



# Fuzzy Preference Modeling and Its Application to Multiobjective Decision Making

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**Abstract**—This paper reflects results of research into the construction and analysis of  $\langle X, R \rangle$  models within the framework of a general approach to solving optimization problems with fuzzy coefficients. This approach involves a modification of traditional mathematical programming methods and is associated with formulating and solving one and the same problem within the framework of mutually interrelated models. The use of the approach allows one to maximally cut off dominated alternatives. The subsequent contraction of the decision uncertainty region is based on reducing the problem to models of multiobjective choosing alternatives in a fuzzy environment with the use of fuzzy preference relation techniques for analyzing these models. Three techniques for fuzzy preference modeling are discussed in the paper. The first technique is based on the construction of membership functions of subsets of nondominated alternatives with simultaneous considering of all criteria (fuzzy preference relations). The second technique is of a lexicographic character and consists of step-by-step introducing of fuzzy preference relations. The third technique is based on aggregating membership functions of subsets of nondominated alternatives corresponding to each preference relation. These techniques have served for developing a corresponding system for multiobjective decision making (MDMS). C++ windows of the MDMS are presented for input, output, and some intermediate procedures. The results of the paper are of a universal character and are already being used to solve power engineering, naval engineering, and management problems. © 2006 Elsevier Ltd. All rights reserved.

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## 1. INTRODUCTION

In the process of posing and solving a wide range of problems related to the design and control of complex systems, one inevitably encounters different kinds of uncertainty [1]. Taking into account the uncertainty in shaping the mathematical models should be inherent to the practice of systems analysis. This serves as a means for increasing the adequacy of these models and, as a result, their credibility and the factual effectiveness of solutions based on their analysis.

At present, investigators have doubts about the validity or, at least, the expediency of including the uncertainty within the framework of models that are shaped by traditional approaches. In general, these approaches do not ensure an adequate or sufficiently rational consideration of the uncertainty throughout the entire spectrum of its manifestations. Considering this, the application of the fuzziness concept to the systems to be studied may play a significant role in overcoming the indicated difficulties. Use of this concept opens a natural path to giving up "excessive" precision, which is inherent in the traditional approaches to constructing models while preserving reasonable rigor. Besides, operation with a fuzzy parameter space allows one not only to be oriented toward the contextual or intuitive aspect of qualitative analysis as a fully substantiated process, but, by means of fuzzy set theory [2,3], also to use this approach as a reliable source for obtaining quantitative information. Finally, fuzzy set theory allows one to reflect more adequately the essence of the decision-making process. In particular, since the "human" factor has a noticeable effect in making decisions, it is expedient to use the important linguistic aspect of fuzzy set theory [2,3].

Nevertheless, in solving problems under condition of uncertainty it is necessary to exert maximum efforts in seeking the possibilities for overcoming the uncertainty. This is done, for example, by using information of informal character (based on experience, knowledge, and intuition of specialists) or, in the general case, by aggregating information arriving from various sources of both formal and informal nature [4]. Here we are essentially speaking of the fact that the characteristic of uncertain information (usually specified by intervals) may and should be supplemented by specifically adopted, well-founded assumptions as to differentiated reliability of different values of uncertain parameters. Such supplementation represents a generalization of the interval specification of information and serves as a technique for removing the uncertainty, but it requires the use of the corresponding apparatus. The apparatus of fuzzy set theory can serve as the latter. Its utilization in problems of complex system optimization offers advantages of both fundamental nature (based on the possibility of validly obtaining the more effective, less "cautious solutions") and computational character [4,5].

When using fuzzy set theory, certain fundamental problems arise in the comparison of alternatives on the basis of fuzzy values of objective functions, consideration of constraints containing fuzzy coefficients, development of fundamental principles, and concrete methods of solving associated optimization problems. Below we discuss some approaches to solving these problems and propose ways of implementing these approaches as applied to discrete programming models with fuzzy coefficients both in the objective function and constraints with the use of modification of the algorithms of discrete optimization and contraction of arising decision uncertainty regions on the basis of procedures of multiobjective choosing alternatives in a fuzzy environment.

## 2. PROBLEM FORMULATION AND TRANSFORMATION

Numerous problems related to the design and control of complex systems may be formulated as follows:

$$\text{maximize } \tilde{f}(x_1, \dots, x_n), \quad (1)$$

subject to the constraints

$$\tilde{g}_j(x_1, \dots, x_n) \subseteq b_j, \quad j = 1, \dots, m, \tag{2}$$

where the objective function (1) and constraints (2) include fuzzy coefficients, as indicated by the tilde ( $\tilde{\phantom{x}}$ ) symbol.

Given a maximization problem (1),(2), we can state the following problem:

$$\text{minimize } \tilde{f}(x_1, \dots, x_n), \tag{3}$$

subject to the same constraints (2).

An approach [1] to handling the constraints such as (2) involves replacing each of them by a finite set of nonfuzzy constraints. Depending on the sense of the problem, it is possible to convert constraints (2) to equivalent nonfuzzy analogs

$$g_j(x_1, \dots, x_n) \leq b_j, \quad j = 1, \dots, d' \geq m, \tag{4}$$

or

$$g_j(x_1, \dots, x_n) \geq b_j, \quad j = 1, \dots, d'' \geq m. \tag{5}$$

Problems with fuzzy coefficients only in the objective functions can be solved by modifying traditional optimization methods. As an example, we consider the analysis of fuzzy discrete optimization models.

### 3. ANALYSIS OF MODELS WITH FUZZY COEFFICIENTS IN OBJECTIVE FUNCTIONS

The desirability of allowing for constraints on the discreteness of variables in the form of increasing or decreasing discrete sequences

$$x_{s_i}, \rho_{s_i}, \tau_{s_i}, \dots, s_i = 1, \dots, r_i \tag{6}$$

has been validated in [6,7]; here  $\rho_{s_i}, \tau_{s_i}, \dots$  are characteristics required to form objective functions, constraints, and their increments that correspond to the  $s^{\text{th}}$  standard value of the variable  $x_i$ . It permits one to formulate the maximization problem as follows.

Assume we are given discrete sequences of type (6). From these sequences it is necessary to choose elements such that the objective

$$\text{maximize } \tilde{f}(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \tag{7}$$

is met while satisfying the constraints

$$g_j(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \leq b_j, \quad j = 1, \dots, d'. \tag{8}$$

Given (6)–(8), we can formulate a problem of minimization as follows:

$$\text{minimize } \tilde{f}(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots), \tag{9}$$

while satisfying the constraints

$$g_j(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \geq b_j, \quad j = 1, \dots, d''. \tag{10}$$

The solution of problems (6)–(8) and (6), (9), (10) is possible on the basis of modifying the generalized algorithms of discrete optimization [6,7]. These algorithms are based on a combination of formal and heuristic procedures and allow one to obtain quasioptimal solutions after a small

number of steps, thus overcoming the *NP*-completeness. Considering that the minimization problem is more difficult than the maximization problem [6,7], we shall describe the algorithm of solving problem (6), (9), (10) on the basis of the results of [7].

Assume that at step  $t$ , variable  $x_i$  is at its discrete level  $x_{s_i}^{(t)}$  with its parameters at the respective levels  $\rho_{s_i}^{(t)}, \tau_{s_i}^{(t)}, \dots$ . Introducing  $\theta_{s_i}^{(t)} = \{x_{s_i}^{(t)}, \rho_{s_i}^{(t)}, \tau_{s_i}^{(t)}, \dots\}$  and assuming that constraints (10) are already normalized as

$$g_j^{(0)}(\theta_{s_1}, \dots, \theta_{s_n}) = g_j(\theta_{s_1}, \dots, \theta_{s_n}) \frac{b}{b_j} \geq b, \quad j = 1, \dots, d'',$$

with the use of a normalized factor  $b > 0$ , we can write the following algorithm.

1. The components of the constraint increment vector  $\{\Delta G_i^{(t)}\}$  are evaluated as

$$\Delta G_i^{(t)} = \sum_j \Delta g_{ji}^{(t)}, \quad i \in I^{(t)}, \quad j \in J^{(t)}. \quad (11)$$

In (11),

$$\Delta g_{ji}^{(t)} = [g_j(\theta_{s_1}, \dots, \theta_{s_{i+1}}, \dots, \theta_{s_n}) - g_j(\theta_{s_1}, \dots, \theta_{s_i}, \dots, \theta_{s_n})] \frac{b_j^{(t1)}}{b}, \quad j \in J^{(t)}, \quad i \in I^{(t)},$$

where  $J^{(t)}$  is the set of the constraints at the  $t^{\text{th}}$  step (for  $t = 1$  we have  $j \in J_{d''}$ ,  $J_{d''}$  is the initial set of constraints);  $I^{(t)}$  is the set of variables at the  $t^{\text{th}}$  step (for  $t = 1$  we have  $i \in I_n$ ,  $I_n$  is the initial set of variables);  $b_j^{(t-1)} = b_j^{(0)} = b_j$ .

2. The components of the increment vector of the objective function  $\{\Delta \tilde{f}_i^{(t)}\}$  are calculated as

$$\Delta \tilde{f}_i^{(t)} = \tilde{f}(\theta_{s_1}^{(t)}, \dots, \theta_{s_{i+1}}^{(t)}, \dots, \theta_{s_n}^{(t)}) - \tilde{f}(\theta_{s_1}^{(t)}, \dots, \theta_{s_i}^{(t)}, \dots, \theta_{s_n}^{(t)}), \quad i \in I^{(t)}.$$

3. The components of a vector  $\{\tilde{v}^{(t)}\}$  are calculated as

$$\tilde{v}_i^{(t)} = \frac{\Delta \tilde{f}_i^{(t)}}{\Delta G_i^{(t)}}, \quad i \in I^{(t)}.$$

4. The index  $i = l$  of the incremented variable is determined from

$$\tilde{v}_i^{(t)} = \min_i \tilde{v}_i^{(t)}, \quad i \in I^{(t)}. \quad (12)$$

5. We recalculate the current values of the quantities

$$x_{s_i}^{(t)} = \begin{cases} x_{s_i}^t, & \text{if } i \neq l, \quad i \in I^{(t)}, \\ x_{s_{i+1}}^t, & \text{if } i = l, \end{cases}$$

$$b_j^{(t)} = b_j^{(t1)} - \Delta g_{jl}^{(t)} \frac{b}{b_j^{(t1)}}, \quad j = 1, \dots, d''.$$

6. If  $J^{(t)} = \{j \mid b_j^{(t)} > 0, j \in J^{(t)}\} \neq \emptyset$ , then go to Operation 7; otherwise the calculations are completed because the solution is obtained.
7. If  $I^{(t)} = \{i \mid s_i < r_i, i \in I^{(t)}\} \neq \emptyset$ , then go to Operation 1, taking  $t := t + 1$ ; otherwise the calculations are completed because the problem has no solution.

To compare alternatives (in essence, to rank fuzzy numbers  $v_i^{(t)}$ ,  $i \in I^{(t)}$ ) on the basis of (12) it is necessary to use corresponding methods, which are considered and analyzed in [8–12]. The authors of [8] classify four groups of methods related to the ordering of fuzzy quantities. Among these groups, the authors of [13] consider the construction of fuzzy preference relations for pairwise comparisons as the most practical and justified way. Taking this into account, it is necessary to distinguish the fuzzy number ranking index [14] based on the concept of a generalized preference relation.

If  $\tilde{v}_1$  and  $\tilde{v}_2$  have the membership functions  $\mu(v_1)$  and  $\mu(v_2)$ , the quantity  $\eta\{\mu(v_1), \mu(v_2)\}$  is the degree of preference  $\mu(v_1) \succcurlyeq \mu(v_2)$ , while  $\eta\{\mu(v_2), \mu(v_1)\}$  is the degree of preference  $\mu(v_2) \succcurlyeq \mu(v_1)$ . Then, the membership functions of the generalized preference relations  $\eta\{\mu(v_1), \mu(v_2)\}$  and  $\eta\{\mu(v_2), \mu(v_1)\}$  take the following form:

$$\eta\{\mu(v_1), \mu(v_2)\} = \sup_{v_1, v_2 \in V} \min\{\mu(v_1), \mu(v_2), \mu_R(v_1, v_2)\}, \tag{13}$$

$$\eta\{\mu(v_2), \mu(v_1)\} = \sup_{v_1, v_2 \in V} \min\{\mu(v_1), \mu(v_2), \mu_R(v_2, v_1)\}, \tag{14}$$

where  $\mu_R(v_1, v_2)$  and  $\mu_R(v_2, v_1)$  are the membership functions of the corresponding fuzzy preference relations.

If  $F$  is the numerical axis on which the values of the minimized objective function, for example, are plotted, and  $R$  is the natural order ( $\leq$ ) along  $F$ , then (13) and (14) reduce to

$$\eta\{\mu(v_1), \mu(v_2)\} = \sup_{\substack{v_1, v_2 \in V \\ v_1 \leq v_2}} \min\{\mu(v_1), \mu(v_2)\}, \tag{15}$$

$$\eta\{\mu(v_2), \mu(v_1)\} = \sup_{\substack{v_1, v_2 \in V \\ v_2 \leq v_1}} \min\{\mu(v_1), \mu(v_2)\}. \tag{16}$$

These agree with the Baas-Kwakernaak [15], Baldwin-Guild [16], and one of the Dubois-Prade [17] fuzzy number ranking indices.

On the basis of the relations between (15) and (16), it is possible to judge the degree of preference of any of the alternatives compared. Utilization of this approach is well founded. However, experience shows that in many cases the membership functions of the alternatives  $\mu(v_1)$  and  $\mu(v_2)$  compared form flat apices (for example, [4,18]), i.e., they are so-called flat or trapezoidal fuzzy numbers [2,3]. In view of this, we can say that for the situation shown in Figure 1 the alternatives  $\tilde{v}_1$  and  $\tilde{v}_2$  are indistinguishable if

$$\eta\{\mu(v_1), \mu(v_2)\} = \eta\{\mu(v_2), \mu(v_1)\} = \sigma. \tag{17}$$

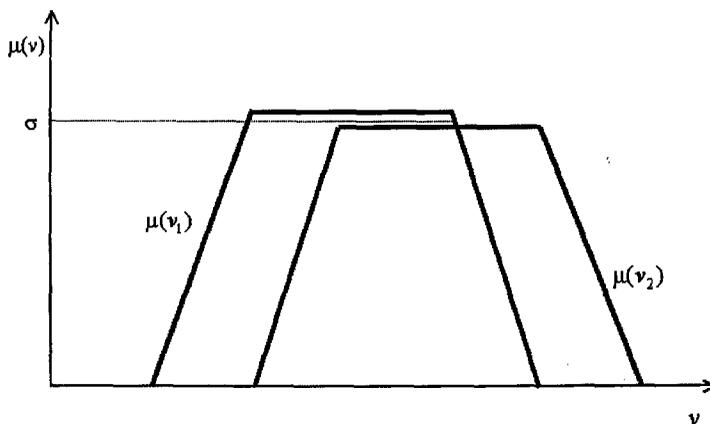


Figure 1. Comparison of alternatives.

In such situations the modified algorithms of discrete optimization do not allow one to obtain a unique solution because they “stop” when conditions like (17) arise. This occurs also with other modifications of traditional mathematical programming methods (this is illustrated in [1,19] by simple examples) because combination of the uncertainty and the relative stability of optimal solutions can produce these so-called decision uncertainty regions. In this connection, other indices may be used as additional means for the ranking of fuzzy numbers.

Wang and Kerre [11] count more than 35 existing fuzzy number ranking indices and indicate the following.

Unlike in the case of real numbers, fuzzy quantities have no natural order. A straightforward idea with the ordering of fuzzy quantities is to convert a fuzzy quantity into a real number and base the comparison of fuzzy quantities on that of real numbers. Each individual conversion way, however, pays attention to a special aspect of fuzzy quantity. As a consequence, each approach suffers from some defects if only one real number is associated with each fuzzy quantity. The authors of [10,20] share this opinion as well. Cheng [20] also indicates that many of the indices produce different rankings for the same problem. The authors of [1,10,20,21] underline that fuzzy number ranking indices occasionally result in choices which appear inconsistent with intuition. Finally, from the substantial point of view it is necessary to indicate the following.

The majority of indices for the ranking of fuzzy quantities has been proposed with the aspiration for obligatory distinguishing of the alternatives. This is not natural because the uncertainty of information creates the decision uncertainty regions.

There actually is another approach that is better validated, natural, and acceptable for the decision-making practice. This approach is associated with transition to multiobjective choosing alternatives in a fuzzy environment because the application of additional criteria (including the criteria of qualitative character, such as “comfort of operation”, “flexibility of development”, “level of maintenance”, etc.) can serve as a convincing means to contract the decision uncertainty regions.

#### 4. MULTIOBJECTIVE CHOICE AND FUZZY PREFERENCE RELATIONS

Before starting to discuss questions of multiobjective decision making in a fuzzy environment, it is necessary to note that considerable contraction of the decision uncertainty regions may be obtained by formulating and solving one and the same problem within the framework of mutually related models [5,22]. For example, it is possible to solve one and the same problem within the framework of the following models:

- (a) the model of maximization (7),(8) with increasing (decreasing) sequences (6);
- (b) the model of minimization (8),(9) with decreasing (increasing) sequences (6).

When using this approach, solutions dominated by the initial objective function are cut off from above as well as from below to the greatest degree [5,22]. It should be stressed that this approach is of a universal character and may also be used in solving continuous problems as well.

Assume we are given a set  $X$  of alternatives (from the decision uncertainty region), which are to be examined by  $q$  criteria of quantitative and/or qualitative nature. The problem of decision making is presented by a pair  $\langle X, R \rangle$  where  $R = \{R_1, \dots, R_q\}$  is a vector fuzzy preference relation [14,23]. In this case, we have

$$R_p = [X \times X, \mu_{R_p}(X_k, X_l)], \quad p = 1, \dots, q, \quad X_k, X_l \in X, \quad (18)$$

where  $\mu_{R_p}(X_k, X_l)$  is a membership function of fuzzy preference relation.

In (18),  $R_p$  (also called a nonstrict fuzzy preference relation, fuzzy binary preference relation, fuzzy weak preference relation, and fuzzy binary relation in literature) is defined as a fuzzy set of all pairs of the Cartesian product  $X \times X$ , such that the membership function  $\mu_{R_p}(X_k, X_l)$

represents the degree to which  $X_k$  weakly dominates  $X_l$ , i.e., the degree to which  $X_k$  is at least as good as  $X_l$  ( $X_k$  is not worse than  $X_l$ ) for the  $p^{\text{th}}$  criterion. In a somewhat loose sense [24],  $\mu_{R_p}(X_k, X_l)$  also represents the degree of truth of the statement “ $X_k$  is preferred over  $X_l$ ”.

It is supposed in [14,23] that the matrices,  $R_p$ ,  $p = 1, \dots, q$  are directly given as expert’s estimates (from the interval  $[0, 1]$ ) denoting the degree of preference of one alternative over the other. However, there is another, more convincing and natural, approach to obtaining these matrices. In particular, the availability of fuzzy or linguistic estimates of alternatives  $f_p(X_k)$ ,  $p = 1, \dots, q$ ,  $X_k \in X$  (constructed on the basis of expert estimation or on the basis of aggregating information arriving from different sources of both formal and informal character [4]) with the membership functions  $\mu[f_p(X_k)]$ ,  $p = 1, \dots, q$ ,  $X_k \in X$  permits one to construct the matrices  $R_p$ ,  $p = 1, \dots, q$  as follows, using expressions (15) and (16):

$$\mu_{R_p}(X_k, X_l) = \sup_{\substack{X_k, X_l \in X \\ f_p(X_k) \leq f_p(X_l)}} \min\{\mu[f_p(X_k)], \mu[f_p(X_l)]\}, \tag{19}$$

$$\mu_{R_p}(X_l, X_k) = \sup_{\substack{X_k, X_l \in X \\ f_p(X_l) \leq f_p(X_k)}} \min\{\mu[f_p(X_k)], \mu[f_p(X_l)]\}. \tag{20}$$

If the  $p^{\text{th}}$  criterion is associated with maximization, then (19) and (20) are written for regions  $f_p(X_k) \geq f_p(X_l)$  and  $f_p(X_l) \geq f_p(X_k)$ , respectively.

Let us consider the situation of setting up a single preference relation  $R$ . It can be represented by the strict  $R^S$  and indifferent  $R^I$  fuzzy preference relations [14,23]. It is possible to use the inverse relation  $R^{-1}$  ( $(X_k, X_l) \in R^{-1}$  is equivalent to  $(X_l, X_k) \in R$ ) to obtain

$$R^S = R \setminus R^{-1}. \tag{21}$$

If  $(X_k, X_l) \in R^S$ , then  $X_k$  dominates  $X_l$ , i.e.,  $X_k \succ X_l$ . The alternative  $X_k \in X$  is nondominated in  $(X, R)$  if  $(X_k, X_l) \in R^S$  for any  $X_l \in X$ .

If we have  $\mu_R(X_k, X_l)$  as a nonstrict fuzzy preference relation, then the value  $\mu_R(X_k, X_l)$  is the degree of preference  $X_k \succcurlyeq X_l$  for any  $X_k, X_l \in X$ . The membership function, which corresponds to (21), is the following:

$$\mu_R^s(X_k, X_l) = \max\{\mu_R(X_k, X_l) - \mu_R(X_l, X_k), 0\}. \tag{22}$$

Expression (22) serves as the basis for the choice procedure introduced by Orlovsky [25]. Many authors have studied this procedure. For instance, it was shown in [26] that the Orlovsky choice procedure possesses many interesting desirable properties. Axiomatic characterization of the Orlovsky choice procedure is given, for example, in [27–29].

The use of (22) permits one to carry out the choice of alternatives. In particular,  $\mu_R^s(X_l, X_k)$  is the membership function of the fuzzy set of all  $X_k$ , which are strictly dominated by  $X_l$ . Its complement by  $1 - \mu_R^s(X_l, X_k)$  gives the fuzzy set of alternatives, which are not dominated by other alternatives from  $X$ . To choose the set of all alternatives, which are not dominated by other alternatives from  $X$ , it is necessary to find the intersection of all  $1 - \mu_R^s(X_l, X_k)$ ,  $X_k \in X$  on all  $X_l \in X$  [14,25]. This intersection is the set of nondominated alternatives with the membership function

$$\mu_R^{\text{ND}}(X_k) = \inf_{X_l \in X} [1 - \mu_R^s(X_l, X_k)] = 1 - \sup_{X_l \in X} \mu_R^s(X_l, X_k). \tag{23}$$

Because  $\mu_R^{\text{ND}}(X_k)$  is the degree of nondominance, it is natural to obtain alternatives providing

$$X^{\text{ND}} = \left\{ X_k^{\text{ND}} \mid X_k^{\text{ND}} \in X, \mu_R^{\text{ND}}(X_k^{\text{ND}}) = \sup_{X_k \in X} \mu_R^{\text{ND}}(X_k) \right\}. \tag{24}$$

Orlovsky [14,25] also introduced the notion of a set of nonfuzzy nondominated alternatives. In particular, if  $\sup_{X_k \in X} \mu_R^{ND}(X_k) = 1$ , then alternatives

$$X^{NFND} = \{X_k^{NFND} \mid X_k^{NFND} \in X, \mu_R^{ND}(X_k^{NFND}) = 1\}$$

are nonfuzzy nondominated and can be considered as a nonfuzzy solution of the fuzzy problem.

If the fuzzy preference relation  $R$  is transitive, then  $X^{NFND} \neq \emptyset$ . Taking this into account, it should be noted that when  $\tilde{f}_p(X_k)$  is quantitatively expressed,  $X^{NFND} \neq \emptyset$ . With qualitative  $\tilde{F}_p(X_k)$  it is possible to have  $X^{NFND} = \emptyset$  under intransitivity of  $R$  [30]. It permits one to detect contradictions in expert's estimates.

When  $R$  is a vector fuzzy preference relation, expressions (22)–(24) can serve as a basis for the first technique for multiobjective decision making in a fuzzy environment if we take  $R = \bigcap_{p=1}^q R_p$ , i.e.,

$$\mu_R(X_k, X_l) = \min_{1 \leq p \leq q} \mu_{R_p}(X_k, X_l), \quad X_k, X_l \in X. \tag{25}$$

When using this intersection, the set  $X^{ND}$  fulfills the role of a Pareto set [14]. Its contraction is possible on the basis of differentiating the importance of  $R_p$ ,  $p = 1, \dots, q$  with the use of the following convolution (aggregation of nonobjective fuzzy preference relations) [14]:

$$\mu_T(X_k, X_l) = \sum_{p=1}^q \lambda_p \mu_{R_p}(X_k, X_l), \quad X_k, X_l \in X,$$

where  $\lambda_p \geq 0$ ,  $p = 1, \dots, q$  are weights (importance factors) for the corresponding criteria normalized as

$$\sum_{p=1}^q \lambda_p = 1. \tag{26}$$

The construction of  $\mu_T(X_k, X_l)$ ,  $X_k, X_l \in X$  allows one to obtain the membership function  $\mu_T^{ND}(X_k)$  of the set of nondominated alternatives according to an expression similar to (23). The intersection of  $\mu_R^{ND}(X_k)$  and  $\mu_T^{ND}(X_k)$  defined as

$$\mu^{ND}(X_k) = \min \{ \mu_R^{ND}(X_k), \mu_T^{ND}(X_k) \}, \quad X_k \in X,$$

provides us with

$$X^{ND} = \left\{ X_k^{ND} \mid X_k^{ND} \in X, \mu^{ND}(X_k^{ND}) = \sup_{X_k \in X} \mu^{ND}(X_k) \right\}.$$

Expressions (23) and (24) can also serve as the basis for building the second technique, which is of a lexicographic character. It is associated with step-by-step introduction of criteria for comparing alternatives. The technique permits one to construct a sequence  $X^1, X^2, \dots, X^q$  so that  $X \supseteq X^1 \supseteq X^2 \supseteq \dots \supseteq X^q$  with the use of the following expressions:

$$\mu_{R_p}^{ND}(X_k) = \inf_{X_l \in X^{p-1}} \left[ 1 - \mu_{R_p}^S(X_l, X_k) \right] = 1 - \sup_{X_l \in X^{p-1}} \mu_{R_p}^S(X_l, X_k), \quad p = 1, \dots, q, \tag{27}$$

$$X^p = \left\{ X_k^{ND,p} \mid X_k^{ND,p} \in X^{p-1}, \mu_{R_p}^{ND}(X_k^{ND,p}) = \sup_{X_l \in X^{p-1}} \mu_{R_p}^{ND}(X_l) \right\}. \tag{28}$$

It should be noted that if  $R_p$  is transitive, we can bypass the pairwise comparison of alternatives at the  $p^{\text{th}}$  step. In this situation, the comparison can be done on a serial basis (the direct use of (19) and (20)) with memorizing the best alternatives.

The described choice techniques have found applications in solving power engineering problems [31]. However, it is possible to propose the third technique to contract the decision uncertainty region.

The use of (23) represented in the form

$$\mu_{R_p}^{ND}(X_k) = 1 - \sup_{X_l \in X} \mu_{R_p}^S(X_l, X_k), \quad p = 1, \dots, q, \quad (29)$$

permits one to construct the membership functions of the set of nondominated alternatives for each fuzzy preference relation.

The membership functions  $\mu_{R_p}^{ND}(X_k)$ ,  $p = 1, \dots, q$  play a role identical to membership functions replacing objective functions  $f_p(X)$ ,  $p = 1, \dots, q$  in solving traditional problems of multiobjective optimization [32] on the basis of the Bellman-Zadeh approach to decision making in a fuzzy environment [2,3]. Therefore, it is possible to construct

$$\mu^{ND}(X_k) = \min_{1 \leq p \leq q} \mu_{R_p}^{ND}(X_k) \quad (30)$$

to obtain  $X^{ND}$ .

If necessary to differentiate the importance of different preference relations, it is possible to transform (30) as

$$\mu^{ND}(X_k) = \min_{1 \leq p \leq q} \left[ \mu_{R_p}^{ND}(X_k) \right]^{\lambda_p}. \quad (31)$$

The utilization of (31) does not require the normalization of  $\lambda_p$ ,  $p = 1, \dots, q$  in the way similar to (26).

It is natural that the use of the second technique may lead to solutions different from results obtained on the basis of the first technique. However, solutions based on the first and third techniques, which have a single generic basis, may at times also be different. At the same time, the third technique is more preferential from the substantial point of view. In particular, the use of the first technique can lead to choosing alternatives with the degree of nondominance equal to one, though these alternatives are not the best ones from the point of view of all preference relations. The third procedure can give this result only for alternatives that are the best solutions from the point of view of all fuzzy preference relations. Taking the above into account, it should be stressed that the fact of the possibility to obtain different solutions on the basis of different approaches is natural, and the choice of the approach is a prerogative of the decision maker.

In actuality, the procedures discussed above suppose the explicit direct or indirect ordering of the criteria. Considering this, it is necessary to distinguish the approach [14], which allows one to present information related to the importance of the criteria in the following form:

$$W = [\lambda \times \lambda, \mu_W(\lambda_p, \lambda_t)], \quad p, t = 1, \dots, q. \quad (32)$$

Using the membership functions of the sets of nondominated alternatives (28) for all preference relations, it is possible to construct the following fuzzy preference relation induced by the preference relations (29) and (32):

$$\mu_{R,W}(X_k, X_l) = \sup_{\lambda_p, \lambda_t \in \lambda} \min_{X_k, X_l \in X} \left\{ \mu_{R_p}^{ND}(X_k), \mu_{R_t}^{ND}(X_l), \mu_W(\lambda_p, \lambda_t) \right\}. \quad (33)$$

The fuzzy preference relation (33) can be considered as a result of aggregating the family of  $\mu_{R_p}(X_k, X_l)$ ,  $p = 1, \dots, q$  with the use of information reflecting the relative importance of criteria given in form (32). Applying procedures (23) and (24) to (33), it is possible to construct the set of nondominated alternatives  $\bar{\mu}_{R,W}^{ND}(X_k)$ . As it is shown in [14], the set  $\bar{\mu}_{R,W}^{ND}(X_k)$  is to be corrected in accordance with the following correlation:

$$\mu_{R,W}^{ND}(X_k) = \min \left\{ \bar{\mu}_{R,W}^{ND}(X_k), \mu_{R,W}(X_k, X_l) \right\}. \quad (34)$$

Applying (24) to (34), it is possible to obtain the problem solution  $X^{ND}$ .

The described techniques are of a universal character and are already being used to solve power engineering [31], naval engineering [33], and management [34] problems and have been implemented within the framework of a corresponding system for multiobjective decision making (MDMS).

#### 4.1. Computing Implementation

The MDMS has been developed in the C++ programming language and is executed in the graphical environment of the *Microsoft Windows*<sup>®</sup> operating system. In this section, we list several typical windows that appear in the process of initial data preparation and, in the next section, we list several typical windows that appear in the process of multiobjective decision making.

The initial window (see Figure 2) permits one to start the decision-making process by indicating the technique used and by defining the number of alternatives and the number of criteria. The screen in Figure 2 reflects input information for an example of multicriteria decision making discussed below.

The next windows (see Figures 3 and 4) permit one to define the character of the problem (maximization or minimization) for the corresponding criterion and to define estimates for alter-

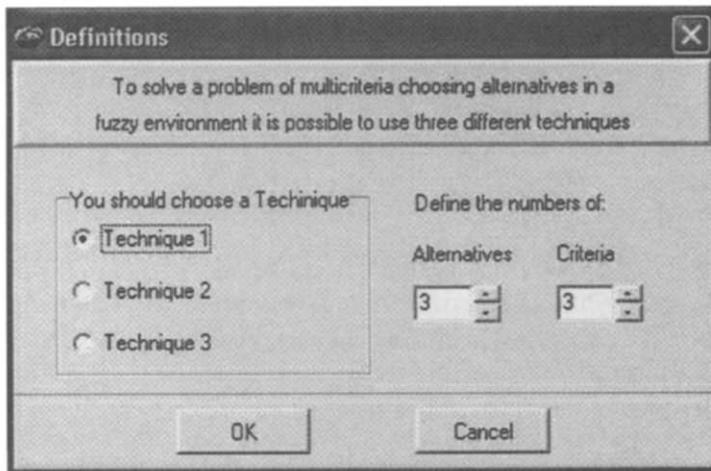


Figure 2. Initial window.

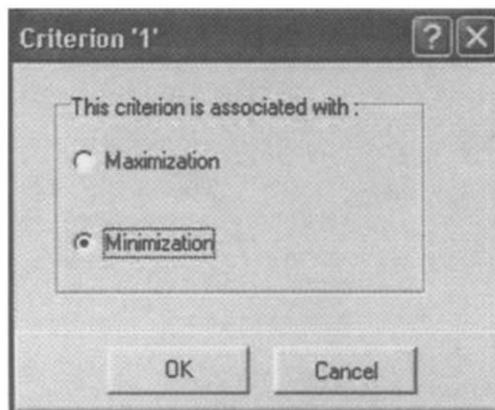


Figure 3. Problem character.

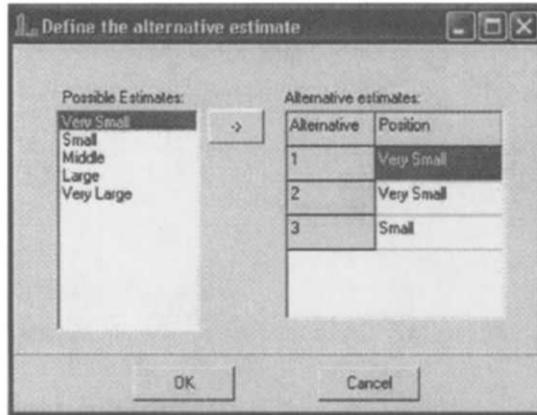


Figure 4. Estimates of alternatives.

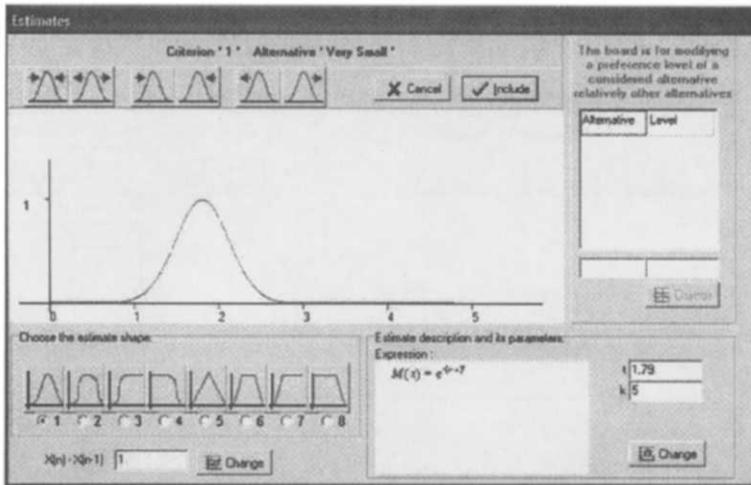


Figure 5. Estimate “very small”.

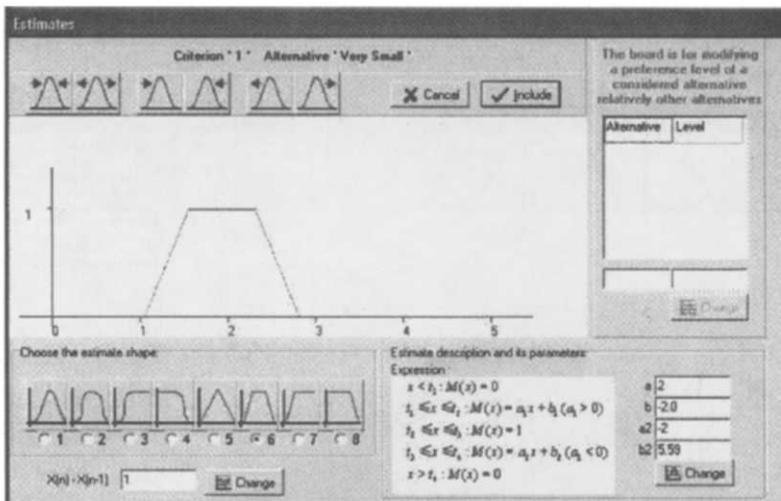


Figure 6. Changed type of estimate “very small”.

natives related to this criterion (the estimates given in Figure 4 correspond to the first criterion for an example discussed below).

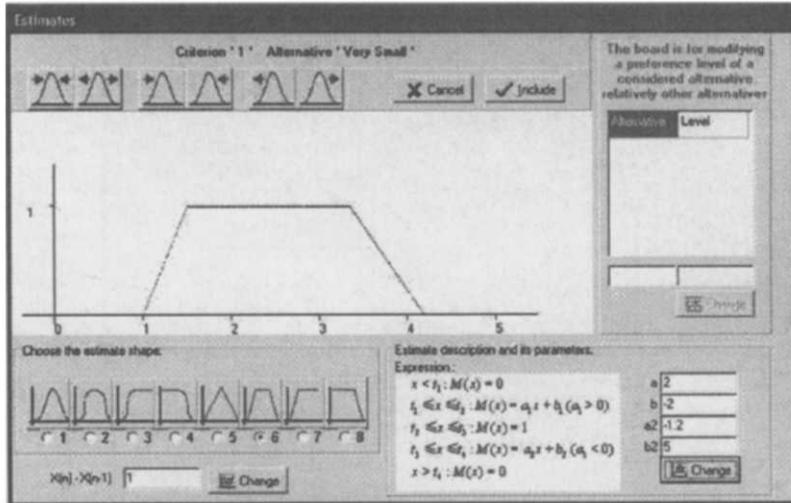


Figure 7. Modified estimate “very small”.

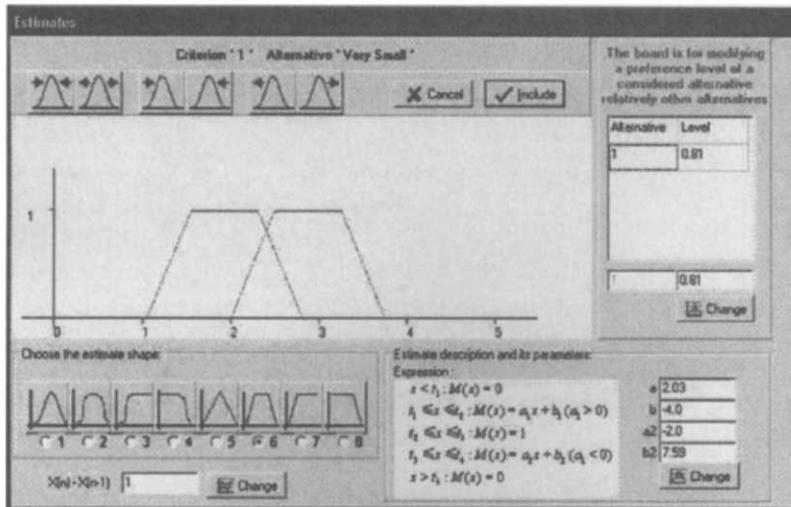


Figure 8. Initial preference level.

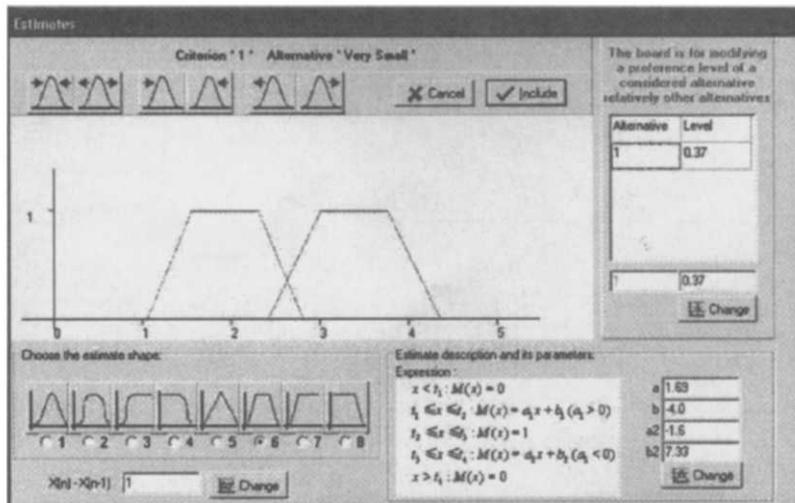


Figure 9. Modified preference level.

The estimate “very small” for the first alternative for the first criterion is shown in Figure 5. The analytical description of the estimate and its parameters are given at “estimate description and its parameters”. The type of the estimate shape (the basic estimate shape) can be changed by the corresponding choice at “choose the estimate shape” (see Figure 5) as it is shown in Figure 6. The alternative shape can be modified by changing parameters at “estimate description and its parameters” (see Figure 5) as it is shown in Figure 7. The alternative shape can also be modified on the experimental basis by clicking one of six buttons in the upper left corner of the screen (see Figure 5). Finally, the MDMS permits one to modify a preference level of a considered alternative relative to other alternatives using “the board is for modifying a preference level of a considered alternative relative to other alternatives” (see Figure 8) as it is shown in Figure 9.

**4.2. Illustrating Example**

Assume we are given the set  $X = \{X_1, X_2, X_3\}$  that must be compared with the application of three criteria. The first criterion ( $p = 1$ ) demands the minimization of  $\tilde{f}_1(X_k)$ . The second criterion ( $p = 2$ ) and the third criterion ( $p = 3$ ) demand the maximization of  $\tilde{f}_2(X_k)$  and  $\tilde{f}_3(X_k)$ , respectively. The alternatives have received the following estimates:  $\tilde{f}_1(X_1) =$  “very small”,  $\tilde{f}_1(X_2) =$  “very small”,  $\tilde{f}_1(X_3) =$  “small”,  $\tilde{f}_2(X_1) =$  “very small”,  $\tilde{f}_2(X_2) =$  “small”,  $\tilde{f}_2(X_3) =$

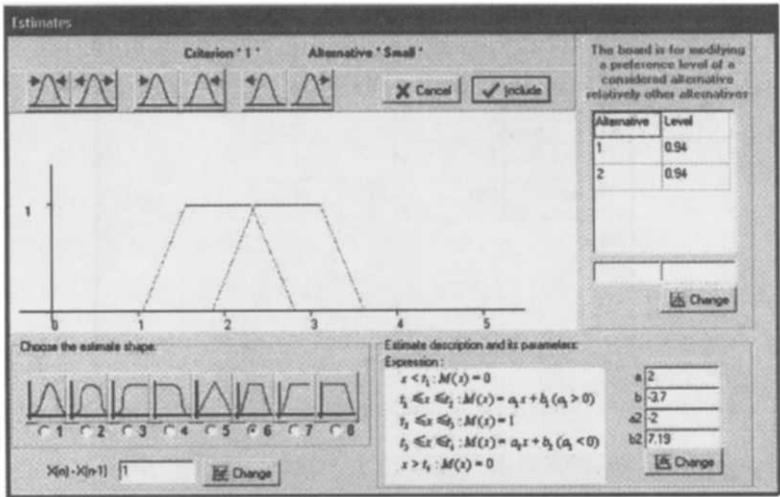


Figure 10. Estimates for the first criterion.

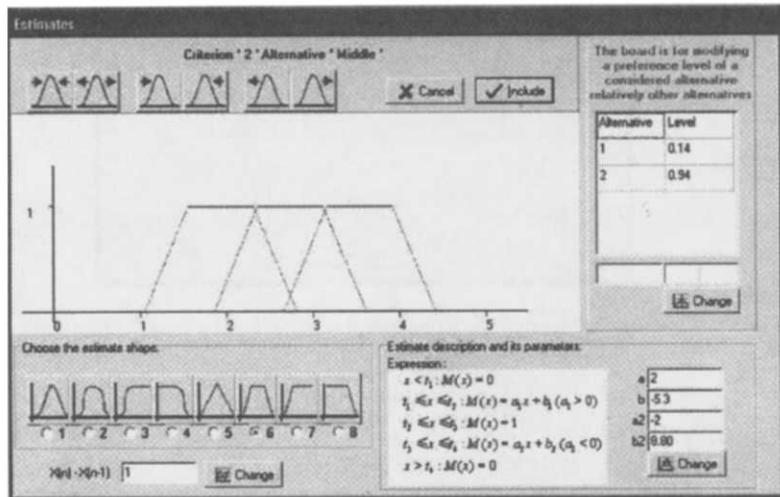


Figure 11. Estimates for the second criterion.

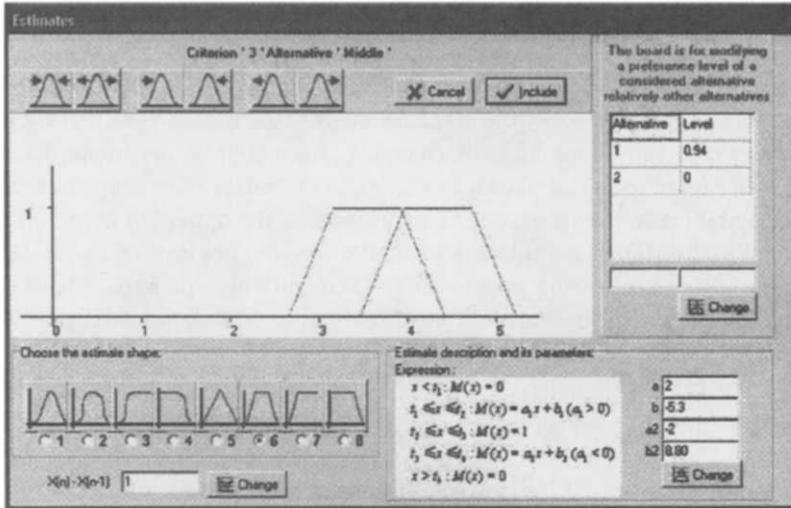


Figure 12. Estimates for the third criterion.

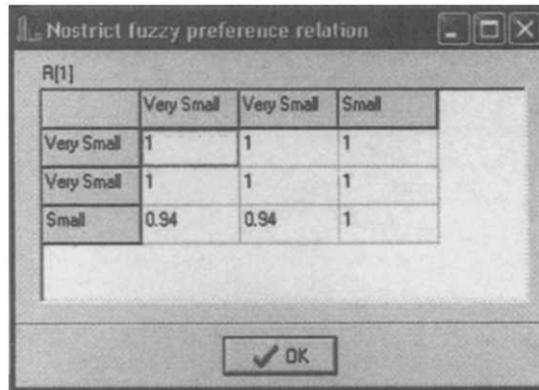


Figure 13. Nonstrict fuzzy preference relation for the first criterion.

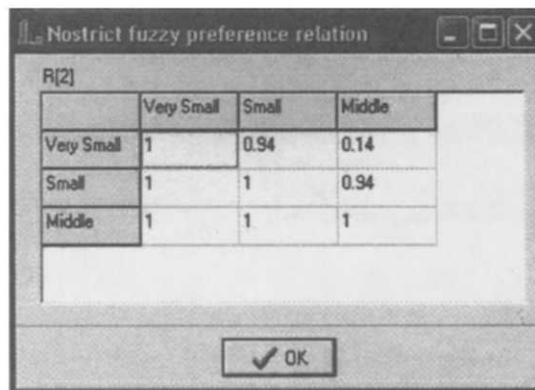


Figure 14. Nonstrict fuzzy preference relation for the second criterion.

“middle”,  $\tilde{f}_3(X_1) = \text{“large”}$ ,  $\tilde{f}_3(X_2) = \text{“middle”}$ , and  $\tilde{f}_3(X_3) = \text{“middle”}$ . The description for all estimates is trapezoidal (see Figure 10 ( $\tilde{f}_1(X_1)$  and  $\tilde{f}_1(X_2)$  coincide), Figure 11, and Figure 12 ( $\tilde{f}_3(X_2)$  and  $\tilde{f}_3(X_3)$  coincide), respectively). The estimates generate the nonstrict fuzzy preference relations given in Figure 13 for  $p = 1$ , in Figure 14 for  $p = 2$ , and in Figure 15 for  $p = 3$ .

	Large	Middle	Middle
Large	1	1	1
Middle	0.94	1	1
Middle	0.94	1	1

Figure 15. Nonstrict fuzzy preference relation for the third criterion.

Let us consider the solution of the problem on the basis of the first technique. The intersection of nonstrict fuzzy preference relations constructed on the basis of (25) is given in Figure 16. The strict fuzzy preference obtained on the basis of (22) is presented in Figure 17 and permits one to find the membership function of the set of nondominated alternatives given in Figure 18. Thus, the problem solution on the basis of the first technique is  $X^{ND} = \{X_2, X_3\}$ .

Let us consider the second approach if the criteria are arranged, for example, in the following order of their importance:  $p = 1$ ,  $p = 2$ , and  $p = 3$ .

Following (22), (27), and (28), we obtain on the basis of the nonstrict fuzzy preference relation given in Figure 13

$$\mu_{R_1}^S(X_k, X_l) = \begin{bmatrix} 0 & 0 & 0.06 \\ 0 & 0 & 0.06 \\ 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$\mu_{R_1}^{ND}(X_k) = [1 \quad 1 \quad 0.94],$$

and  $X^1 = \{X_1, X_2\}$ . Thus, the alternatives  $X_1$  and  $X_2$  are to be considered for a subsequent analysis, and we can construct for the second step (see Figure 14)

$$\mu_{R_2}^2(X_k, X_l) = \begin{bmatrix} 1 & 0.94 \\ 1 & 1 \end{bmatrix},$$

$$\mu_{R_2}^S(X_k, X_l) = \begin{bmatrix} 0 & 0 \\ 0.06 & 0 \end{bmatrix},$$

$$\mu_{R_2}^{ND}(X_k) = [0.94 \quad 1],$$

and  $X^{ND} = \{X_2\}$ .

	1	2	3
1	1	0.94	0.14
2	0.94	1	0.94
3	0.94	0.94	1

Figure 16. Intersection of nonstrict fuzzy preference relations.

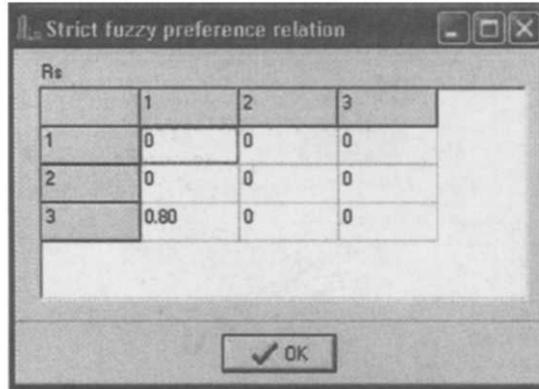


Figure 17. Strict fuzzy preference relation.

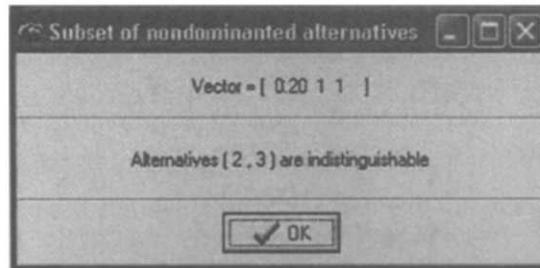


Figure 18. Membership function of the set of nondominated alternatives.

Finally, let us consider the application of the third approach. The membership function of the subset of nondominated alternatives for the first fuzzy preference relation  $\mu_{R_1}^{ND}(X_k)$  is (35). The nonstrict fuzzy preference relation given in Figure 14 leads to

$$\mu_{R_2}^{ND}(X_k) = [0.14 \quad 0.94 \quad 1] \tag{36}$$

and the nonstrict fuzzy preference relation given in Figure 15 leads to

$$\mu_{R_3}^{ND}(X_k) = [1 \quad 0.94 \quad 0.94]. \tag{37}$$

The intersection of (35)–(37) in accordance with (30) permits us to construct

$$\mu^{ND}(X_k) = [0.14 \quad 0.92 \quad 0.92]$$

to get  $X^{ND} = \{X_2, X_3\}$ .

### 5. CONCLUSIONS

In this paper, the construction and analysis of  $\langle X, R \rangle$  models within the framework of a general approach to solving optimization problems with fuzzy coefficients has been considered. This approach is associated with modifying traditional mathematical programming methods and consists of formulating and solving one and the same problem within the framework of mutually interrelated models. The use of the approach allows one to maximally cut off dominated alternatives. The subsequent contraction of the decision uncertainty region is based on reducing the problem to models of multiobjective choosing alternatives in a fuzzy environment with the use of fuzzy preference relation techniques for analyzing these models. Three techniques for fuzzy preference modeling are discussed in the paper. The first technique is based on the construction

of membership functions of subsets of nondominated alternatives with simultaneous considering all criteria (fuzzy preference relations). The second technique is of a lexicographic character and consists of step-by-step introducing of fuzzy preference relations. The third technique is based on aggregating membership functions of subsets of nondominated alternatives corresponding to each preference relation. The choice of the technique is a prerogative of a decision maker. The techniques have served for developing a corresponding system for multiobjective decision making (MDMS). C++ windows of the MDMS have been presented for input, output, and some intermediate procedures. All techniques have been illustrated by considering an example. The results of the paper are of a universal character and are already being used to solve power engineering, naval engineering, and management problems.

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