot, \cdot),$ the $H(\Omega) \times H_{\text{div}}(\Omega)$-continuity (2.34), the $H^1(\Omega) \times H_{\text{div}}(\Omega)$-ellipticity (2.33), and inequality (2.35), the following result is an immediate consequence of the Lax-Milgram theorem.
THEOREM 2. Assume that \( f \in L^2(\Omega) \) and that (i)–(iii) hold true. Then problem (2.8) has a unique solution \((u, \phi) \in V_0 \times S_0\).

In addition to the inequalities we proved so far, notice that the definition of \( B(\cdot, \cdot) \) immediately implies the following inequality:

\[
|B((u, \phi), (v, \psi))| \leq \|| (u, \phi) || (\| A\nabla u - \nabla \phi \|_0 + \| - \text{div} \psi + q v \|_0). \tag{2.36}
\]

As we already pointed out, the inequalities we proved so far show that the most natural setting for problem (2.8) is obtained by posing it on a subspace of \( H^1(\Omega) \times H_{\text{div}}(\Omega) \), with the norm \( \| \cdot \| \) defined by (2.32). This is also the approach taken in [5]. Another approach is possible by posing the problem on \( H^1(\Omega) \times (H^1(\Omega))^n \), which assumes extra smoothness conditions (see [3] and [4]).

In the next section, we make a finite element approximation of problem (2.10) and derive error estimates for it. The first series of results refer to basic error estimates, like the ones obtained in [5] when \( A \) is a matrix and \( q = 0 \).

A second series of results will lead to an \( L^2 \)-error estimate for \( u \). In the last part of Section 3, we use the technique of [3] and [4] to sharpen these \( L^2 \)-error estimates for \( \phi \), provided the finite element spaces use special types of grids on the domain \( \Omega \).

3. FINITE ELEMENT APPROXIMATION

Let \( h > 0 \) and \( \delta > 0 \) be discretization parameters, and let \( V^h_0 \) and \( S^h_0 \) be finite-dimensional subspaces of \( V_0 \) and \( S_0 \), respectively. We shall assume that both these spaces are associated with quasi-uniform grids on \( \Omega \) [9].

A finite element approximation of problems (2.10) can then be formulated as follows.

Find \((u_h, \phi_h) \in V^h_0 \times S^h_0\) such that

\[
B((u_h, \phi_h), (v^h, \psi^h)) = (f, -\text{div} \psi^h + q v^h)_0, \quad \text{for all } (v^h, \psi^h) \in V^h_0 \times S^h_0. \tag{3.1}
\]

As in the previous section, if \( f \in L^2(\Omega) \) and (i)–(iii) hold, then inequalities (2.11) and (2.12) show that problem (3.1) has a unique solution. Let

\[
e_h := u - u_h \tag{3.2}
\]

and

\[
e_\delta := \phi - \phi_h, \tag{3.3}
\]

where \((u, \phi)\) is the solution of problem (2.10), and \((u_h, \phi_h)\) is the solution of problem (3.1). As we already mentioned, if \( A \) is a symmetric matrix, optimal rates of convergence have been obtained for \( \| e_h \|_1 + \| e_\delta \|_{\text{div}} \) [5]. Nevertheless, a major improvement was made in [3] for the case where \( A = I \), by using grids with special properties, and thus, improving the order of \( \| e_\delta \|_0 \) over the order of \( \| \nabla e_h \|_0 \). The same technique was used in [4] for the case where \( q = 0 \). Notice that improving the order of convergence of \( \| e_\delta \|_0 \) over the order of \( \| \nabla e_h \|_0 \) is very important. It can be seen from the definition of the residual functional \( J \) that \( \phi_h \) is another approximation of \( A\nabla u \), so improving the order of \( \| e_\delta \|_0 \) means a better approximation of \( \nabla u \) than that accomplished by \( \nabla u_h \).

In what follows, we cover the following outline for the case where Hypotheses (i)–(iii) are satisfied:

- derive optimal error estimates for \( \| e_h \|_1 \) and for \( \| e_\delta \|_{\text{div}} \);
- derive optimal error estimates for \( \| e_h \|_0 \);
- assuming the grids on \( \Omega \) have special properties, derive optimal error estimates for \( \| e_\delta \|_0 \), that will improve the order of convergence of \( \| e_\delta \|_0 \) over that of \( \| \nabla e_h \|_0 \).

Numerical results that support these conclusions will be provided in Section 4.

To start the analysis, we first prove a result showing that solving equations (3.1) gives the best approximation to \((u, \phi)\), in the norm \( \| \cdot \| \), over the space \( V^h_0 \times S^h_0 \) (see also [3]).
THEOREM 3. Assume that (i)-(iii) hold true. Then
\[ ||(e_h, e_\delta)|| \leq ||(u - u^h, \phi - \psi^\delta)||, \quad \text{for all} \quad (u^h, \psi^\delta) \in V_0^h \times S_0^\delta. \] (3.4)

PROOF. To prove this inequality, we show that \((e_h, e_\delta)\) is orthogonal to the space \(V_0^h \times S_0^\delta\) in the space \(V_0 \times S_0\), with respect to the bilinear form \(B(, )\). For this, using the standard technique \[9\], first consider equation (2.10) for \((v, \psi) = (v^h, \psi^\delta) \in V_0^h \times S_0^\delta\) (this is possible, because \(V_0 \supset V_0^h\) and \(S_0 \supset S_0^\delta\)) and obtain
\[ B((v^h, \psi^\delta), (uh, +)) = (f, -\text{div} \psi^\delta + qv^h)_0, \quad \text{for all} \quad (v^h, \psi^\delta) \in V_0^h \times S_0^\delta. \] (3.5)

Now subtract (3.1) from (3.5), to obtain
\[ B((e_h, e_\delta), (uh, +)) = 0, \quad \text{for all} \quad (v^h, \psi^\delta) \in V_0^h \times S_0^\delta, \] (3.6)
and the proof is complete.

This result and the equivalence of the norms \((v, \psi) \mapsto B((v, \psi), (v, \psi))^{1/2}\) and \((v, \psi) \mapsto (\|v\|_1^2 + \|\psi\|_{\text{div}}^2)^{1/2}\) also imply the following result.

THEOREM 4. Assume that (i)-(iii) hold true. Then
\[ ||e_h||_1 + ||e_\delta||_{\text{div}} \leq C_4 \left( ||u - u^h||_1 + ||\phi - \psi^\delta||_{\text{div}} \right), \quad \text{for all} \quad (v^h, \psi^\delta) \in V_0^h \times S_0^\delta, \] (3.7)
where \(C_4 = (2C_2/C_1)^{1/2}\).

Assume that the spaces \(V_0^h\) and \(S_0^\delta\) have standard approximation properties, as follows.
(iv) There exist integers \(k \geq 1, l \geq 1\), and there exists a constant \(C_A\), independent of \(h\) and \(\delta\), such that for each \((v, \psi) \in V_0 \times S_0\), there exists a \((\tilde{v}^h, \tilde{\psi}^\delta) \in V_0^h \times S_0^\delta\), such that
\[ ||v - \tilde{v}^h||_t \leq C_A h^{k-t} ||v||_k, \quad t = 0, 1, \] (3.8)
\[ ||\psi - \tilde{\psi}^\delta||_0 \leq C_A \delta^l \|\psi\|_l, \] (3.9)
and
\[ ||\psi - \tilde{\psi}^\delta||_{\text{div}} \leq C_A \delta^{l-1} \||\psi||_l. \] (3.10)

For example, these inequalities are satisfied if \(V_0^h\) and \(S_0^\delta\) are spaces of piecewise polynomials of order \(k\) and \(l\), respectively, associated with the grids on \(\Omega\).

Now the following basic error estimates are a direct consequence of Theorem 4 and the approximation properties (see also [3,5]).

THEOREM 5. Assume that (i)-(iv) hold true. Then
\[ ||e_h||_1 + ||e_\delta||_{\text{div}} \leq K \left( h^{k-1} ||u||_k + \delta^{l-1} \|\phi\|_l \right) \] (3.11)
and
\[ ||(e_h, e_\delta)|| \leq K \left( h^{k-1} ||u||_k + \delta^{l-1} \|\phi\|_l \right), \] (3.12)
where \(K\) is a constant that does not depend on \(h\) or \(\delta\).

It is obvious that method (3.1) is highly practical when \(h = \delta\) and when the grids and the finite elements used for defining \(V_0^h\) and \(S_0^\delta\) are the same, because this makes the implementation and programming very easy. For example, if \(h = \delta\) and \(k = l\), inequality (3.11) becomes
\[ ||e_h||_1 + ||e_\delta||_{\text{div}} < K h^{k-1} (||u||_k + \|\phi\|_k); \] (3.13)
i.e., \( \| (e_h, e_h) \| \) is of order \( O(h^{k-1}) \). In what follows, we show that the optimal order of convergence of \( \| e_h \|_0 \) is improved by 1 over that of \( \| e_h \|_1 \). In addition, we show that if \( S_0^d \) satisfies the grid decomposition property (GDP), then the optimal order of convergence of \( \| e_\delta \|_0 \) is improved by 1 over the that of \( \| e_\delta \|_{\text{div}} \).

The following is a regularity hypothesis.

(v) Assume that \( A \) is such that if \( f \in L^2(\Omega) \) and \( u \) is the solution of (1.1)-(1.3), there exists a constant \( C_R > 0 \) such that the following inequality holds:

\[
\| u \|_1 \leq C_R \| f \|_0. \tag{3.14}
\]

Also assume that the same inequalities hold true when \( A \) is replaced by \( A^* \), and \( u \) is the solution of problem (1.1)-(1.3) with \( A \) replaced by \( A^* \).

First we prove a technical result.

**Theorem 6.** Assume that (i)-(v) hold true. Then

\[
\| -\text{div} e_\delta + qe_h \|_{-1} \leq C_6 \| (e_h, e_\delta) \| \left( h^{k-1} + \delta^{l-1} \right), \tag{3.15}
\]

where \( C_6 \) is a constant that does not depend on \( h \) or \( \delta \).

**Proof.** The proof follows the ideas of [3] and [4]. Let \( \theta \in H_0^1(\Omega) \) be arbitrary such that \( \| \theta \|_1 = 1 \). Let \( \xi \in V_0 \) be the solution of problem

\[
-\text{div}(A\nabla \xi) + q\xi = \theta, \quad \text{in } \Omega, \tag{3.16}
\]
\[
\xi = 0, \quad \text{on } \Gamma_D, \tag{3.17}
\]
\[
(A\nabla \xi) \cdot \nu = 0, \quad \text{on } \Gamma_N. \tag{3.18}
\]

The following identity follows immediately from the definition of \( B(\cdot, \cdot) \) and equation (3.16):

\[
B((e_h, e_\delta), (\xi, A\nabla \xi)) = (-\text{div} e_\delta + qe_h, \theta)_0. \tag{3.19}
\]

In addition, the orthogonality equation (3.6) and the bilinearity of \( B(\cdot, \cdot) \) imply that for all \( (\xi^h, \psi^\delta) \in V_0^h \times S_0^d \), the following holds:

\[
B((e_h, e_\delta), (\xi - \xi^h, A\nabla \xi - \psi^\delta)) = (-\text{div} e_\delta + qe_h, \theta)_0. \tag{3.20}
\]

Now using the last identity and (2.36), we obtain

\[
|(-\text{div} e_\delta + qe_h, \theta)_0| = |B((e_h, e_\delta), (\xi - \xi^h, A\nabla \xi - \psi^\delta))| \leq \| (e_h, e_\delta) \| \| A\nabla (\xi - \xi^h) - (A\nabla \xi - \psi^\delta) \|_0 \tag{3.21}
\]
\[
+ \| -\text{div} (A\nabla \xi - \psi^\delta) + q(\xi - \xi^h) \|_0.
\]

Now triangle inequalities, (2.4) and (2.6), give

\[
\| A\nabla (\xi - \xi^h) - (A\nabla \xi - \psi^\delta) \|_0 + \| -\text{div} (A\nabla \xi - \psi^\delta) + q(\xi - \xi^h) \|_0 \\
\leq \beta \| \xi - \xi^h \|_0 + \| A\nabla (\xi - \xi^h) \|_0 + \| (A\nabla \xi - \psi^\delta) \|_0 + \| \text{div} (A\nabla \xi - \psi^\delta) \|_0 \tag{3.22}
\]
\[
\leq 2\beta \| \xi - \xi^h \|_1 + 2 \| A\nabla \xi - \psi^\delta \|_{\text{div}}.
\]

Combining the last inequality with (3.21), taking infimum over \( \xi^h \in V_0^h \) and \( \psi^\delta \in S_0^d \), and using the approximation Properties (iv), we obtain

\[
|(-\text{div} e_\delta + qe_h, \theta)_0| \leq 2C_4 \| (e_h, e_\delta) \| \left( \beta h^{k-1} \| \xi \|_k + \delta^{l-1} \| A\nabla \xi \|_l \right), \tag{3.23}
\]

for all \( \theta \in H_0^1(\Omega) \), with \( \| \theta \|_1 = 1 \).
and therefore,
\[ \| \text{div} \epsilon_h + q_{eh}, \theta \|_0 \leq C_0 \| (e_{eh}, \epsilon_h) \| \left( h^{k-1} + \delta^{l-1} \right), \]
for all \( \theta \in H^1_0(\Omega) \), with \( \| \theta \|_1 = 1 \),

\[ (3.24) \]

where
\[ C_0 = 2C_A \max \{ \beta \| \xi \|_k, \| A \nabla \xi \|_i \}. \]

\[ (3.25) \]

Now take supremum over \( \theta \in H^1_0(\Omega) \) with \( \| \theta \|_1 = 1 \) in (3.24), to obtain (3.15).

We shall also use the following boundedness assumption on \( A \).

(vi) There exists a constant \( C_B > 0 \) such that the following inequality holds:

\[ \| \text{div} A^* \psi \|_{-1} \leq C_B \| \text{div} \psi \|_{-1}, \quad \text{for all } \psi \in \mathcal{S}_0, \]

\[ (3.26) \]

which can also be written as

\[ \| (\psi, A \nabla v)_0 \| \leq C_B \| (\psi, \nabla v)_0 \|, \quad \text{for all } v \in H^1_0(\Omega) \text{ and all } \psi \in \mathcal{S}_0. \]

\[ (3.26') \]

The following inequality is a technical result.

**Theorem 7.** Assume that (i)-(vi) hold true. Then

\[ B((e_{eh}, \epsilon_h), (v, \psi)) \leq C_T \|(e_{eh}, \epsilon_h)\| \| \text{div} (A \nabla v - \psi) \|_{-1} + \| - \text{div} \psi + q_{eh} \|_0, \]

for all \((v, \psi) \in \mathcal{V}_0 \times \mathcal{S}_0,\)

\[ (3.27) \]

where \( C_T \) is a positive constant that does not depend on \( h \) or \( \delta \).

**Proof.** Taking into account the definition of \( B(\cdot, \cdot) \), it is sufficient to show the following inequality:

\[ |(A \nabla e_{eh} - \epsilon_h, \psi)_0| \leq C_T \| (e_{eh}, \epsilon_h) \| \| \text{div} \psi \|_{-1}, \quad \text{for all } \psi \in \mathcal{S}_0, \]

\[ (3.28) \]

where \( C_T \) is a positive constant. For this, let \( \psi \in \mathcal{S}_0 \). Decompose \( A^* \psi \) as

\[ A^* = \psi \nabla p + \mu, \]

\[ (3.29) \]

where \( p \in \mathcal{V}_0, \mu \in \mathcal{S}_0, \text{div} \mu = 0, \) and

\[ \| \nabla p \|_0 \leq C_{R} \| \text{div} A^* \psi \|_{-1} \]

\[ (3.30) \]

(this can be done by solving the problem: \( -\text{div} \nabla p = \text{div} A^* \psi \) in \( \Omega, \) \( p = 0 \) on \( \Gamma_D, \text{div} \nabla p \cdot \nu = 0 \) on \( \Gamma_N, \) and then taking \( \mu := A^* \psi - \nabla p \)). We may assume without loss of generality that the constant \( C_{R} \) is the same one that appears in (v). Then we have

\[ (A \nabla e_{eh} - \epsilon_h, \psi)_0 = (A \nabla e_{eh} - \epsilon_h, A^{-1} (\nabla p + \mu)_0), \]

\[ (3.31) \]

and taking into account Theorem 3, this implies

\[ (A \nabla e_{eh} - \epsilon_h, \psi)_0 = (A \nabla e_{eh} - \epsilon_h, A^{-1} \nabla p)_0. \]

\[ (3.32) \]

Therefore,

\[ |(A \nabla e_{eh} - \epsilon_h, \psi)_0| \leq \| A \nabla e_{eh} - \epsilon_h \|_0 \| A^{-1} \nabla p \|_0. \]

\[ (3.33) \]

Using (ii), (3.30), and the definition of \( B \), the last inequality implies

\[ |(A \nabla e_{eh} - \epsilon_h, \psi)_0| \leq \beta C_{R} \| (c_h, c_{eh}) \| \| \text{div} A^* \psi \|_{-1}. \]

\[ (3.34) \]

This last inequality and (vi) imply (3.27), with \( C_T = \beta C_{B} C_{R}. \)
THEOREM 8. Assume that (i)-(iii) and (vi) hold true. Then
\[ (\varepsilon_\delta, \psi_\delta)_0 = 0, \quad \text{for all } \psi_\delta \in \mathcal{S}_0^\delta, \quad \text{with } \nabla \psi_\delta = 0. \tag{3.35} \]

PROOF. Orthogonality (3.6) for \( \nu^h = 0 \) and \( \psi_\delta \in \mathcal{S}_0^\delta \) with \( \nabla \psi_\delta = 0 \) gives
\[ (A\nabla e_h - \varepsilon_\delta, \psi_\delta)_0 = 0, \tag{3.36} \]
and therefore,
\[ |(\varepsilon_\delta, \psi_\delta)_0| = |(A\nabla e_h, \psi_\delta)_0| = |(\nabla e_h, A^* \psi_\delta)_0| \]
\[ = |(e_h, \nabla A^* \psi_\delta)_0| \leq \|e_h\|_1 \|\nabla A^* \psi_\delta\|_{-1}, \tag{3.37} \]
and using (vi) we have
\[ |(\varepsilon_\delta, \psi_\delta)_0| \leq C_B \|e_h\|_1 \|\nabla \psi_\delta\|_{-1} = 0, \tag{3.38} \]
so that (3.35) is true.

OBSERVATION. Notice that, in general, if Condition (vi) does not hold, the orthogonality relationship \( (\varepsilon_\delta, A^{-1*} \psi_\delta)_0 = 0 \) for all \( \psi_\delta \in \mathcal{S}_0^\delta \) with \( \nabla \psi_\delta = 0 \) cannot be derived from (3.6) by letting \( v \in \mathcal{V}_0 \) solve \( -\nabla(AVv) + qv = -\nabla A^{-1*} \psi_\delta \) in \( \Omega \), \( v = 0 \) on \( \Gamma_D \), \( (AVv) \cdot \nu = 0 \) on \( \Gamma_N \) (where \( \nabla \psi_\delta = 0 \)), and letting \( \psi := AVv - A^{-1*} \mu_\delta \), simply because this couple \( (v, \psi) \) is not guaranteed to be in \( \mathcal{V}_0^h \times \mathcal{S}_0^\delta \), but only in \( \mathcal{V}_0 \times \mathcal{S}_0 \) (see [4]).

(vii) Assume that matrix \( A \) is such that for all \( f \in L^2(\Omega) \), the following problem has a unique solution \( \zeta \):
\[ \begin{align*}
\nabla A \nabla \xi &= f, & \text{in } \Omega, \\
\xi &= 0, & \text{on } \Gamma_D, \\
(A \nabla \xi) \cdot \nu &= 0, & \text{on } \Gamma_N,
\end{align*} \tag{3.39} \]
and, in addition,
\[ \|\xi\|_2 \leq C_R \|f\|_0, \tag{3.40} \]
where we may assume that \( C_R \) is the same constant with the one that appears in Hypothesis (v).

For simplicity, in what follows we shall assume that \( k \geq 2 \) and \( l \geq 2 \), even though the same analysis can be carried out for \( k \geq 2 \) and \( l \geq 1 \).

THEOREM 9. Assume (i)-(vii) hold true. Then
\[ \|e_h\|_0 \leq K_1(h + \delta) \left( h^{k-1}\|u\|_k + \delta^{l-1}\|\phi\|_l \right), \tag{3.43} \]
where \( K_1 \) is a positive constant that does not depend on \( h \) or \( \delta \).

PROOF. Let \( \eta \in H^1(\Omega) \) be the solution of the adjoint problem
\[ \begin{align*}
-\nabla A^* \nabla \eta + q \eta &= e_h, & \text{in } \Omega, \\
\eta &= 0, & \text{on } \Gamma_D, \\
(A^* \nabla \eta) \cdot \nu &= 0, & \text{on } \Gamma_N.
\end{align*} \tag{3.44} \]

Then (v) implies
\[ \|\eta\|_1 \leq C_R \|e_h\|_0. \tag{3.47} \]
Now let $\xi \in H^1(\Omega)$ be the solution of problem (3.39)-(3.42) with $f = \text{div}\nabla$, i.e.,

$$\text{div}A\nabla\xi = \text{div}\nabla, \quad \text{in } \Omega, \quad (3.48)$$
$$\xi = 0, \quad \text{on } \Gamma_d, \quad (3.49)$$
$$(A\nabla\xi) \cdot \nu = 0, \quad \text{on } \Gamma_N, \quad (3.50)$$

so, taking into account (i), we have

$$\alpha \|\nabla\xi\|_0 \leq \|\nabla\nabla\|_0. \quad (3.51)$$

Using the Poincaré-Friedrichs inequality (1.11), the following inequality can also be obtained:

$$\|\xi\|_1 \leq \frac{(C_F + 1)C_R}{\alpha} \|e_h\|_0. \quad (3.52)$$

Now notice that the following identity holds:

$$B((e_h, e_\delta), (\xi, A\nabla\xi - \nabla\eta)) = \|e_h\|_0^2 + (-\text{div}e_\delta + qe_h, q\xi - \eta)_0. \quad (3.53)$$

Hypothesis (iii) implies

$$|(-\text{div}e_\delta + qe_h, q\xi - \eta)_0| \leq \beta \|\text{div}e_\delta + qe_h, \xi\|_0 + \|(-\text{div}e_\delta + qe_h, \eta)_0\| \quad (3.54)$$
$$\leq \beta \|\text{div}e_\delta + qe_h\|_{-1} + \|\xi\|_1 + \|\eta\|_1,$$

so that, using inequalities (3.47) and (3.52), we obtain

$$|(-\text{div}e_\delta + qe_h, q\xi - \eta)_0| \leq C_R \left(\frac{\beta}{\alpha}(C_F + 1) + 1\right) \|\text{div}e_\delta + qe_h\|_{-1} \|e_h\|_0. \quad (3.55)$$

On the other side, the orthogonality relationship (3.6) and the bilinearity of $B$ imply that for all $\xi^h \in V^h_0$ we have

$$B((e_h, e_\delta), (\xi - \xi^h, A\nabla\xi - \nabla\eta)) = B((e_h, e_\delta), (\xi, A\nabla\xi - \nabla\eta)). \quad (3.56)$$

Now this identity and Theorem 7 yield

$$|B((e_h, e_\delta), (\xi - \xi^h, A\nabla\xi - \nabla\eta))| \leq C_T \|(e_h, e_\delta)\| \left(\|\text{div}(A\nabla(\xi - \xi^h) - (A\nabla\xi - \nabla\eta))\|_{-1} + \|\text{div}(A\nabla\xi - \nabla\eta) + q(\xi - \xi^h)\|_0\right), \quad (3.57)$$

and since (3.48) and (iii) hold, this shows that

$$|B((e_h, e_\delta), (\xi - \xi^h, A\nabla\xi - \nabla\eta))| \leq C_T \|(e_h, e_\delta)\| \left(\|\text{div}(\nabla\eta - A\nabla\xi^h)\|_{-1} + \beta \|\xi - \xi^h\|_0\right), \quad \text{all } \xi^h \in V^h_0. \quad (3.58)$$

The next step is to show that

$$\|\text{div}(\nabla\eta - A\nabla\xi^h)\|_{-1} \leq \|A\nabla(\xi - \xi^h)\|_0. \quad (3.59)$$

Let $\theta \in H^1_0(\Omega)$. Then (3.48) implies

$$(\text{div}(\nabla\eta - A\nabla\xi^h), \theta)_0 = (\text{div}(A\nabla\xi - A\nabla\xi^h), \theta)_0 = \|A\nabla(\xi - \xi^h, \nabla\theta)_0, \quad (3.60)$$
so that
\[
|\text{div} (\nabla \eta - A \nabla \xi^h)\cdot \theta_0| \leq \|A \nabla (\xi - \xi^h)\|_0 \|\nabla \theta\|_0 \leq \|A \nabla (\xi - \xi^h)\|_0 \|\theta\|_1. \tag{3.61}
\]

Now taking the supremum over \(\theta \in H_0^1(\Omega), \|\theta\|_1 = 1\), we obtain (3.59). Combining (3.58), (3.59), and (ii) yields
\[
B ((e_h, e_\delta), (\xi - \xi^h, A \nabla \xi - \nabla \eta)) \leq \beta C_T \|\| (e_h, e_\delta)\|\| (\|\nabla (\xi - \xi^h)\|_0 + \|\xi - \xi^h\|_0), \tag{3.62}
\]
so that the approximation Properties (iv) further give
\[
B ((e_h, e_\delta), (\xi - \xi^h, A \nabla \xi - \nabla \eta)) \leq \beta C_T C_A \|\| (e_h, e_\delta)\|\| \|\xi\|_2, \tag{3.63}
\]
and therefore, since (3.42) holds, we have
\[
B ((e_h, e_\delta), (\xi - \xi^h, A \nabla \xi - \nabla \eta)) \leq \beta C_T C_A C_R \|\| (e_h, e_\delta)\|\| \|\eta\|_2. \tag{3.64}
\]

Now combining this last inequality with (3.47), we obtain
\[
B ((e_h, e_\delta), (\xi - \xi^h, A \nabla \xi - \nabla \eta)) \leq \beta C_T C_A C_R \|\| (e_h, e_\delta)\|\| \|e_h\|_0. \tag{3.65}
\]

Putting together (3.53), (3.55), and (3.65), we obtain
\[
\|e_h\|_0^2 \leq \left\{ C_R \left( \frac{\beta}{\alpha} (C_F + 1) + 1 \right) \| - \text{div} e_\delta + q e_h \|_{-1} + 2 \beta C_T C_A C_R \| (e_h, e_\delta)\| \right\} \|e_h\|_0, \tag{3.66}
\]
so that
\[
\|e_h\|_0 \leq C_R \left( \frac{\beta}{\alpha} (C_F + 1) \| - \text{div} e_\delta + q e_h \|_{-1} + 2 \beta C_T C_A C_R \| (e_h, e_\delta)\| \right). \tag{3.67}
\]

Now the last inequality and Theorem 6 imply
\[
\|e_h\|_0 \leq C (h + \delta) \|\| (e_h, e_\delta)\|\|, \tag{3.68}
\]
where \(C\) is a combination of \(\alpha, \beta, C_R, C_F, C_A, C_0, \) and \(C_T\), so, taking into account Theorem 5, we obtain (3.43) with \(K_1 = CK\).

The error estimates that follow will use the following hypothesis [3].

\(\text{(viii)}\) Assume that the space \(S_0^\delta\) has the grid decomposition property (GDP), which means that there exists a positive constant \(C_G\) such that for every \(\psi^\delta \in S_0^\delta\), there exist \(\lambda^\delta, \mu^\delta \in S_0^\delta\) satisfying
\[
\psi^\delta = \lambda^\delta + \mu^\delta, \tag{3.69}
\]
\[
\text{div} \mu^\delta = 0, \tag{3.70}
\]
\[
(\lambda^\delta, \mu^\delta)_{0, \Omega} = 0, \tag{3.71}
\]
\[
\|\lambda^\delta\|_0 \leq C_G \|\| \text{div} \psi^\delta\|_{-1}. \tag{3.72}
\]

The simplest example of finite element spaces having the GDP, given in [10], is the space of piecewise linear functions associated with a criss-cross grid on a two-dimensional domain \(\Omega\). Notice that the space of piecewise linear functions associated with a directional grid does not have this property [10].
THEOREM 10. Assume that (i)-(viii) hold. Then the following inequality holds true:

$$\|\epsilon_\delta\|_0 \leq K_2(h + \delta) \left( h^{k-1} + \delta^{l-1} \right) \|\phi_\delta\|_0,$$

where $K_2$ is a positive constant that does not depend on $h$ or $\delta$.

PROOF. We follow the idea of proof in [3]. Let $(u_h, \phi_\delta)$ be the solution of equation (3.5), and let $\hat{\phi}_\delta \in S^0_\delta$ be the best approximation to $\phi_\delta$, in the sense that (3.9) holds true. Then we have $\phi_\delta - \hat{\phi}_\delta \in S_0$. Since GDP is satisfied, there exist $\lambda_\delta, \mu_\delta \in S^0_\delta$ such that

$$\phi_\delta - \hat{\phi}_\delta = \lambda_\delta + \mu_\delta,$$

$$\text{div} \mu_\delta = 0,$$

$$\langle \lambda_\delta, \mu_\delta \rangle_0 = 0,$$

and

$$\|\lambda_\delta\|_0 \leq C_G \left\| \text{div} \left( \phi_\delta - \hat{\phi}_\delta \right) \right\|_{-1}.$$

Now (3.77), triangle inequalities, and the fact that

$$\|\text{div} \psi\|_{-1} \leq \|\psi\|_0$$

imply that

$$\|\lambda_\delta\|_0 \leq C_G \left( \|\text{div} \epsilon_\delta + qe_h\|_{-1} + \|qe_h\|_{-1} + \|\phi - \hat{\phi}_\delta\|_0 \right).$$

Theorem 6, Hypothesis (iii), and the approximation Properties (iv) further imply

$$\|\lambda_\delta\|_0 \leq C_G \left( \|\epsilon_h(\epsilon_h, \epsilon_\delta)\| \left( h^{k-1} + \delta^{l-1} \right) + \|qe_h\|_{-1} + h \delta \|\phi\|_1 \right).$$

Now the embedding inequality

$$\|qe_h\|_{-1} \leq C_I \|qe_h\|_0$$

(where $C_I$ is a positive constant), (iii), Theorem 5, and Theorem 9 imply the inequality

$$\|\lambda_\delta\|_0 \leq C(h + \delta) \left( h^{k-1} + \delta^{l-1} \right) \|\phi_\delta\|_0,$$

where $C$ is a positive constant (a combination of $\beta, C_I, C_\lambda, C_G, C_\delta, C_\epsilon, K_1$).

We now find a similar type of estimate for $\mu_\delta$. Since $\text{div} \mu_\delta = 0$, Theorem 8 gives

$$\langle \epsilon_\delta, \mu_\delta \rangle_0 = 0.$$ (3.83)

Now (3.74) and (3.83) give

$$\|\mu_\delta\|_0^2 = \langle \phi_\delta - \hat{\phi}_\delta - \lambda_\delta, A^{-1} \mu_\delta \rangle_0 = \langle \phi - \hat{\phi}_\delta, \mu_\delta \rangle_0 - \langle \lambda_\delta, \mu_\delta \rangle_0,$$

so that

$$\|\mu_\delta\|_0^2 \leq \|\mu_\delta\|_0 \left( \|\phi - \hat{\phi}_\delta\|_0 + \|\lambda_\delta\|_0 \right),$$

which implies

$$\|\mu_\delta\|_0 \leq \left( \|\phi - \hat{\phi}_\delta\|_0 + \|\lambda_\delta\|_0 \right).$$

Therefore, triangle inequalities, (3.74), and the last inequality imply

$$\|\epsilon_\delta\|_0 \leq \|\phi - \hat{\phi}_\delta\|_0 + \|\phi_\delta - \hat{\phi}_\delta\|_0 \leq \|\phi - \hat{\phi}_\delta\|_0 + \|\lambda_\delta\|_0 + \|\mu_\delta\|_0 \leq 2 \left( \|\phi - \hat{\phi}_\delta\|_0 + \|\lambda_\delta\|_0 \right).$$

Finally, using the approximation Properties (iv) and inequality (3.82), we obtain the desired inequality.

For example, if $k = l$ and $h = \delta$, we obtain the fact that $\|\epsilon_h\|_0$ is of order $O(h^k)$, which agrees with the numerical results provided in [3], and will also be demonstrated by our numerical results in Section 4.

In the next section, we present numerical results that support Theorems 5, 9, and 10.
4. NUMERICAL RESULTS

A number of results already confirm the error estimates in Section 3 for particular cases. For example, the numerical results obtained in [3] for $A = I, q = -k^2$ (a strictly negative constant), and $\Omega$ a square in $\mathbb{R}^2$; these results agree with Theorems 9 and 10 in Section 3. Also, the results in [4] (where $A = I, q = 0$, and $\Omega$ is a domain in $\mathbb{R}^2$ with a circular hole) confirm the fact that the method converges; and the results in [5] (where $A$ is a $2 \times 2$ diagonal matrix with equal diagonal entries, $q = 0$, and $\Omega$ is a square in $\mathbb{R}^2$) support the estimates of Theorem 5.

In what follows, we present additional numerical results we obtained for a wider range of examples, which come in support of Theorems 5, 9, and 10. The exact solution is $u = \sin(\pi x) \sin(\pi y)$ on $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$. We present below the results obtained for Dirichlet boundary conditions; for mixed boundary conditions, similar results have been obtained. We took $h = \delta$ and we used the same type of basis functions for approximating $u$, as well as each component of $\phi$ (for example, if $u_h$ is a sum of piecewise linears on a directional triangular grid, each of the two components of $\phi_h$ is also a sum of piecewise linears on a directional triangular grid).

A first set of examples we studied refers to the case where $A = I$ and $q$ varies, and the finite element spaces consist of piecewise linear functions on a union-jack grid (Figure 1b), so that $k = l = 2$. These results can be seen in Figure 2, containing $\|e_h\|_0$, $\|e_h\|_1$, $\|e_h\|_{\text{div}}$, respectively, plotted on a logarithmic scale, for different functions $q$, like $q = 1, 0, -1/8, -1, -5, -10$. It can be seen that the convergence of $\|e_h\|_1$ and $\|e_h\|_{\text{div}}$ in all these graphs agrees with Theorem 5 (i.e., the slopes are $-2$), convergence of $\|e_h\|_0$ agrees with Theorem 9 (i.e., the slopes are $-1$), and convergence of $\|e_h\|_{\text{div}}$ agrees with Theorem 10 (i.e., the slopes are $-2$). As $q$ approaches the eigenvalue $-2\pi^2$ (like $q = -10$), it can be seen that the convergence rate is attained slower. For $q = -2\pi^2$ the method does not converge. Another observation is that the rates obtained for the case where the inequality $\gamma < \alpha/C_F^2$ is satisfied (like $q = 1$, where $\alpha = 1, \gamma = 0$) have also been obtained for some cases where this inequality does not hold (like $q = -1/8$, where $\alpha = 1, \gamma = 1/8, C_F = 2\sqrt{2}$, and this is in some sense a "limit case", because $\gamma = \alpha/C_F^2$; but also $q = -1$, where $\alpha = 1, \gamma = 1, C_F = 2\sqrt{2}$, so $\gamma > \alpha/C_F^2$). This suggests the fact that condition $\gamma < \alpha/C_F^2$ can probably be improved, in the sense that Hypothesis (iii) can probably be replaced by a less restrictive one.

A second set of examples uses $A = I$, a nonconstant $q$, and different types of finite element spaces. The results are reported in Figure 3. They were obtained for $q = xy + 1$ and six different finite element spaces of continuous functions on grids like the ones shown in Figure 1:

- piecewise linear functions on directional triangles (ldt);
- piecewise linear functions on union-jack triangles (lut);
- piecewise bilinear functions on rectangles (blr);
- piecewise quadratic functions on directional triangles (qdt);
- piecewise quadratic functions on union-jack triangles (qut);
- piecewise biquadratic functions on rectangles (bqr).
Figure 2. The errors for $A = I$, piecewise linears on union-jack triangles ($k = l = 2$ and GDP satisfied).
Figure 3. The errors for $A = I$, $q = xy + 1$, different grids.

(a) $\|\epsilon_h\|_0$.  
(b) $\|\epsilon_h\|_1$.  
(c) $\|\epsilon_h\|_0$.  
(d) $\|\epsilon_h\|_{div}$.  

Error vs. $1/h$ for different grids.
Table 1. The errors and error rates for $A = (a_{ij})_{1 \leq i,j \leq 2}$, $a_{11} = x + 1, a_{12} = 1, a_{21} = -1, a_{22} = y + 1, q = 1$, for piecewise linears on union-jack triangles ($k = l = 2$ and GDP is satisfied).

<table>
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<th>Rate</th>
<th>$|e_h|_1$</th>
<th>Rate</th>
<th>$|\varepsilon|_0$</th>
<th>Rate</th>
<th>$|\varepsilon|_{div}$</th>
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<td>0.68424</td>
<td>1.96</td>
<td>0.39772(+1)</td>
<td>0.99</td>
</tr>
<tr>
<td>8</td>
<td>0.28455(-1)</td>
<td>1.98</td>
<td>0.41336</td>
<td>1.00</td>
<td>0.17753</td>
<td>2.00</td>
<td>0.20007(+1)</td>
<td>1.00</td>
</tr>
<tr>
<td>16</td>
<td>0.71959(-2)</td>
<td>2.00</td>
<td>0.20665</td>
<td>2.00</td>
<td>0.043763(-1)</td>
<td>2.00</td>
<td>0.99983</td>
<td></td>
</tr>
</tbody>
</table>

Of all these spaces, the space of piecewise linears on union-jack triangles is the only one that has the GDP. The graphs of $\|e_h\|_1$, $\|\varepsilon\|_{div}$, and $\|\varepsilon\|_0$ show that in all six cases the numerical results agree with Theorems 5 and 9 (i.e., the slopes for $\|e_h\|_1$ and $\|\varepsilon\|_{div}$ are $-1$ in the case of linears and bilinears, and $-2$ in the case of quadratics and biquadratics; the slopes for $\|\varepsilon\|_0$ are $-2$ in the case of linears and bilinears, and $-3$ in the case of quadratics and biquadratics). The graph of $\|\varepsilon\|_0$ agrees with Theorem 10 (i.e., the slopes are $-2$ in the case of linears on directional triangles and bilinears, and $-3$ in the case of linears on union-jack triangles, as well as in the case of quadratics and biquadratics) and shows that the condition on the finite element space to have the GDP is essential in deriving this estimate, since the rate of convergence stated by Theorem 10 is obtained only for the space of piecewise linears on union-jack triangles (the only space that has the GDP).

A third set of examples refers to the case where $A$ is a $2 \times 2$ matrix whose entries are functions, and $q$ is a function. We present the results for the space of piecewise linears on union-jack triangles for $A = (a_{ij})_{1 \leq i,j \leq 2}$, $a_{11} = x + 1, a_{12} = 1, a_{21} = -1, a_{22} = y + 1, q = 1$ in Table 1 and Figure 4a. The results demonstrate the validity of Theorems 5, 9, and 10.
For the latter choice of $A$ and $q$, the difference of results obtained by using linears on union- jack triangles and linears on directional triangles can be seen by comparing Figures 4a and 4b, respectively. In Figure 4a, the slopes for $\|e_h\|_1$ and $\|e_\delta\|_{div}$ are $-1$, and the slopes for $\|e_h\|_0$ and $\|e_\delta\|_0$ are $-2$, while in Figure 4b only the slope for $\|e_h\|_0$ is $-2$, and the other three slopes are $-1$.

5. CONCLUSIONS AND FUTURE WORK

The error analysis of a least squares finite element method for solving second-order problems has been made for certain elliptic cases. The analysis extends and improves previous work made in [3-6], and refers to partial differential equations with homogeneous boundary conditions. The numerical results presented here support the theoretical conclusions, and extend previous numerical work. Similar numerical experiments yield the same conclusions for problems with nonhomogeneous boundary conditions, suggesting the fact that this analysis could be extended to nonhomogeneous problems of this type. Also, the results presented here suggest that condition (2.5) is too restrictive, and the analysis could be valid in a larger context. These issues will be the object of future work.

REFERENCES