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Edge-Ends in Countable Graphs

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We introduce the notion of an edge-end and characterize those countable graphs which have edge-end-faithful spanning trees. We also prove that for a natural class of graphs, there always exists a tree which is faithful on the undominated ends and rayless over the dominated does. © 1997 Academic Press

1. INTRODUCTION

The notion of ends—equivalence classes on the set of rays (one-way infinite paths)—of a graph is one of the most studied topics in infinite graph theory. An introduction to this theory and basic results can be found in Halin [3]. Halin defined two rays to be equivalent if no finite set of vertices can separate an infinite part of the first ray from an infinite part of the second one. In particular, Halin proved that in a countable connected graph G, the end-structure can be represented by a kind of spanning tree that he called faithful. Such a tree is defined by the property that from any given end of G it contains exactly one ray originating at x, for any $x \in V(G)$).

A natural and, as will be seen in this paper, very useful property of ends is the domination property. An end α is dominated if for some ray R (and so for all rays) in α there exists a vertex x which cannot be separated from an infinite part of R by any finite set of vertices. Intuitively, undominated rays are those one normally has in mind when thinking about infinite paths as "going to infinity," whereas dominated rays appear to be "trapped" by the vertices that dominate them (and thus are not "really" rays).

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In this paper we study the end-structure in the case where ray equivalence is defined in terms of edge-separation instead of vertex-separation. That is, two rays will be edge-equivalent if no finite set of edges can separate an infinite part of the first one from an infinite part of the second one. The resulting equivalence classes will be called edge-ends (or $\mathscr{E}$-ends). In order to distinguish edge-ends from the ends defined by Halin, we shall refer to the latter as vertex-ends (or $\mathscr{V}$-ends). Further, we shall speak of $\mathscr{V}$-domination instead of domination, and, by analogy, we shall define $\mathscr{E}$-domination in terms of separation by finite sets of edges rather than vertices. Edge-ends appear to have a more "stable" structure with respect to the domination property in the sense that unlike in the case of $\mathscr{V}$-domination and vertex-ends, there is an intimate relationship between $\mathscr{E}$-equivalent rays and their $\mathscr{E}$-dominating vertices.

Unfortunately, even for countable graphs, edge-end structure cannot always be represented by an $\mathscr{E}$-faithful spanning tree (see Figure 1 for a counterexample). The main result of this paper (Theorem 5) gives a characterization of graphs having an $\mathscr{E}$-faithful spanning tree. In fact, we give three equivalent characterizations of such graphs, the most natural one being what we call end-correlatedness, meaning that the relations of $\mathscr{V}$-equivalence and $\mathscr{E}$-equivalence coincide on $\mathscr{V}$-undominated rays. As for the proof of our main result, we have been unable to make use of the type of argument used by Halin in the vertex case. We have shown instead that the existence of an $\mathscr{E}$-faithful spanning tree is closely related to the existence of another type of spanning tree (called $\mathscr{U}$-faithful) in graphs containing no $\mathscr{E}$-dominated $\mathscr{V}$-undominated rays. Such spanning trees represent the vertex-end structure of the $\mathscr{V}$-undominated rays only; no $\mathscr{V}$-dominated end may have a ray in a $\mathscr{U}$-faithful subgraph.

Ends which are $\mathscr{V}$-undominated are in a sense inevitable: we show that any spanning tree of any connected graph (possibly uncountable) must contain at least one ray from every $\mathscr{V}$-undominated vertex-end. $\mathscr{U}$-faithful spanning trees are therefore the "as rayless as possible" spanning subgraphs.

Again, not all countable graphs have $\mathscr{U}$-faithful spanning trees (see [4] for counterexamples). However, we prove that they exist for countable graphs having no $\mathscr{E}$-dominated $\mathscr{V}$-undominated rays (Theorem 4). In view of the connection between $\mathscr{E}$-faithful spanning trees and $\mathscr{E}$-faithful spanning trees mentioned earlier, Theorem 4 is the key element in the proof of Theorem 5.

2. PRELIMINARIES

For the purposes of this paper we assume all graphs to be infinite, connected and simple, unless otherwise stated. A *ray* in a graph is a one-way infinite path $\{a_i: i<\omega\}$. Each sub-path $\{a_i: j\leqslant i<\omega\}$ will be called a *tail* of the ray.

Let G be an infinite graph and let P and Q be two rays in G. We say that P and Q are *vertex-equivalent* [*edge-equivalent*], denoted by $P\sim_v [P\sim_E q]$, if for any finite set of vertices [edges] S, some tails of P and Q lie in the same component of $G-S[G\backslash S]$. Here $G-S$ is the graph obtained from G by the removal of the vertices of S and all incident edges in the case where S is a set of vertices; while $G\backslash S$ is obtained by the removal of all edges of S (retaining all vertices), in the case where S is a set of edges. For a subgraph H of G, $G-H$ and $G\backslash H$ denote $G-V(H)$ and $G\backslash E(H)$, respectively.

We note that both $\sim_v$ and $\sim_e$ are equivalence relations. The equivalence classes are called *ends* if no confusion is likely, otherwise we speak of *vertex-ends* and *edge-ends*. The set of vertex-ends of a graph G is denoted by $\mathscr{V}(G)$, the set of edge-ends by $\mathscr{E}(G)$. If P and Q are not equivalent, then there is a finite set S of vertices [or edges] which separates tails of P and Q. In this case we also say that S *separates* P and Q.

We note that two vertex-equivalent rays are also edge-equivalent (and, hence, every edge-end is a union of vertex-ends) but that the converse is usually false, see Figure 1. There is a close relationship between

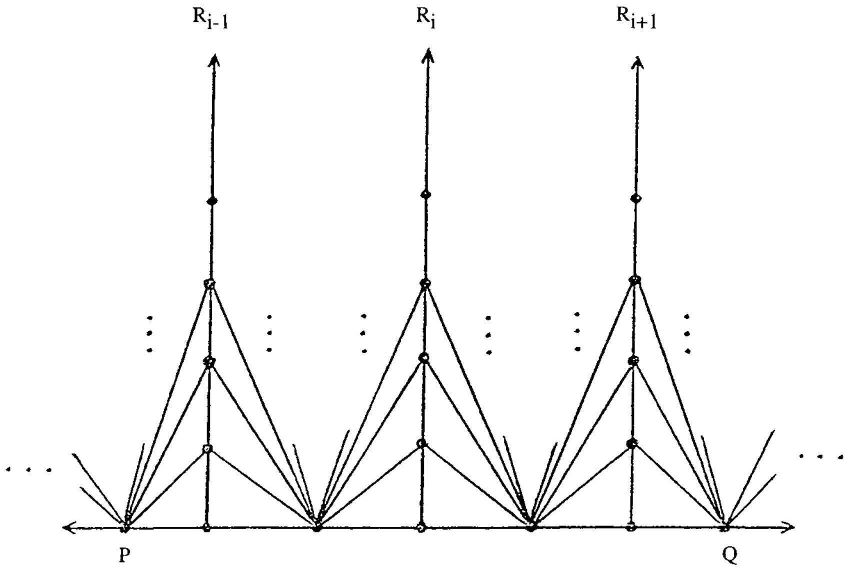

FIG. 1. The rays P, Q and the R_i's are edge-equivalent but not vertex-equivalent. Such a graph cannot have an $\mathscr{E}$-faithful spanning tree since any spanning tree must contain two disjoint edge-equivalent rays, the first one being vertex-equivalent in G to the ray P and the second one vertex-equivalent to the ray Q.

edge-equivalence in G and vertex-equivalence in its line-graph, as given by the following lemma. Let $\mathscr{L}(G)$ be the line-graph of G and let P be a ray in G. Then $\mathscr{L}(P)$ is the ray in $\mathscr{L}(G)$ defined by the edges of P.

LEMMA 1. *Let P and Q be rays in an infinite graph G and let $\mathscr{L}(G)$ be the line-graph of G. Then P and Q are edge-equivalent in G if and only if $\mathscr{L}(P)$ and $\mathscr{L}(Q)$ are vertex-equivalent in $\mathscr{L}(G)$.*

Proof. Clearly a finite set S of edges of G separates tails of P and Q if and only if S, viewed as a set of vertices of $\mathscr{L}(G)$, separates tails of $\mathscr{L}(P)$ and $\mathscr{L}(Q)$. ∎

Vertex-equivalence is a concept much studied since its introduction by Halin [3]. Halin's paper gives—among many other things—a characterization of vertex-equivalent rays. Two rays P and Q are vertex-equivalent in G if and only if there are infinitely many pairwise (vertex-) disjoint paths connecting them; trivial (one-vertex) paths are allowed. We shall give an analogous characterization of edge-equivalence.

Let X and Y be disjoint sets of vertices in a graph G. An XY-path P is a path whose one endpoint lies in X and the other in Y. A *linking* $L(X, Y)$ between X and Y is an infinite set of pairwise edge-disjoint XY-paths. A linking $L(X, Y)$ is X-*strong* (respectively Y-*strong*) if all endpoints of its paths which belong to X (respectively Y) are distinct. If a linking is both X-strong and Y-strong, we simply say that it is a *strong linking*. Clearly, if $L(X, Y)$ is a strong linking then both X and Y are infinite. For convenience, we shall abreviate $L(V(H), V(K))$ by $L(H, K)$ when H and K are two subgraphs of G and $\{x\}$ by x.

Strong linkings are related to edge-equivalence in the following way.

LEMMA 2. *Two rays P and Q in G are edge-equivalent if and only if there is a strong linking $L(P, Q)$.*

Proof. It is obvious that $P \sim_e Q$ when there is a strong linking $L(P, Q)$. On the other hand, the construction of such a linking is straightforward when $P \sim_e Q$. First take any PQ-path, say W_1. Since $P \sim_e Q$, we have that the tails of P and Q lie in the same component of $G \backslash W_1$; let us take in $G \backslash W_1$ any PQ-path, say W_2 which does not have the same endpoints as W_1. Again, $E(W_1 \cup W_2)$ cannot separate tails of P and Q, and so we may choose any PQ-path W_3 in $G \backslash (W_1 \cup W_2)$ which does not have the same endpoints as W_1 and W_2. Continuing in this manner ad infinitum will give the desired strong linking. ∎

One might think that problems of edge-equivalence reduce simply to those of vertex-equivalence. This, however, is not the case: there are rays in line-graphs which do not correspond to rays in the original graphs. Consider, for example, an infinite star and its line graph.

The ideas of quotient of a graph and of vertex-domination, as used in [4] and [8] for problems on vertex-equivalence, turn out to be useful in the present setting together with the notion corresponding to domination for edge-end.

Let H be a spanning subgraph of G (possibly with isolated vertices), and let K any subgraph of G. We denote by K/H the graph whose vertex set is the set of all connected components of H meeting K and where $H_0 H_1$ is an edge of K/H if and only if $H_0 \neq H_1$ and there exists an edge of K incident with both a vertex of H_0 and a vertex of H_1. The graph K/H is called the *quotient graph of K by H* (or, as in [8], the *contraction of K along H*). Note that we will not always require that H be spanning since the completion to a spanning subgraph will canonically be assumed by adding isolated vertices to H.

A ray P is said to be $\mathscr{V}$*-dominated* [$\mathscr{E}$*-dominated*] in G if there exists a vertex $x \in V(G)$ which cannot be separated from a tail of P by the removal of finitely many vertices of $V(G)\backslash\{x\}$ [finitely many edges of G] or, equivalently, if there is a linking $L(x, P)$ whose paths pairwise intersect in x only [which is strong on $V(P)$]. If P is a ray $\mathscr{V}$-dominated [$\mathscr{E}$-dominated] by x and if $Q \sim_v P$ [$Q \sim_e P$], then so is Q. This allows us to say that a (vertex- or edge-) end is dominated whenever one ray of it is. We shall also say that a vertex-end is *strictly edge-dominated*, or, simply, *strictly dominated*, if it is $\mathscr{E}$-dominated but not $\mathscr{V}$-dominated.

Note that it is immediate from the definition of $\mathscr{V}$- and $\mathscr{E}$-domination that if a vertex $\mathscr{V}$-dominates some vertex-end α, then it will $\mathscr{E}$-dominate the edge-end which contains α.

Remark 1. There is an important distinction between $\mathscr{V}$-domination and $\mathscr{E}$-domination in the sense that a vertex can $\mathscr{E}$-dominate *at most one* edge-end, whereas the number of $\mathscr{V}$-dominated vertex ends can be arbitrarily large. The reason for this is an underlying transitivity between $\mathscr{E}$-equivalent rays, their $\mathscr{E}$-dominating vertices and the vertices infinitely linked to them.

The close relationship between edge-equivalence in a graph G and vertex-equivalence in its line graph also extends to $\mathscr{E}$-domination in G and $\mathscr{V}$-domination in $\mathscr{L}(G)$, as shown in the following lemma.

LEMMA 3. *Let G be a graph and P a ray of G. Then P is $\mathscr{E}$-dominated in G if and only if $\mathscr{L}(P)$ is $\mathscr{V}$-dominated in $\mathscr{L}(G)$.*

Proof. If a vertex x $\mathscr{E}$-dominates P in G, then it is easy to see that any edge e incident with x will $\mathscr{V}$-dominate $\mathscr{L}(P)$ in $\mathscr{L}(G)$. Indeed, if we cannot separate x from a tail of P by deleting finitely many edges of G we will also be unable to separate the vertex e from a tail of $\mathscr{L}(P)$ by deleting finitely many vertices of $\mathscr{L}(G)$. On the other hand, if $\mathscr{L}(P)$ is $\mathscr{V}$-dominated by some $e = xy \in V(\mathscr{L}(G))$, any infinite family of $e\,\mathscr{L}(P)$-paths of $\mathscr{L}(G)$, pairwise intersecting in e only, will induce an infinite family of finite pairwise edge-disjoint connected subgraphs of G, each of which contains at least one of the vertices x, y and at least one edge of P. Without loss of generality suppose that x belongs to infinitely many of these subgraphs. Construct a linking $L(X, P)$ which is strong on P by choosing a path from each of infinitely many of these subgraphs in such a way that their endpoints on P are distinct. ∎

The end structure of a graph (see [1] for an excellent survey) is best studied by considering a faithful representation of it in a simpler subgraph. More precisely, a subgraph H of a graph G is called *$\mathscr{V}$-faithful* [*$\mathscr{E}$-faithful*] if there is a bijection $f\colon \mathscr{V}(H) \to \mathscr{V}(G)[g\colon \mathscr{E}(H) \to \mathscr{E}(G)]$ such that for every $\alpha \in \mathscr{V}(H)$ $[\beta \in \mathscr{E}(H)]$ we have $\alpha \subseteq f(\alpha)[\beta \subseteq g(\beta)]$. The faithful subgraphs most frequently studied are trees.

For vertex-ends, Halin showed that for countable graphs such a representation in simpler subgraphs always exists.

THEOREM (Halin [3]). *Every countable graph has a $\mathscr{V}$-faithful spanning tree.*

There is also the following result which characterizes the existence of a rayless spanning tree (i.e., a subgraph representing no end at all).

THEOREM (Polat [5] and Širáň [8]). *A countable graph G has a rayless spanning tree if and only if every ray in G is $\mathscr{V}$-dominated.*

3. DOMINATION PROPERTIES

As we shall see, rays that are not $\mathscr{V}$-dominated (shortly, $\mathscr{V}$-undominated rays) play a prominent part in the existence of spanning trees with some specific properties. One of the best examples of this fact is the following lemma of [4]. We give here a different, more direct, proof.

LEMMA 4. *Let G be a connected graph, T a spanning tree of G. Then T contains a ray of any given $\mathscr{V}$-undominated vertex-end of G.*

Proof. Fix $x_0 \in V(G)$, take any $\mathscr{V}$-undominated ray $R = x_0x_1\ldots$ of G and define T_0 as the subtree of T which is the union of all x_0R-paths in T. We claim that T_0 is locally finite. By way of contradiction, suppose that there exists a vertex x of infinite degree in T_0 and let $T_1 = T_0 - \{x\}$. Since T_0 is a tree, T_1 must have infinitely many connected components, one for each edge of T_0 incident with x. By the construction of T_0 we have that every component of T_1 must contain a vertex of R. Hence it is easy to construct a linking $L(x, R)$ whose paths pairwise intersect on x only, which contradicts the fact that R is not $\mathscr{V}$-dominated. The tree T_0 is therefore locally finite and, since it is infinite, it must contain a ray, say R_0. Now, $R_0 \sim_v R$ in G, because by the construction of T_0, any finite subset of $V(T_0)$ can be included in a finite union of xR-paths in T. Thus such a finite subset can only separate the vertices of a tail of R from a finite subgraph of T_0 whereas R_0 is an infinite subgraph of it. ▮

An interesting and useful consequence of Lemma 4 is:

COROLLARY 1. *Let A be an infinite set of vertices of a connected graph G. Then there exists either a vertex $x \in V(G)$ and a linking $L(x, A)$ intersecting at x only, or a ray R and a linking $L(R, A)$ consisting of pairwise vertex-disjoint (possibly trivial) paths.*

Proof. Construct a graph H by adding to G a ray Q whose vertex set is contained in A and which is edge-disjoint from G. Note that H might have multiple edges. Let T be a spanning tree of G or equivalently a spanning tree of H containing no edge of Q. If Q is $\mathscr{V}$-dominated by a vertex x in H, there exists a linking $L(x, Q)$ by paths of G pairwise intersecting in x only, and since $V(Q) \subset A$ we are done. On the other hand, if Q is not $\mathscr{V}$-dominated in H, then, applying Lemma 4, we have that T must contain a ray R which is vertex-equivalent to Q in H, and hence there must exist a linking $L(R, Q)$ consisting of pairwise disjoint paths of G. ▮

Restricting graphs to those having only one edge-end, Lemma 4 and the theorem of Polat–Širáň [5, 8] provide a solution to our $\mathscr{E}$-faithful spanning tree problem.

PROPOSITION 1. *Let G be a countable connected graph with precisely one edge-end. Then G has an $\mathscr{E}$-faithful spanning tree if and only if all $\mathscr{V}$-undominated rays are vertex-equivalent.*

Proof. If G contains two $\mathscr{V}$-undominated rays which are not vertex-equivalent, then by Lemma 4 any spanning tree has two disjoint rays and hence cannot be $\mathscr{E}$-faithful in a one-edge-ended graph. On the other hand, take any ray P ($\mathscr{V}$-undominated if such exist, arbitrary otherwise), and observe that in G/P all rays are $\mathscr{V}$-dominated. By the theorem of Polat–Širáň [5, 8], G/P has a rayless spanning tree $\tilde{T}$. Let $x = P/P$ and for each edge $xy \in E(\tilde{T})$ let z_y be one of the neighbours of y in P. The spanning tree

T of G obtained from $\tilde{T}$ by replacing x by P and each edge of the form xy in $\tilde{T}$ by the edge $z_y y$ in G, has the required properties.

It is important to note that the above proposition does not extend to uncountable graphs. This is witnessed by the example constructed by Seymour and Thomas [7] which provides a negative solution to Halin's problem [3] of the existence of a $\mathscr{V}$-faithful spanning tree in general (see also Thomassen [9]).

If we now try to generalize the condition of Proposition 1 to arbitrary graphs we are naturally led to the following definition.

DEFINITION 1. A graph G is *end-correlated* if any two edge-equivalent $\mathscr{V}$-undominated rays of G are vertex-equivalent; that is, if

$$P \sim_e Q \Leftrightarrow P \sim_v Q$$

for any two $\mathscr{V}$-undominated rays P and Q of G.

In other words, G is end-correlated if and only if every edge-end contains at most one $\mathscr{V}$-undominated vertex-end.

As an immediate consequence of Lemma 4, we have the following.

PROPOSITION 2. *A graph containing an $\mathscr{E}$-faithful spanning tree is end-correlated.* ▮

In fact (as we will show in Theorem 5), for countable connected graphs, end-correlation is not only a necessary condition but it is also sufficient. However, before proving Theorem 5, we introduce some further concepts which will turn out to be equivalent to end-correlation. The first of these generalizes the example of Figure 1.

A *caterpillar* is a connected graph G containing a sequence $\{H_i\}_{i \in \mathscr{Z}}$ of (not necessarily connected) subgraphs whose union is G and such that for any two distinct integers $i, j \in \mathscr{Z}$

1. $S_i = V(H_i) \cap V(H_{i+1})$ is a finite set;
2. $V(H_i) \cap V(H_j) = \varnothing$ if $|i - j| \geqslant 2$;
3. there is a sequence of vertices $(x_i)_{i \in \mathscr{Z}}$, $x_i \in S_i$ such that each H_i contains a linking $L(x_{i-1}, x_i)$.

If G contains such a sequence indexed only by *non-negative* integers we call it a *half-caterpillar*. Clearly, a caterpillar is also a half-caterpillar but the converse is not true. Observe that since G is connected the sets S_i must be non-empty and hence are necessarily cutsets separating $\bigcup_{j \leqslant i} V(H_j)$ from $\bigcup_{j > i} V(H_j)$.

The following result relates caterpillars to contraction and domination.

PROPOSITION 3. *Let P and Q be disjoint edge-equivalent rays in a graph G such that P is not $\mathscr{V}$-dominated in G/Q. Then G is a half-caterpillar.*

Proof. Observe first that P and Q cannot be in the same vertex-end, otherwise the vertex x_Q obtained by contracting Q $\mathscr{V}$-dominates P in G/Q. Therefore, there is a finite vertex cutset S_0 which separates P and Q. Let G_1 be the component of $G - S_0$ that contains a tail of P and let G_1^* be the subgraph of G induced by the vertex set $V(G_1) \cup S_0$. Moreover let $H_0 = G - G_1$; observe that $S_0 = V(H_0) \cap V(G_1^*)$.

Since $P \sim_e Q$, there is a strong linking $L(Q, P)$, by Lemma 2. This linking squeezes infinitely many edge-disjoint paths through a finite set S_0 of vertices. Consequently, there exists a vertex $x_0 \in S_0$ and a linking $L_0 = L(x_0, P)$ (considered as a subgraph of $L(Q, P)$) in G_1^* such that the set of its endpoints on P is infinite. Let $W_1 \subset V(G_1)$ be the set of all neighbours (that belong to G_1) of vertices of S_0. This set is also infinite, since L_0 is (and our graphs do not have multiple edges). We may suppose that W_1 is disjoint from a tail of P since otherwise some vertex in S_0 would $\mathscr{V}$-dominate P.

We observe that in G_1 there is a finite cutset S_1 separating W_1 from a tail of P (indeed, in the contrary case we would have an infinite set of mutually vertex-disjoint W_1P-paths in G_1, and consequently some vertex in S_0 would $\mathscr{V}$-dominate P). Now, the linking $L_0 = L(x_0, P)$ squeezes an infinite number of edge-disjoint x_0P-paths through the finite set S_1. It follows that there exists a vertex $x_1 \in S_1$ and a linking $L(x_0, x_1)$ which is a subgraph of L_0 (and hence a subgraph of G_1^*).

Let G_2 be the component of $G_1 - S_1$ containing a tail of P and let $H_1 = G_1^* - G_2$. (Note that $S_0 = V(H_0) \cap V(H_1)$.) Clearly, the linking $L(x_0, x_1)$ is contained in H_1. At the same time we see that, by the finiteness of S_1, the linking L_0 necessarily contains (as a subgraph) linking $L_1 = L(x_1 P)$ with an infinite set of endpoints on P.

This construction can be repeated, mutatis mutandis, to obtain the subgraphs H_i, the finite cutsets $S_i = V(H_i) \cap V(H_{i+1})$ and the linkings $L(x_{i-1}, x_i)$, $x_i \in S_i$ for $i = 2, 3, \ldots$ starting with S_{i-1} in place of S_0 and defining $L_{i-1} = L(x_{i-1}, P)$, W_i, H_i, G_i^* and the linkings $L(x_{i-1}, x_i)$ along the way so that all the conditions from the definition of a half-caterpillar are satisfied. Thus, G is a half-caterpillar, as claimed. (Moreover, as a by-product we see that our construction guarantees that the ray P is vertex-equivalent to any ray that passes through the vertices x_i for infinitely many $i > 0$.) ∎

We will say that a graph G has the *symmetric domination property* if for any two disjoint edge-equivalent rays P and Q either P is $\mathscr{V}$-dominated in G/Q or Q is $\mathscr{V}$-dominated in G/P.

PROPOSITION 4. *Let G be a graph with the symmetric domination property. Then in every edge-end α of G there is a ray Q_α such that every ray in α disjoint from Q_α is $\mathscr{V}$-dominated in G/Q_α.*

Proof. Let α be an edge-end of G and let $Q \in \alpha$ be a ray. Assume that there is another ray $P \in \alpha$, disjoint from Q, which is not $\mathscr{V}$-dominated in G/Q. Fix such Q and P; by the symmetric domination property, Q is $\mathscr{V}$-dominated in G/P. Let R be an arbitrary ray in α disjoint from P. We claim that R is $\mathscr{V}$-dominated in G/P, which will prove the Proposition with $Q_\alpha = P$. Clearly, the claim is true if $R \sim_v Q$ because Q, an hence R, is $\mathscr{V}$-dominated in G/P.

The facts that $P, Q \in \alpha$ and that P is not $\mathscr{V}$-dominated in G/Q imply (by Proposition 3) that G is a half-caterpillar. A half-caterpillar admits infinitely many ways of choosing the subgraphs, cutsets and linkings for its description. In what follows we assume that the subgraphs H_i and the finite cutsets $S_i = V(H_i) \cap V(H_{i+1})$, $i \geqslant 0$ are *exactly* the ones given by the construction in the proof of Proposition 3. Thus, we assume that the ray Q is contained in H_0 whereas the say P is the one which intersects every S_i, $i > 0$. We will consider two cases.

1. First assume that the vertices of a tail of R lie in some H_i, $i \geqslant 0$. If R is not $\mathscr{V}$-dominated in G/P then, by the symmetric domination property, P is $\mathscr{V}$-dominated in G/R. But then, since P intersects every S_i, the infinitely many paths of domination would have to pass through the finitely many vertices of the two cutsets S_{i-1} and S_i (or, just S_0 if $i = 0$), which is impossible.

2. If no tail of R lies in some H_i then R intersects infinitely many of the H_i's. Then R also intersects infinitely many cutsets S_i. Now, by the construction from the proof of Proposition 3, no finite cutset can separate R from P, and so P and R are vertex-equivalent. But then clearly R is $\mathscr{V}$-dominated in G/P (assuming, of course, that P and R are disjoint).

This completes the proof. ∎

4. END-CORRELATION, CATERPILLARS, AND SYMMETRIC DOMINATION

In this section we will establish equivalence between the different properties we have defined previously. As a first result, let us show that end-correlation, not being a caterpillar and symmetric domination are three aspects of the same thing, the first being stated in terms of $\mathscr{V}$-dominating vertices, the second in terms of separators and edge-connectivity, and the third in terms of quotient graphs.

THEOREM 1. *For a connected graph G, the following statements are equivalent.*

1. *G is end-correlated*;
2. *G is not a caterpillar*;
3. *G has the symmetric domination property.*

Note that no assumption is made concerning the cardinality of G.

Proof. $1 \Rightarrow 2$. We prove the converse. Let G be a caterpillar and let $(x_i)_{i \in \mathcal{Z}}$ be the sequence referred to in the definition of a caterpillar. Let P and Q be two rays in G such that P contains all the x_i for $i > 0$ and Q contains all the x_i for $i < 0$. Invoking the properties of a caterpillar, it is easy to see that P and Q belong to the same edge-end but to different vertex-ends, and neither of these two vertex-ends is $\mathcal{V}$-dominated in G.

$2 \Rightarrow 3$. Assume that there are two disjoint edge-equivalent rays P and Q such that neither is $\mathcal{V}$-dominated by the other in the appropriate contracted graph. But if P is not $\mathcal{V}$-dominated in G/Q, then G is a half-caterpillar whose structure (i.e., the subgraphs H_i, cutsets $S_i = V(H_i) \cap V(H_{i+1})$ and linkings $L(x_i, x_{i+1})$ in H_{i+1}, $i \geqslant 0$) has been described in the proof of Proposition 3. In particular, adopting the same notation as in that proof, we may assume that the ray Q is contained in H_0 while P intersects all the S_i for i sufficiently large. Since now we also assume that Q is not $\mathcal{V}$-dominated in G/P, we may switch the roles of P and Q. Using now the non-positive integers as subscripts, the construction in the proof of Proposition 3 now endows G with another half-caterpillar structure with subgraphs H'_i, $i \leqslant 0$, where H'_0 is obtained by deleting from G all vertices of the component of $G - S_0$ that contains a tail of Q (note that S_0 is the initial cutset separating P from Q). It is a matter of routine to check that the two half-caterpillars can be combined to form a caterpillar, which contradicts 2.

$3 \Rightarrow 1$. Let α be an edge-end of G. For a contradiction assume that α contains two disjoint rays P and Q such that $P \nsim_v Q$ and neither is $\mathcal{V}$-dominated in G. By the symmetric domination property and Proposition 4, α contains a ray R such that every ray in α disjoint from R is $\mathcal{V}$-dominated in G/R. If some tails of both P and Q are disjoint from R, then both P and Q must be $\mathcal{V}$-dominated in G/R by the vertex obtained by contracting R. But then P and Q are vertex-equivalent in G or one of them is $\mathcal{V}$-dominated already in G, a contradiction. If, on the other hand and without loss of generality, P intersects R infinitely often (and hence $P \sim_v R$), then a tail of Q may be assumed to be disjoint from R (otherwise $Q \sim_v R \sim_v P$, contrary to our assumption). Since Q is $\mathcal{V}$-dominated in G/R

and $P \not\sim_v R$, Q is also $\mathscr{V}$-dominated in G/P. But then, $P \sim_v Q$ implies that Q must already be $\mathscr{V}$-dominated by some vertex on P, again a contradiction. The proof is complete. ∎

We also have a similar result involving half-caterpillars.

THEOREM 2. *A connected graph G is a half-caterpillar if and only if it contains a strictly dominated end.*

Proof. *Necessity.* Let G be a half-caterpillar and $(x_i)_{i \geqslant 0}$ the sequence referred to in the definition. Let P be a ray in G passing through all vertices x_i. Invoking the properties of a half-caterpillar it is easily shown that P is $\mathscr{E}$-dominated but not $\mathscr{V}$-dominated in G.

Sufficiency. Suppose G is not a half-caterpillar but contains a strictly dominated ray P. Let x be a vertex which $\mathscr{E}$-dominates P in G. Now take a ray Q vertex-disjoint from G and construct a new graph $\hat{G}$ by attaching Q to G as follows:

$$V(\hat{G}) = V(G) \cup V(Q),$$
$$E(\hat{G}) = E(G) \cup E(Q) \cup \{ax : a \in V(Q)\}.$$

Note that since G is not a half-caterpillar, neither is $\hat{G}$. Moreover, since in $\hat{G}$ both P and Q are $\mathscr{E}$-dominated by x, we have $P \sim_e Q$ in $\hat{G}$. Now applying Theorem 1 and Proposition 4 we obtain that P must be $\mathscr{V}$-dominated in $\hat{G}/Q$, which is a contradiction since, by construction of $\hat{G}$, $\hat{G}/Q$ is isomorphic to the union of G with some edge $[x, q]$, $q \neq V(G)$. ∎

5. FAITHFULNESS ON $\mathscr{V}$-UNDOMINATED ENDS

A subgraph H of a graph G is called *end-preserving* if there is an injective function $f\colon \mathscr{V}(H) \to \mathscr{V}(G)$, necessarily unique, such that $\alpha \subset f(\alpha)$ for every $\alpha \in \mathscr{V}(H)$. This notion is a generalization of the notion of $\mathscr{V}$-faithfulness, an end-preserving subgraph being $\mathscr{V}$-faithful if f is also surjective. Both concepts were introduced by Halin [3]. It follows that an $\mathscr{E}$-faithful subgraph H of G is an end-preserving subgraph such that $f(\mathscr{V}(H))$ contains exactly one vertex-end from each of the edge-ends of G. As in [4] an end-preserving subgraph H will be called *$\mathscr{A}$-faithful*, where $\mathscr{A} \subset \mathscr{V}(G)$, if $f(\mathscr{V}(H)) = \mathscr{A}$, i.e., the ends of G represented in H are precisely those belonging to $\mathscr{A}$. By Lemma 4 it is easy to see that a necessary condition for a graph to have an $\mathscr{A}$-faithful spanning tree is that each vertex-end not in $\mathscr{A}$ be $\mathscr{V}$-dominated. The most interesting case, however, occurs when $\mathscr{A}$ is the set of *all* $\mathscr{V}$-undominated vertex-ends, which we denote by $\mathscr{U}$. A $\mathscr{U}$-faithful spanning tree is in some sense "as rayless as possible."

Unfortunately, such spanning trees do not always exist even in the countable case (see [4] for counterexamples). However, as we will prove in Theorem 4, they exist in countable graphs having no strictly dominated ends. This fact will lead us to our main theorem, the characterization of countable graphs having an $\mathscr{E}$-faithful spanning tree (Section 6).

In this section, given any $A \subset V(G)$, $[A, \bar{A}]$ will denote the set of all edges having one endvertex in A and the other in $\bar{A} = V(G) \backslash A$.

DEFINITION 2. Given any subgraph H of a connected graph G, a *crown* of H is a family $(K_i)_{i \in I}$ of pairwise vertex-disjoint connected subgraphs of G such that

(i) H is vertex-disjoint from K_i for any i;

(ii) $C_i := [V(K_i), \overline{V(K_i)}]$ is finite for any i;

(iii) any ray of G which is not $\mathscr{E}$-dominated in G by a vertex of H and not edge-equivalent in G to a ray in H, has a tail in some K_i.

Note that in the preceding definition we did not insist that the set I be non-empty, and thus we allow also empty crowns (where applicable). For example, if G is a (connected) graph with no finite edge-cuts and if H is any connected subgraph of G, then the empty family is a crown of H in G (because in this case any vertex of H $\mathscr{E}$-dominates any ray in G).

If a crown of a subgraph H exists we say that H is *crownable*. Intuitively, a crown presents a means for "isolating H from" the edge-ends not in $\mathscr{E}(H)$ which are not $\mathscr{E}$-dominated by any vertex of H. As we will show in the next result, for connected subgraphs such an "isolation" is always possible.

THEOREM 3. *Any connected subgraph of a connected graph is crownable.*

Proof. Let G be a connected graph and H a connected subgraph of G. We distinguish two cases.

Case 1. H consists of just one isolated vertex u. Let J be the set of all vertices $v \in V(G)$ which are infinitely edge-connected to u (that is, there is a linking $L(u, v)$), and J^+ the set J together with all its neighbours. Let $\mathscr{R}$ be a maximal set (w.r. to inclusion) of rays emanating from u which are $\mathscr{E}$-dominated in G by u, and uJ^+-paths such that the intersection of any two members of $\mathscr{R}$ is an initial segment of both (the existence of such a maximal set is obvious). Finally let $\hat{H}$ be the subgraph of G induced by u and the vertices of all paths and rays in $\mathscr{R}$. Let $(K_i)_{i \in I}$ be the set of all non-trivial components of $G \backslash \hat{H}$ (that is, no K_i is an isolated vertex). We claim that $(K_i)_{i \in I}$ is a crown of $\{u\}$.

Note that it is clear that the K_i's are pairwise disjoint connected subgraphs of G, and that condition (i) of the definition of crowns is satisfied, because by construction $\hat{H}$ contains all the edges incident to u implying that u is an isolated vertex of $G\backslash\hat{H}$.

Now, suppose for a moment that we have a finite cut set $[A, \bar{A}]$ of G with $u \in \bar{A}$. Then there must exist a finite subset $\mathscr{B}$ of $\mathscr{R}$ such that

$$V\left(\bigcup_{R \in \mathscr{B}} R\right) \supseteq V(\hat{H})\backslash\bar{A}.$$

Consequently (as it can be easily seen), the set $V(\hat{H})\backslash\bar{A}$ is necessarily finite. This analysis quickly implies that for any infinite subset $U \subseteq V(\hat{H})$ there is a U-strong linking $L(u, U)$. Hence, condition (iii) of the definition of a crown is satisfied, because any ray not $\mathscr{E}$-dominated by u in G will have a tail disjoint from $\hat{H}$.

It remains to show that condition (ii) is also satisfied. Suppose by way of contradiction that there exists an $i \in I$ such that $C_i := [V(K_i), \overline{V(K_i)}]$ is infinite. Let X (resp. Y) be the set of vertices in $V(K_i)$ (resp. $\overline{V(K_i)}$) incident with an edge of C_i (so that $C_i = [X, Y]$). The fact that K_i is a component of $G\backslash\hat{H}$ implies that C_i is contained in $E(\hat{H})$ and hence $X, Y \subseteq V(\hat{H})$. Since C_i is assumed to be infinite, and G has no multiple edges, either X or Y must be infinite. Observe that since K_i is a non trivial connected component of $G\backslash\hat{H}$ and since $\hat{H}$ is an induced subgraph of G, each vertex of X is adjacent in K_i to a vertex not in $\hat{H}$. Hence, no vertex of X may belong to J; otherwise it would be possible to add to $\mathscr{R}$ another xJ^+-path, contradicting the maximality. This all implies that X must be infinite. Indeed, if not, then Y would be infinite and hence there would be a Y-strong linking $L(Y, u)$ (cf. the remark at the end of the proof of condition (iii)). Due to finiteness of X, one of its vertices, say y would have infinitely many neighbours among the endpoints of $L(Y, u)$. Thus y would be infinitely connected to u, a contradiction to X and J being disjoint.

Because of the fact that X is an infinite subset of $V(\hat{H})$, there exists an X-strong linking $L(u, X)$. Let X' be the set of all endpoints (in X) of paths of $L(u, X)$. As $X' \subseteq V(K_i)$, Corollary 1 implies that in K_i there is either a vertex z and an X'-strong linking $L(X', z)$, or a ray R and a strong linking $L(X', R)$. In the first case, z must be infinitely edge-connected to u, because one can construct a linking $L(u, z)$ from $L(u, X')$ and $L(X', z)$. In the second case, the ray R must be $\mathscr{E}$-dominated by u, a $V(R)$-strong linking $L(u, R)$ being constructible from $L(u, X')$ and $L(X', R)$. In either case we obtain a contradiction with maximality of $\mathscr{R}$. This shows that (ii) is satisfied, as claimed.

Hence, $(K_i)_{i \in I}$ is a crown of u.

Case 2. H is any connected subgraph of G. W.l.o.g. we may suppose that no vertex of $G-H$ is adjacent to more than one vertex of H. (If this is not the case, just subdivide each edge in $[H, \bar{H}]$ into a path of length two, obtaining a new graph G'. A crown in G' of the graph H' induced by H and the newly added degree 2 vertices will yield a crown of H in G.) Now let $(\tilde{K}_i)_{i \in I}$ be a crown of the vertex $\tilde{u} = H/H$ in G/H. Let $(K_i)_{i \in I}$ be a family of subgraphs of G such that $K_i/H = \tilde{K}_i$. It is easy to see that the K_i's are pairwise vertex-disjoint since so are the $\tilde{K}_i$'s. Let us show that $(K_i)_{i \in I}$ is a crown of H in G.

Condition (i) is clearly satisfied for any i because H and K_i are already vertex-disjoint since, the $\tilde{K}_i$'s being a crown of $\tilde{u} = H \backslash H$, we have $\tilde{u} \notin V(\tilde{K}_i)$ for any i.

(ii) C_i is finite since by construction no vertex of K_i is incident with more than one vertex of H, whence there is a canonical bijection from $C_i = [V(K_i), \overline{V(K_i)}]$ to the finite subset $[V(\tilde{K}_i), \overline{V(\tilde{K}_i)}]$ of G/H.

(iii) Given a ray R with no tail in any K_i, let us show that R is either $\mathscr{E}$-dominated by a vertex of H or edge-equivalent to a ray in H. If R meets H infinitely often, we are done by Corollary 1. Otherwise we may suppose that R is disjoint from H, so that the ray R/H is $\mathscr{E}$-dominated by $\tilde{u}$ in G/H, since R/H has no tail contained in a $\tilde{K}_i$. Let $L(\tilde{u}, R/H)$ be a $V(R/H)$-strong linking and let $L(H, R)$ be the corresponding linking in G. Then it is easy to see that if there is an $x \in V(H)$ which is the endpoint of infinitely many paths of $L(H, R)$, then x $\mathscr{E}$-dominates R; if there is no such vertex, then the set B of endpoints of $L(H, B)$ is infinite and hence by Corollary 1, we have either a B-strong linking $L(y, B)$ in H for some $y \in V(H)$ or a strong linking $L(Q, B)$ for some ray Q in H. In the first case, y must $\mathscr{E}$-dominate R in G; in the second case, Q must be edge-equivalent to R in G. This completes the proof. ∎

Before proceeding, let us state a lemma which slightly generalizes the Polat–Širáň theorem [5, 8]; we omit the proof which is very similar to the ones given in [5, 8].

LEMMA 5. *Let G be a countable graph in which every ray is dominated. Then each rayless tree of G can be extended to a rayless spanning tree of G.* ∎

The next result can be viewed as a generalization of the preceding lemma.

LEMMA 6. *Let G be a countable graph with no strictly dominated ends. Let T be any rayless tree of G and let* $(K_i)_{i \in I}$ *be a crown of T. Then there exists a rayless tree T′ containing T and such that*

1. $\bigcap_{i \in I} \overline{V(K_i)} \subseteq V(T')$;

2. *all endvertices of edges in* $C_i := [V(K_i), \overline{V(K_i)}]$ *are in* T' *for any* i;

3. *for any* $i \in I$ *there exists a unique edge* $e_i \in C_i$ *which separates* $T' \cap K_i$ *from* T *in* T'.

Proof. Choose a spanning tree F_i in each K_i and let F be the spanning forest of G for which $E(F) = \bigcup_{i \in I} E(F_i)$ (that is, all vertices of $V(G) \setminus \bigcup_{i \in I} V(K_i)$ are isolated in F). Because T is vertex-disjoint from every K_i, the contracted graph T/F is isomorphic to T. Moreover, since T is rayless and crowned by $(K_i)_{i \in I}$, any ray of G having no tail in any K_i must be $\mathscr{E}$-dominated (and hence $\mathscr{V}$-dominated) in G. This implies that every ray of G/F is $\mathscr{V}$-dominated, and so, by Lemma 5, G/F contains a rayless spanning tree, say $\tilde{U}$, such that $T/F \subseteq \tilde{U}$.

Now for any $xy \in E(G/F)$ fix an edge e_{xy} of $E(G)$ connecting the component x of F to the component y of F, and let U be the subgraph induced by $\{e_{xy} : xy \in E(\tilde{U})\}$. Note that $T \subseteq U$ since $T/F \subseteq \tilde{U}$, and that U contains no edges of the K_i's. Finally for any $i \in I$ take a finite $T_i \subseteq K_i$ containing all the vertices incident with C_i and let $T' = U \cup \bigcup_{i \in I} T_i$.

Observe that each vertex K_i of $\tilde{U}$ lifts to the corresponding T_i in T' in such a way that acyclicity and connectedness is carried over from $\tilde{U}$ to T'. Due to the fact that the T_i's are finite (and hence rayless), no rays are introduced by the lifting, i.e., T' is rayless as well.

For the same reasons, conditions 1 and 2 are satisfied. Finally, if condition 3 fails for some $i \in I$, then, as T is connected, there would be a cycle in $\tilde{U}$ containing the vertex K_i and intersecting T/F, a contradiction. ∎

Before presenting the last (and most important) result of this section, we recall that we denoted by $\mathscr{U}$ the set of all $\mathscr{V}$-undominated vertex-ends of a graph G.

THEOREM 4. *Any countable connected graph G without strictly dominated rays has a* $\mathscr{U}$*-faithful spanning tree.*

Proof. By induction, we construct two sequences $(T_n)_{n \geqslant 0}$, $(T_n^+)_{n \geqslant 0}$ of rayless trees in G such that $T_n \subseteq T_n^+$, as well as a crown $\mathscr{K}_n$ of T_n^+, for each n.

Pick any vertex x_0 of G. Let $T_0 = \{x_0\}$, let T_0^+ be the (rayless) tree whose vertices are x_0 and its neighbours in G, and let $\mathscr{K}_0$ be any crown of T_0^+. Suppose T_n, T_n^+ and $\mathscr{K}_n$ have already been constructed. Let T_{n+1} be any tree containing T_n^+ and having properties 1, 2 and 3 of Lemma 6 with respect to $\mathscr{K}_n$, and let T_{n+1}^+ be any rayless tree containing T_{n+1} such that $V(T_{n+1})$ consists of the vertices of T_n and all their neighbours. Let $\mathscr{K}_{n+1}$ be a crown of T_{n+1}^+. It is clear from the construction that $(T_n)_{n \geqslant 0}$ is a nested sequence. We claim that $T := \bigcup_{n \geqslant 0} T_n$ is a $\mathscr{U}$-faithful spanning tree.

Note that, by construction, T is clearly a spanning tree, hence we only have to show that

(a) any $\mathscr{V}$-undominated end of G has a ray in T;

(b) no ray of T is $\mathscr{V}$-dominated in G;

(c) T is end-preserving; since T is a tree, this is equivalent to showing that any two disjoint rays of T are vertex-inequivalent.

(a) This a direct consequence of Lemma 4 (because T is a spanning tree).

(b) Let α be a $\mathscr{V}$-dominated end and suppose by way of contradiction that α contains a ray R which belongs to T. Let x be a vertex that $\mathscr{V}$-dominates R, and let n be any integer such that $x \in V(T_n)$. Note that $x \notin V(K)$ for any $K \in \mathscr{K}_n$. Therefore, and because $C_K := [V(K), \overline{V(K)}]$ is finite, for any $K \in \mathscr{K}_n$ each $\bar{K}$ contains some tail of R. Hence R meets T_{n+1} infinitely often, which implies that $R \cup T_{n+1}$ will contain a cycle, since obviously R cannot lie in the rayless tree T_{n+1}. Therefore T must also contain a cycle since it contains both R and T_{n+1}, a contradiction.

(c) Let P, P' be two disjoint rays of T and let T_n be any tree of the nested sequence, containing initial segments of both P and P'. As proved in (b), P and P' are not $\mathscr{V}$-dominated in G, and since $\mathscr{K}_n$ is a crown of T_n^+, a tail of P (resp. P') is contained in some $K \in \mathscr{K}_n$ (resp. $K' \in \mathscr{K}_n$). Note that by our construction there is a unique edge in $C_K = [V(K), \overline{V(K)}]$ (resp. $C_{K'} = [V(K'), \overline{V(K')}]$) separating T_n from $T_{n+1} \cap K$ (resp. $T_{n+1} \cap K'$) in T_{n+1}. This, together with the fact that the endpoints of the edges in C_K and $C_{K'}$ are in $V(T_{n+1})$, and the disjointness of P and P', implies that $K \neq K'$. Therefore C_K separates a tail of P from a tail of P' in G, forcing their edge-inequivalence and a fortiori their vertex-inequivalence. The proof is complete. ∎

6. $\mathscr{E}$-FAITHFUL SPANNING TREES

We can now prove the main result of the paper.

THEOREM 5. *Let G be a countable connected graph. Then the following are equivalent.*

1. *G has an $\mathscr{E}$-faithful spanning tree*;
2. *G is end-correlated*;
3. *G has the symmetric domination property*;
4. *G is not a caterpillar.*

Proof. By Theorem 1 and Proposition 2, we only have to show that $2 \Rightarrow 1$. Assume that G is end-correlated and let $(\alpha_i)_{i \in I}$ be the set of all $\mathscr{E}$-dominated edge-ends of G. Note that since G is countable and since (by Remark 1) a vertex can $\mathscr{E}$-dominate at most one edge-end, I must also be countable. Moreover, as G is end-correlated, at most one vertex-end of each α_i is not $\mathscr{V}$-dominated. Let $(R_i)_{i \in I}$ be a family of pairwise vertex-disjoint rays such that $R_i \in \alpha_i$ for any i, and choose R_i to be $\mathscr{V}$-undominated whenever $\mathscr{V}$-undominated rays exist in α_i.

For each $i \in I$ introduce a new vertex z_i ($\notin V(G)$) and define a graph $\hat{G}$ as follows:

$$V(\hat{G}) = V(G) \cup \{z_i \colon i \in I\},$$
$$E(\hat{G}) = E(G) \cup \{xz_i \colon i \in I \text{ and } x \in V(R_i)\}.$$

Claim 1. *The inclusion* $G \subseteq \hat{G}$ is *$\mathscr{V}$-faithful. In other words,*

(a) any ray in $\hat{G}$ is vertex-equivalent to some ray in G, and

(b) two rays in G are vertex-equivalent in G if and only if they are vertex-equivalent in $\hat{G}$.

To prove (a) note that for any ray $\hat{R}$ in $\hat{G}$, the subgraph R of G obtained from $\hat{R}$ by replacing for each $z_i \in V(\hat{R})$ the two edges xz_i and $z_i y$ of $E(\hat{R})$ by the xy-path in R_i, is a connected infinite, locally finite graph and any ray contained in R intersects $\hat{R}$ infinitely often.

The sufficiency part of (b) is a direct consequence of the fact that $G \subseteq \hat{G}$. To prove the necessity, take any two rays R and Q in G, which are vertex-equivalent in $\hat{G}$, and let $L(R, Q)$ be a linking consisting of pairwise disjoint paths of $\hat{G}$. Denote the paths of the linking by $(\hat{P}_k)_{k \geqslant 0}$. Let P_k be the subgraph of G obtained from $\hat{P}_k$ by replacing, for each $z_i \in V(\hat{P}_k)$, the two edges xz_i and $z_i y$ of $E(\hat{P}_k)$ by the xy-path in R_i. Note that P_k is connected and contains the endpoints of $\hat{P}_k$ since $\hat{P}_k$ is a $(V(R), V(Q))$-path and R, $Q \subseteq G$. Therefore, there exists a path Q_k in P_k which has the same endpoints as $\hat{P}_k$. Note that the Q_k's are not necessarily pairwise disjoint but that at most two of them can meet in any given $x \in V(G)$, and, moreover, that the common vertex x must belong to some R_i (this follows immediately from the construction of the Q_k's and the fact that the $\hat{P}_k$'s are pairwise disjoint). It is now a matter of routine to construct (out of the Q_k's) a linking $L'(R, Q)$ of pairwise vertex-disjoint paths of G, which establishes $\mathscr{V}$-equivalence of R and Q in the graph G.

Claim 2. *No Vertex-End of $\hat{G}$ Is Strictly Dominated.* Suppose this is not the case. As we have already shown that G is $\mathscr{V}$-faithful in $\hat{G}$, and since all rays in the $\mathscr{E}$-dominated ends in G (the α_i's) are $\mathscr{V}$-dominated in $\hat{G}$, a strictly dominated end in $\hat{G}$ must be $\mathscr{E}$-undominated in G. Take a strictly dominated ray R in $\hat{G}$ (which, as we have just seen, is necessarily $\mathscr{E}$-undominated in G). By the $\mathscr{V}$-faithfulness of G we may suppose $R \subseteq G$. Let $L(x, R)$ be any $V(R)$-strong linking in $\hat{G}$.

If $x \in V(G)$, then as in the proof of the preceding claim, we can construct a $V(R)$-strong linking $L'(x, R)$ in G, contradicting the hypothesis that R is $\mathscr{E}$-undominated in G. On the other hand, if $x = z_i$ for some $i \in I$, then in view of the definition of $\hat{G}$, we obtain a strong linking $L(R_i, R)$ by removing the vertex z_i from each path of the linking $L(z_i, R)$. Hence $R_i \sim_e R$, and by the $\mathscr{V}$-faithfulness of G in $\hat{G}$, we have that $R \in \alpha_i$, again a contradiction with the fact that R is not $\mathscr{E}$-dominated in G. This completes the proof of Claim 2.

By Theorem 4, $\hat{G}$ contains a $\mathscr{U}$-faithful spanning tree $\hat{T}$, where $\mathscr{U}$ is now the set of all $\mathscr{V}$-undominated vertex-ends of $\hat{G}$. Let $T := \hat{T} \cap G$ and $H := T \cup \bigcup_{i \in I} R_i$.

To complete this proof we only have to show that H is an $\mathscr{E}$-faithful connected spanning subgraph of G, because then any $\mathscr{V}$-faithful spanning tree of H is also an $\mathscr{E}$-faithful spanning tree of G. Note that a $\mathscr{V}$-faithful spanning tree in H always exists since H is countable; see Halin [3]. Since H is clearly a connected spanning subgraph of G, we only have to show that every edge-end has a ray in H and that any two rays of H that are edge-equivalent in G are edge-equivalent in H. Instead, we prove the following assertion (which appears to be stronger compared to what we need but actually is equivalent to it, see Diestel [2]):

every edge-end β has a ray in H that meets all the other rays of β that are in H.

Case 1. *$\beta = \alpha_j$ for some $j \in I$.* By way of contradiction, suppose there exists a ray R in H that is edge-equivalent to R_j but disjoint from it. Since such an R is edge-inequivalent to all R_i's, $i \in I \setminus \{j\}$, R can meet each of these R_i's at most finitely many times. Observe that each edge $e \in E(R) \setminus E(\hat{T})$ belongs to a unique R_i for some $i = i(e)$. Let $\hat{R}$ be the subgraph of $\hat{T}$ obtained from R by replacing each edge $e = xy \in E(R) \setminus E(\hat{T})$ by the two edges xz_i and $z_i y$ where $i = i(e)$. Clearly, $\hat{R}$ is connected, infinite and locally finite, and moreover, by the construction of $\hat{G}$ and $\hat{T}$, any ray $R' \subseteq \hat{R}$ is vertex-equivalent to R and disjoint from R_j. Since R' is contained in $\hat{T}$, which is $\mathscr{U}$-faithful in $\hat{G}$, R' and hence R are $\mathscr{V}$-undominated in $\hat{G}$ and a fortiori so is R in G.

By our assumptions, R and R_j are edge-equivalent in G. Moreover, as has just been proved, R is $\mathscr{V}$-undominated in G. However, by the choice of the R_i's, the ray R_j is $\mathscr{V}$-undominated in G as well. Invoking end-correlation of G, it follows that $R \sim_v R_j$ in G and therefore also $R' \sim_v R_j$ in $\hat{G}$. But then R' would be $\mathscr{V}$-dominated in $\hat{G}$ by z_j, contrary to what we obtained in the preceding paragraph.

Case 2. *$\beta \neq \alpha_i$ for all $i \in I$.* First note that by Lemma 4, β must contains a ray R in H because as $\beta \neq \alpha_i$ for all $i \in I$, it can not be $\mathscr{E}$-dominated in G and therefore it does not contains $\mathscr{V}$-dominated vertex-end. By way of contradiction, suppose that β contains an other ray Q, disjoint from R but still contained in H. Note that since in this case is $\mathscr{E}$-undominated in G, both R and Q are $\mathscr{E}$-undominated and a fortiori $\mathscr{V}$-undominated in G. Therefore, since G is end-correlated, we have that $R \sim_v Q$ in G.

Similarly to Case 1, let $\hat{R}$ (resp. $\hat{Q}$) be the subgraph of $\hat{T}$ obtained by replacing each edge xy in $E(\hat{R})\backslash E(\hat{T})$ (resp. $E(\hat{Q})\backslash E(\hat{T})$) by the edges xz_i and $z_i y$. Take two rays $R' \subseteq \hat{R}$ and $Q' \subseteq \hat{Q}$. As in the preceding case we have $R' \sim_v Q'$, $Q \sim_v Q$ (in $\hat{G}$). Therefore, since $R \sim_v Q$ in G (and hence in $\hat{G}$) we have $R' \sim_v Q'$ in $\hat{G}$. Since R' and Q' are contained in the $\mathscr{U}$-faithful tree $\hat{T}$, it follows that a tail of Q' is contained in R'. But according to our construction, R' and Q' can meet in the z_i's only, and hence must be edge-disjoint, the z_i's being by construction pairwise non-adjacent. This contradiction completes the proof. ∎

REFERENCES

1. R. Diestel, The end structure of a graph: Recent results and open problems, *Discrete Math.* **100** (1992), 313–327.
2. R. Diestel, Strongly end-faithful spanning trees in countable graphs, preprint, University of Cambridge.
3. R. Halin, Über unendliche Wege in Graphen, *Math. Ann.* **157** (1964), 125–137.
4. F. Laviolette and N. Polat, Spanning trees of countable graphs omitting sets of dominated ends, preprint, Université de Montréal, 1995.
5. N. Polat, Développements terminaux des graphes infinis. III. Arbres maximaux sans rayon. Cardinalité maximum des ensembles disjoints de rayons, *Math. Nachr.* **115** (1984), 337–352.
6. N. Polat, Topological aspects of infinite graphs, *in* "Cycles and Rays" (G. Hahn *et al.*, Eds.), NATO ASI Ser. C, pp. 197–220, Kluwer, Dordrecht, 1990.
7. P. Seymour and R. Thomas, An end-faithful spanning tree counterexample, *Discrete Math.* **95** (1991), 321–330.
8. J. Širáň, End-faithful forests and spanning trees in infinite graphs, *Discrete Math.* **95** (1991), 331–340.
9. C. Thomassen, Infinite connected graphs with no end-preserving spanning trees, *J. Combin. Theory Ser. B* **54** (1992), 322–324.