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Symmetry, Singularities, and Integrability in Complex Dynamics
II. Rescaling and Time-Translation in Two-Dimensional Systems

P. G. L. Leach,[1] S. Cotsakis, and G. P. Flessas

Geodysyc, Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece

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The explicit integrability of second-order ordinary differential equations invariant under time-translation and rescaling is investigated. Quadratic systems generated from the linearisable version of this class of equations are analysed to determine the relationship between the Painlevé and singularity properties of the different systems. The transformation contains a parameter and for critical values, intimately related to the possession of the Painlevé property in the parent second-order equation, one finds a difference from the generic behaviour. This study is a prelude to a full discussion of the class of transformations which preserve the Painlevé property in the construction of quadratic systems from scalar nth-order ordinary differential equations invariant under time invariant under time translation and rescaling.

1. SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS INVARIANT UNDER TIME-TRANSLATION AND RESCALING

The general form of the second-order ordinary differential equation invariant under the two symmetries representing invariance under time translation and rescaling, *viz.*,

$$G_1 = \partial_t, \qquad G_2 = -t\partial_t + x\partial_x, \tag{1.1}$$

[1] Permanent address: School of Mathematical and Statistical Sciences, University of Natal, Durban, South Africa 4041.

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is

$$\frac{\ddot{x}}{x^3} + f\left(\frac{\dot{x}}{x^2}\right) = 0, \tag{1.2}$$

where f is an arbitrary function of its argument. As a second-order equation with two symmetries, (1.2) can always be reduced to quadratures. In the particular case where f is linear with a specific form, i.e.,

$$f = 3b\left(\frac{\dot{x}}{x^2}\right) + b^2, \tag{1.3}$$

giving the equation

$$\ddot{x} + 3bx\dot{x} + b^2x^3 = 0, \tag{1.4}$$

the equation possesses eight Lie point symmetries, naturally with the algebra $sl(3, R)$, and so is linearisable [12].

Not only does (1.4) possess the Painlevé property but it has both left and right Painlevé series [11]. The equation is a representative of the Riccati hierarchy [9, Section 6.39, p. 550] and is integrable via transformation to a linear third-order equation using a generalised Riccati transformation. We write

$$x = \alpha\frac{\dot{w}}{w}, \tag{1.5}$$

where α is a constant to be determined. Equation (1.4) becomes

$$\frac{\dddot{w}}{w} + 3(\alpha b - 1)\frac{\dot{w}\ddot{w}}{w^2} + (b^2\alpha^2 - 3b\alpha + 2)\frac{\dot{w}^3}{w^3} = 0. \tag{1.6}$$

Equation (1.6) is a third-order linear differential equation in the case where both

$$\alpha b - 1 = 0 \qquad \text{and} \qquad b^2\alpha^2 - 3b\alpha + 2 = 0; \tag{1.7}$$

i.e., $\alpha b = 1$. The solution is

$$w = A + 2Bt + Ct^2, \tag{1.8}$$

where A, B, and C are arbitrary constants. Consequently,

$$x(t) = \frac{2}{b}\,\frac{t + k_1}{t^2 + 2k_1t + k_2}, \tag{1.9}$$

where k_1 and k_2 are arbitrary constants. The second of (1.7) is zero if $\alpha b = 2$. In this case (1.6) becomes

$$w\dddot{w} + 3\dot{w}\ddot{w} = 0 \tag{1.10}$$

which is a linear third-order equation in the variable w^2 with the same solution as (1.8) with the exception that the w is replaced by w^2. In either case the solution of (1.4) is the same when the transformation (1.5) is applied. It is amusing to note that if (1.10) is multiplied by the integrating factor w^2 and integrated, we obtain

$$\ddot{w} = \frac{K}{w^3}, \tag{1.11}$$

where K is the constant of integration, which is a particular instance of the Ermakov–Pinney equation [3, 14]. Thus one can easily see the relationship between the solutions of the two third-order equations. The solution of the Ermakov–Pinney equation,

$$\ddot{x} + \omega^2 x = \frac{1}{x^3}, \tag{1.12}$$

is given by

$$x = \sqrt{Au^2 + 2Buv + Cv^2}, \qquad AC - B^2 = W, \tag{1.13}$$

where u and v are two linearly independent solutions of

$$\ddot{x} + \omega^2 x = 0 \tag{1.14}$$

and W is the constant value of their Wronskian. Without the constraint on the constants of integration the expression in the square root is simply the general solution of a third-order equation of maximal symmetry, the solution of which can be expressed in terms of the three independent quadratic terms obtainable from the solution set of the corresponding second-order equation [13].

We can transform the second-order differential equation (1.4) into a two-dimensional system of first-order equations by a transformation which preserves the symmetries in (1.1). We write

$$y = x\left(\frac{1}{\beta}\frac{\dot{x}}{x^2} - \frac{\alpha}{\beta}\right) \tag{1.15}$$

to obtain the system

$$\dot{x} = \alpha x^2 + \beta xy \tag{1.16}$$

$$\dot{y} = -\frac{1}{\beta}(\alpha + b)(2\alpha + b)x^2 - 3(\alpha + b)xy - \beta y^2 \tag{1.17}$$

in which the constants α and β are arbitrary.

The generality of (1.4), (1.16), and (1.17) is merely apparent. In (1.4) the parameter b may be scaled out of the equation by the transformation

$bx \to x$ or, alternatively, by a rescaling of the time variable. In the case of Eqs. (1.16) and (1.17) we can remove the parameter β by rescaling the new variable y according to the transformation $\beta y \to y$. Consequently the analysis of these equations can be performed without loss of generality on the second-order ordinary differential equation

$$\ddot{x} + 3x\dot{x} + x^3 = 0 \tag{1.18}$$

and the system of two first-order equations

$$\dot{x} = ax^2 + xy \tag{1.19}$$

$$\dot{y} = -(a+1)(2a+1)x^2 - 3(a+1)xy - y^2 \tag{1.20}$$

obtained from (1.18) by the transformation

$$y = \frac{\dot{x}}{x^2} - ax, \tag{1.21}$$

in which we have replaced the parameter α with the parameter a to avoid confusion when we use the former in the Painlevé analysis. Note that G_2 of the original second-order equation becomes

$$G_2 = -t\partial_t + x\partial_x + y\partial_y \tag{1.22}$$

for the two-dimensional system.

The two-dimensional system, (1.19) and (1.20), is a particular instance of a class of quadratic systems integrable by the factorisation method of Adler–Kostant–Symes and its generalisation [5, 6] which have recently been studied from the point of view of their symmetry and singularity properties [10].

2. EXPLICIT INTEGRATION OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS OF VARIOUS CLASSES

We consider certain specific forms of (1.2), *viz*,

$$\frac{\ddot{x}}{x^3} + f\left(\frac{\dot{x}}{x^2}\right) = 0, \qquad x = x(t). \tag{2.1}$$

2.1. *Class* (A): $f(\dot{x}/x^2) = 2a\dot{x}/x^2$, *a Real*

Equation (2.1) becomes

$$\ddot{x} + 2ax\dot{x} = 0 \tag{2.2}$$

and, since (2.2) can be written as

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -2ax, \qquad p(x) = \dot{x}, \tag{2.3}$$

by carrying out the trivial integration of (2.3) we finally obtain for $x(t)$

$$x(t) = \sqrt{\frac{k}{a}}\left(\frac{k_1 e^{2t\sqrt{ak}} - 1}{k_1 e^{2t\sqrt{ak}} + 1}\right). \tag{2.4}$$

Equation (2.4) has a simple pole as a singularity.

2.2. *Case* (B): $f(\dot{x}/x^2) = 2a\dot{x}/x^2 + 2b$, *a and b Real*

In this case (2.1) becomes

$$\ddot{x} + 2ax\dot{x} + 2bx^3 = 0. \tag{2.5}$$

Equation (2.5) appears in Kamke [9, Section 6.43, p. 551], but the method used there leads, even for special values of a and b, to integrals which are beyond any hope of calculation. Therefore we resort to transforming (2.5) into a third-order equation by means of the Riccati Ansatz [11],

$$x = \alpha\frac{\dot{w}}{w}, \qquad w = w(t), \tag{2.6}$$

in which case (2.5) becomes

$$\frac{\dddot{w}}{w} + \frac{\ddot{w}\dot{w}}{w^2}(-3 + 2a\alpha) + \frac{\dot{w}^3}{w^3}(2 - 2a\alpha + 2b\alpha^2) = 0. \tag{2.7}$$

Choosing in (2.7)

$$\alpha = \frac{3}{2a}, \qquad b = \frac{2a^2}{9}, \tag{2.8}$$

we finally obtain from (2.7) and (2.6) that the general solution of (2.5) is

$$x(t) = \frac{3}{a}\left(\frac{t + k_1}{t^2 + 2k_1 t + k_2}\right) \tag{2.9}$$

provided that

$$b = \frac{2a^2}{9}, \tag{2.10}$$

where k_1 and k_2 are constants of integration. Equation (2.9) has poles as singularities. The results (2.9) and (2.10) for (2.5) are known [11] and are identical to (1.9) for (1.4) provided that we identify the a in (2.5) with $3b/2$ in (1.4).

In the following we drop the condition (2.10) and investigate the resulting third-order equation (2.7), i.e.,

$$w^2\,\dddot{w} + \dot{w}^3 c = 0, \qquad c = \frac{9b}{2a^2} - 1. \tag{2.11}$$

We reduce the order of (2.11) by using the standard substitution

$$\dot{w} = p, \qquad p = p(w). \tag{2.12}$$

Hence, from (2.11) and (2.12), we deduce that

$$w^2 pp'' + w^2 p'^2 + cp^2 = 0, \tag{2.13}$$

which is a linear second-order ordinary differential equation in the variable p^2 of Euler type. Consequently (2.13) has eight Lie point symmetries, which is an increase from the two of the original second-order ordinary differential equation. The technique of increasing and decreasing the order of an equation using different symmetries has often been found to be very useful [1, 7].

Equation (2.13) is treated in Kamke [9, Section 6.184, p. 585] and the result is

$$\begin{gathered} p(w) = \left[C_1 w^{\rho_1} + C_2 w^{\rho_2}\right]^{1/2} \\ \rho_1 = \frac{1}{2}\left(1 - \sqrt{1-8c}\right), \quad \rho_2 = \frac{1}{2}\left(1 + \sqrt{1-8c}\right), \quad c \neq 1/8. \end{gathered} \tag{2.14}$$

Equations (2.12) and (2.14) yield

$$\int w^{-\rho_1/2}\left[C_1 + C_2 w^{\rho_2-\rho_1}\right]^{-1/2} dw = t + C, \tag{2.15}$$

where C_1, C_2, and C are constants of integration. By means of the substitution

$$w^{\rho_2-\rho_1} = v^\gamma, \qquad \gamma = \frac{\rho_2-\rho_1}{1-\rho_1/2}, \qquad \sqrt{1-8c} = \frac{3\gamma}{4-\gamma} \tag{2.16}$$

the integral, I, on the left-hand side of (2.15) becomes

$$I = \frac{2}{2-\rho_1}\int\left[C_1 + C_2 v^\gamma\right]^{-1/2} \mathrm{d}v. \tag{2.17}$$

Now, I in (2.17) can be transformed into an integral of rational functions [8, pp. 70–71] if $1/\gamma = \text{integer} = k$. The case $k = 1$ leads, due to (2.14) and (2.16), to $\sqrt{1-8c} = 1$, i.e., $c = 0$, which in conjunction with (2.10) and (2.11) produces the already known result, (2.9). Let then $k \neq 1$ and set

$$C_1 + C_2 v^\gamma = u^2. \tag{2.18}$$

By means of (2.18) the integral in (2.17) is written as

$$I = \frac{4}{C_2\gamma(2-\rho_1)}\int\left(\frac{u^2-C_1}{C_2}\right)^{k-1} \mathrm{d}u. \tag{2.19}$$

In (2.19) $k-1=$ integer. By virtue of (2.16) $0<\gamma<4$. Thus from $1/\gamma=$ integer $=k$ we have $k=1,2,3,\ldots$. Let $k=2$, i.e., $\gamma=1/2$. Then by (2.11) and (2.16)

$$b=\frac{12a^2}{49}, \tag{2.20}$$

and from (2.19), on inserting ρ_1 from (2.14) and (2.16) with $\gamma=1/2$ and utilising (2.15), we obtain

$$u^3-3C_1u-\frac{9(t+C)C_2^2}{14}=0, \qquad u=u(t). \tag{2.21}$$

Owing to (2.6), (2.16), and (2.18) we obtain, since $\gamma=1/2$,

$$x(t)=\frac{7u\dot{u}}{a\,(u^2-C_1)}, \tag{2.22}$$

where $u(t)$ is given implicitly by (2.21).

Equation (2.21) can be solved analytically for $u=u(t)$. To this end we introduce the notations [2]

$$S_1=\left\{\frac{9}{28}C_2^2(t+C)+\left[-C_1^3+\frac{9^2}{28^2}C_2^4(t+C)^2\right]^{1/2}\right\}^{1/3} \tag{2.23}$$

and

$$S_2=\left\{\frac{9}{28}C_2^2(t+C)-\left[-C_1^3+\frac{9^2}{28^2}C_2^4(t+C)^2\right]^{1/2}\right\}^{1/3}. \tag{2.24}$$

Then, from (2.21),

$$\begin{aligned} u&=S_1+S_2,\\ u&=-\frac{1}{2}(S_1+S_2)+\frac{i\sqrt{3}}{2}(S_1-S_2),\\ u&=-\frac{1}{2}(S_1+S_2)-\frac{i\sqrt{3}}{2}(S_1-S_2), \end{aligned} \tag{2.25}$$

and, after a modicum of calculation, from (2.22)–(2.25) we deduce, considering first that $u=S_1+S_2$ in (2.25),

$$x(t)=\frac{7Y(t)}{3a(u_1^2-1)\sqrt{-1+k^2(t+C)^2}}, \tag{2.26}$$

where $k=\frac{9}{28}C_2^2/C_1^{3/2}$ and C are constants of integration, and

$$\begin{aligned} s_1&=\left\{k(t+C)+[-1+k^2(t+C)^2]^{1/2}\right\}^{1/3},\\ s_2&=\left\{k(t+C)-[-1+k^2(t+C)^2]^{1/2}\right\}^{1/3},\\ u_1&=s_1+s_2, \end{aligned} \tag{2.27}$$

$$Y(t) = k\left(s_1^2 - s_2^2\right). \tag{2.28}$$

It can be easily verified that $u_1^2 \neq 1$. Consequently, the exact solution, (2.26) and (2.27), of (2.5), proviso (2.20), possesses branch points as singularities. We have restricted ourselves to one of the three possible solutions $u(t)$ of (2.21) appearing in (2.25) in order to gain some insight into the structure of the solutions of (2.5) without unduly complicating the respective calculations, provided that (2.20) holds. This clearly implies that in the initial value problem consisting of (2.5) endowed with the initial conditions $x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0$, we take into account the initial conditions such that the unique solution $x(t)$, deriving of course from the basic equation (2.21), can also be extracted from $u = S_1 + S_2$. The remaining possibilities for u in (2.25) will most probably lead, owing to the form of S_1 and S_2, to $x(t)$ possessing branch points or essential singularities or both.

From (2.19) it is obvious that for $k = 3, 4, \ldots$ the $x(t)$, which now holds for all a and b fulfilling $\sqrt{1-8c} = 3/(4k-1)$, $c = (9b - 2a^2)/(2a^2)$, is still being given by (2.22), while $u = u(t)$ is determined implicitly through an algebraic equation of degree $2k - 1$, the solution of which for $u = u(t)$ cannot be effected analytically. Nevertheless the aforementioned $x(t)$ and $u = u(t)$ do represent an exact solution of (2.5). The remaining case for which I in (2.17) can be transformed into an integral of rational functions [8, pp. 70–71] is $1/\gamma - 1/2 =$ integer $= k$. If $k = 0$, then $\gamma = 2$ and by (2.16) $b = 0$, which is simply Case (A) above. As $0 < \gamma < 4$, we have $k = 1, 2, 3, \ldots$ and we set

$$C_1 + C_2 v^\gamma = u^2 v^\gamma, \tag{2.29}$$

so that (2.15)–(2.17) yield

$$I = -\frac{4C_1^k}{\gamma(2-\rho_1)} \int \left[u^2 - C_2\right]^{-(k+1)} \mathrm{d}u = t + C, \tag{2.30}$$

while the equivalent to (2.22) is now

$$x(t) = \frac{3u\dot{u}}{a\,(C_2 - u^2)\sqrt{1-8c}}. \tag{2.31}$$

Clearly the integral in (2.30) is in principle calculable, but even for $k = 1$ the resulting defining equation for $u = u(t)$ is not invertible. However, we do have in implicit form, i.e., (2.30) and (2.31), an exact solution to (2.5) provided $\sqrt{1-8c} = 3/(4k+1)$, $c = (9b - 2a^2)/(2a^2)$. We do not think it is worthwhile to pursue this further, since the characteristic features of the structure of the solutions to (2.5) are borne out by the cases analytically calculated above. Note also that, either by setting one of the two constants

C_1 and C_2 in (2.15) equal to zero or directly from (2.5), one immediately obtains the particular solution to (2.5), *viz.*,

$$x(t) = \frac{\lambda}{t+C}, \qquad b\lambda^2 - a\lambda + 1 = 0, \tag{2.32}$$

with one of the λ-values in (2.32) for $C = 0$ corresponding to (2.9) and (2.10) for $k_1 = k_2 = 0$. The solution (2.32) has a simple pole as singularity.

As can be expected, nothing is gained by interchanging ρ_1 and ρ_2 in (2.14), while $c = 1/8$ in (2.14) leads to the appearance of $\log w$ in the integral in (2.15), thus rendering it not calculable. This coincides with the result found when the Painlevé test is applied to (2.5) [11].

We conclude the discussion of (2.5) by noting that for $a = 1/2$ and $b = -\frac{1}{2}$ (2.5) possesses the general solution [9, Sect. 6.32, p. 548]

$$x(t) = C_1 \frac{\dot{\mathcal{P}}(C_1 t + C_2; 0, 1)}{\mathcal{P}(C_1 t + C_2; 0, 1)}, \tag{2.32a}$$

$\mathcal{P}(C_1 t + C_2; 0, 1)$ being the Weierstrass elliptic function [8, p. 917], where the prime denotes differentiation with respect to $C_1 t + C_2$, and C_1 and C_2 are constants of integration. By using the series expansion of $\mathcal{P}(C_1 t + C_2; 0, 1)$ it is seen that $x(t)$ in (2.32a) has a simple pole at $C_1 t + C_2 = 0$.

Finally, if in (2.5) we set

$$x = u^{-1/2}, \tag{2.32b}$$

(2.5) becomes

$$\ddot{u} - \frac{3}{2}\dot{u}^2 u^{-1} + 2a\dot{u}u^{-1/2} - 4b = 0. \tag{2.32c}$$

Then due to

$$\dot{u} = p, \qquad p = p(u) \tag{2.32d}$$

we deduce from (2.32c) that

$$up\dot{p} - \frac{3}{2}p^2 + 2apu^{1/2} - 4bu = 0. \tag{2.32e}$$

Equation (2.32e) is an Abel equation of the second kind and can be treated according to Kamke [9, pp. 25–28]. The final result obtained after some calculations is

$$\begin{gathered} (-2a)^{-1/2} \int z^{-1-r_1/2} \left(C_1 + C_2 z^{r_2 - r_1}\right)^{-1/2} \mathrm{d}z = t + C_3 \\ u = -2a\left(C_1 z^{r_1} + C_2 z^{r_2}\right)^{-1}, \qquad z = z(t) \\ r_1 = \left(1 - \sqrt{1 - 4b/a^2}\right)/2, \qquad r_2 = \left(1 + \sqrt{1 - 4b/a^2}\right)/2. \end{gathered} \tag{2.32f}$$

There are two independent constants in (2.32f) due to (2.32c). Equations (2.32f) are essentially equivalent to (2.16)–(2.19) and one obtains the same results as previously. Therefore transformation (2.32b) does not lead to any new results other than indicating another way to obtain the results already known. If instead of (2.32b) we consider $x = -u^{-1/2}$, equivalent results are extracted.

2.3. (C) $f(\dot{x}/x^2) = a(\dot{x}/x^2)^2 + b\dot{x}/x^2 + c$, *with* a, b, *and* c *real and* $a \neq 0$

Now (2.1) becomes

$$x\ddot{x} + a(\dot{x})^2 + bx^2\dot{x} + cx^4 = 0. \tag{2.33}$$

Equation (2.33) appears in Kamke [9, Sect. 6.130, p. 574] and we employ the method described there at the end of this paragraph. In the following, by means of (2.6), we obtain the solution from (2.33) by choosing

$$\alpha = \frac{2a+3}{b}, \tag{2.34}$$

$$\dddot{w} + a\frac{\ddot{w}^2}{\dot{w}} + d\frac{\dot{w}^3}{w^2} = 0, \qquad d = a + 2 + c\left(\frac{2a+3}{b}\right)^2 - (2a+3). \tag{2.35}$$

We first assume that

$$d = 0, \tag{2.36}$$

which implies one relation between the coefficients a, b, and c. Then (2.35) gives

$$\dot{w} = y, \qquad \ddot{y} + a\frac{\dot{y}^2}{y} = 0, \tag{2.37}$$

the second of which is linear in y^{a+1}.

Equation (2.37) is treated in Kamke [9, Section 6.128, p. 574]. We obtain, by taking into account (2.6) after some straightforward calculation,

$$x(t) = \frac{\big((2a+3)/b\big)\big((a+2)/(a+1)\big)}{k_2(t+k_1)^{-1/(a+1)} + (t+k_1)}, \qquad a \neq -1, \tag{2.38}$$

as the solution to (2.33) if (2.36) is valid, k_1 and k_2 being constants of integration. If $1/(a+1) = -k$, $k = 1, 2, \ldots$, or $1/(a+1) = k$, $k = 1, 2, \ldots$, the $x(t)$ in (2.38) possesses poles of order k at the most or $k+1$ at the most, respectively, as singularities. For all other values of $1/(a+1)$ (2.38) may possess all kinds of singularities, for example, a pole and an essential singularity if $-1 - 1/(a+1) \neq$ integer > 0, an essential singularity and a branch point if $1/(a+1) + 1 \neq$ integer > 0, and so on. In the case $a = -1$, Eq. (2.38) becomes

$$x(t) = \frac{k_1 e^{k_1 t}}{b\,(k_2 + e^{k_1 t})}, \qquad a = -1, \tag{2.39}$$

which has a simple pole as singularity.

In what follows we discard the condition (2.36) and examine the full equation (2.35). Then

$$\begin{gathered} \dot{w} = p, \qquad p = p(w), \\ w^2 pp'' + w^2 p'^2 (a+1) + dp^2 = 0, \end{gathered} \tag{2.40}$$

which is an equation of Euler type linear in the variable w^{a+2} provided $a \neq -2$. Equation (2.40) is precisely of the form (2.13) treated in Case B above. Before proceeding further we remark that the case $a = -2$ in (2.40) mentioned in Kamke [9, Section 6.184, p. 585] leads to $p(w) = kw^d \exp(C_1 w) = \dot{w}$ and the resulting relation cannot be inverted for $w = w(t)$. Equation (2.40) is solved and, as in (2.14), we obtain

$$\begin{aligned} p(w) &= \left[C_1 w^{\rho_1} + C_2 w^{\rho_2}\right]^{1/(a+2)} \\ \rho_1 &= \tfrac{1}{2}\left(1 - \sqrt{1 - 4(a+2)d}\right), \\ \rho_2 &= \tfrac{1}{2}\left(1 + \sqrt{1 - 4(a+2)d}\right), \qquad 4(a+2)d \neq 1. \end{aligned} \tag{2.41}$$

Consequently, due to (2.40),

$$I = \int w^{-\rho_1/(a+2)} \left[C_1 + C_2 w^{\rho_2 - \rho_1}\right]^{-1/(a+2)} \mathrm{d}w = t + C. \tag{2.42}$$

If we set

$$\begin{gathered} w^{\rho_2 - \rho_1} = v^\gamma, \qquad \gamma = \tfrac{(\rho_2 - \rho_1)(a+2)}{a+2-\rho_1}, \\ \sqrt{1 - 4(a+2)d} = \tfrac{\gamma(a+3/2)}{a+2-\gamma/2}, \end{gathered} \tag{2.43}$$

the integral, I, in (2.42) takes the form

$$I = \frac{a+2}{a+2-\rho_1} \int \left[C_1 + C_2^\gamma\right]^{-1/(a+2)} \mathrm{d}v = t + C \tag{2.44}$$

and, by virtue of the substitution

$$C_1 + C_2 v^\gamma = u^{a+2}, \tag{2.45}$$

(2.44) yields

$$I = \frac{a+2}{C_2\sqrt{1-4(a+2)d}} \int u^a \left(\frac{u^2 - C_1}{C_2}\right)^{r-1} \mathrm{d}u = t + C, \quad 1/\gamma = r. \tag{2.46}$$

The only possibility existing for I in (2.46) to be computable is when both a and r are rational and in particular [8],

(i) $r =$ integer,

(ii) $(a+1)/2 =$ integer,

(iii) $(a+1)/2 + r =$ integer.

If $r = 1$, we arrive at the known cases $a = -2$ or $d = 0$. The next simplest case is $a = 1$ and $r = 2$, i.e., $\gamma = 1/2$. Then from (2.35), (2.43), and (2.46)

$$\begin{gathered} I = \frac{3}{C_2\sqrt{1-12d}} \int u \left(\frac{u^2 - C_1}{C_2}\right) \mathrm{d}u = t + C \\ d = \frac{8}{121}, \quad d = -2 + 25c/b^2, \quad a = 1. \end{gathered} \tag{2.47}$$

From (2.6), (2.34), (2.43), (2.45), and (2.47) with $a = 1$ and $\gamma = 1/2$ we deduce that

$$u^4 - 2u^2 C_1 - \frac{20}{33} C_2^2 (t + C) = 0, \tag{2.48}$$

$$x(t) = \frac{11}{b}\left(\frac{3u^2\dot{u}}{u^3 - C_1}\right). \tag{2.49}$$

Relations (2.48) and (2.49) constitute an exact solution of (2.33) provided the conditions (2.47) are satisfied. Equation (2.48) can be solved for $u(t)$ and along the same lines as in Case (B) above we arrive at a singularity pattern similar to that of (2.22), i.e., branch points and essential singularities.

We think that there is no essence in pursuing this further by examining other cases in (i)–(iii) above since we have already obtained general singularity behaviour of the solutions to (2.33) through (2.48) and (2.49).

Now, in contrast to Case (B) above, some further exact results can be extracted for (2.33) by following Kamke [*loc. cit*] and we obtain for (2.33)

$$\begin{gathered} p = \dot{x}, \quad p = p(x) = x^2 u_1, \quad u_1 = u_1(t_1), \quad t_1 = \ln x, \\ u_1' u_1 + u_1^2(a+2) + b u_1 + c = 0, \end{gathered} \tag{2.50}$$

whereas, after a straightforward calculation,

$$a' = \frac{b}{a+2}, \qquad b' = \frac{c}{a+2}$$

$$\log\left[(u_1^2 + a'u_1 + b')^{1/2}\right] - \log\left[\left(\frac{\sqrt{a'^2 - 4b'} - a' - 2u_1}{a' + 2u_1 + \sqrt{a'^2 - 4b'}}\right)^{a'/[2\sqrt{a'^2-4b'}]}\right] = \log\left[(C_1/x)^{a+2}\right], \tag{2.51}$$

$$C_1 = \text{constant}.$$

For (2.52) to be of any use at all the quantity $a'/[2\sqrt{a'^2 - 4b'}]$ has to be an integer. We consider in the following the simple case

$$\frac{1}{2}a' = \sqrt{a'^2 - 4b'} \tag{2.52}$$

and, therefore, (2.52) finally yields

$$\begin{gathered} u^3 - uw + \frac{1}{2}a' = 0 \\ u = u_1 + \frac{3}{4}a', \qquad w = (C_1/x)^{2a+4}. \end{gathered} \tag{2.53}$$

Along the lines of Case (B) above and by applying (2.50) we deduce

$$x' = x^2 \left\{ \begin{aligned} &-\tfrac{3a'}{4} + \left[-\tfrac{a'}{4} + \left(-\tfrac{C_1^{6a+12}}{9x^{6a+12}} + \tfrac{(a')^2}{16}\right)^{1/2}\right]^{1/3} \\ &\quad + \left[-\tfrac{a'}{4} - \left(-\tfrac{C_1^{6a+12}}{9x^{6a+12}} + \tfrac{(a')^2}{16}\right)^{1/2}\right]^{1/3} \end{aligned} \right\}. \tag{2.54}$$

On integrating (2.54) we obtain

$$-\int \left\{ \begin{aligned} &-\tfrac{3a'}{4} + \left[-\tfrac{a'}{4} + \left(lz^{6a+12} + \tfrac{(a')^2}{16}\right)^{1/2}\right]^{1/3} \\ &\quad + \left[-\tfrac{a'}{4} - \left(lz^{6a+12} + \tfrac{(a')^2}{16}\right)^{1/2}\right]^{1/3} \end{aligned} \right\}^{-1} dz = t + C_2, \tag{2.55}$$

$$z = 1/x, \qquad l = -C_1^{6a+12}/9.$$

The integral in (2.55) can be brought into a form more amenable to treatment by setting

$$\left[-\frac{a'}{4} + \left(lz^{6a+12} + \frac{(a')^2}{16}\right)^{1/2}\right]^{1/3}$$

$$+\left[-\frac{a'}{4}-\left(lz^{6a+12}+\frac{(a')^2}{16}\right)^{1/2}\right]^{1/3}=y, \tag{2.56}$$

whence

$$-(3l^{1/3})^{-1/(2a+4)}\left(\frac{1}{2a+4}\right)\left\{\int\left[\left(-y^3-\frac{a'}{2}\right)^{-(2a+3)/(2a+4)}y^{(4a+7)/(2a+4)}\right.\right.$$
$$\left.\left.\times(-3)^{-1/(2a+4)}-\left(-y^3-\frac{a'}{2}\right)^{1/(2a+4)}y^{-(2a+5)/(2a+4)}\right]\frac{dy}{y-3a'/4}\right\}$$
$$=t+C_2. \tag{2.57}$$

If we reflect a little on (2.57) we conclude that only for specific a-values can we attempt to carry out the integration in (2.57). For example, if $a = -3/2$, then (2.52), (2.52) and (2.55)–(2.57) yield

$$\frac{2y}{3}+\frac{2}{3y}+\log\left[\left|\frac{(y-3b/2)^{b+4/9b}}{y^{4/9b}}\right|\right]=k(t+C_2)$$
$$x(t)=\frac{y}{k(y^3+b)},\qquad y=y(t) \tag{2.58}$$

as an exact solution to (2.33), where C_2 and k are constants, provided

$$a=-3/2,\qquad c=3b^2/8,\qquad b>0. \tag{2.59}$$

We remark that if we replace (2.52) with $\frac{1}{2}a' = -\sqrt{a'^2-4b'}$ we again arrive at (2.58)–(2.59), the only difference being that (2.59) also holds for $b < 0$. From (2.58) it also follows that $x(t)$ has a singularity at $t = t_s$, t_s being calculated from (2.58) by setting $y = (-b)^{1/3}$.

Evidently we can find other a-values, for instance, $a = -11/6$, for which the integration in (2.57) is possible and we would arrive at formulae similar to (2.58). Likewise we may take into account other possibilities in place of (2.52). Nevertheless there is little point in carrying this investigation further since it is almost certain that the resulting relations equivalent to (2.58) will not be invertible for the auxilliary function $y(t)$.

3. PAINLEVÉ ANALYSIS OF THE SECOND-ORDER EQUATION

We determine the leading order behaviour of (1.18) by means of the substitution $x = \alpha\tau^p$ and find that the value of p is -1 in accordance with the coefficients of the rescaling symmetry, G_2, and that the coefficient, α, is a solution of the equation

$$\alpha^2-3\alpha+2=0, \tag{3.1}$$

i.e., $\alpha = 1, 2$. To find the resonances we make the substitution $x = \alpha x^{-1} + \beta x^{r-1}$. The condition that β be arbitrary is

$$r^2 + 3(\alpha - 1)r + 3\alpha - 4 = 0, \tag{3.2}$$

i.e., $r = -1, 4 - 3\alpha$. With the two possible values of α we have $r = -1, 1$ for $\alpha = 1$ and $r = -1, -2$ for $\alpha = 2$. The former gives the standard right Painlevé series, representing a Laurent series in the neighbourhood of the moveable singularity, and the latter gives a left Painlevé series, representing a Laurent expansion valid over the exterior of a disc surrounding the moveable singularity [4].

In terms of $\tau = t - t_0$ the explicit expressions for the series are

$$x(\tau) = \frac{1}{\tau}\left(1 + a_1\tau - a_1^2\tau^2 + a_1^3\tau^3 - a_1^4\tau^4 + \cdots\right) \tag{3.3}$$

for the right Painlevé series and

$$x(t) = 2t^{-1} + a_1 t^{-2} + a_2 t^{-3} - \frac{1}{2}a_1\left(a_1^2 - 3a_2\right)t^{-4} + \frac{1}{2}\left(a_1^2 - a_2\right)^2 t^{-5} \tag{3.4}$$

for the left Painlevé series. For the latter we have written the expansion in terms of t instead of τ because of the asymptotic nature of the expansion [4].

We can reconcile the two Laurent series, (3.3) and (3.4), with the explicit solution, (1.9) with $b = 1$, given in the previous section. In the case of the right Painlevé series we have

$$\begin{aligned} x(t) &= \frac{1}{\tau}\left[2 - \left(1 - a_1\tau + a_1^2\tau^2 - a_1^3\tau^3 - a_1^4\tau^4 + \cdots\right)\right] \\ &= \frac{1}{\tau}\left[2 - \frac{1}{1 + a_1\tau}\right] \\ &= \frac{1 + 2a_1\tau}{\tau(1 + 2a_1\tau)}, \end{aligned} \tag{3.5}$$

which is just (1.9) written slightly differently. If we now expand this as

$$\begin{aligned} x(t) &= \frac{2}{t}\frac{\left[1 - \frac{1}{t}\left(t_0 - \frac{1}{2a_1}\right)\right]}{\left(1 - \frac{t_0}{t}\right)\left[1 - \frac{1}{t}\left(t_0 - \frac{1}{a_1}\right)\right]} \\ &= 2t^{-1}\left(2t_0 - \frac{1}{a_1}\right)t^{-2} + \left(2t_0^2 - \frac{2t_0}{a_1} + \frac{1}{a_1^2}\right)t^{-3} \\ &\quad + \left(2t_0^3 - \frac{3t_0^2}{a_1} + \frac{3t_0}{a_1^2} - \frac{1}{a_1^3}\right)t^{-4} + \cdots, \end{aligned} \tag{3.6}$$

we can verify that the expansion is identical to that in (4.11) when we rename the coefficients of t^{-2} and t^{-3} in (3.6).

4. PAINLEVÉ ANALYSIS OF THE TWO-DIMENSIONAL SYSTEM

To determine the leading order behaviour of the system (1.19) and (1.20), we make the substitutions

$$x = \alpha\tau^{p}, \qquad y = \beta\tau^{q} \tag{4.1}$$

to obtain

$$\begin{aligned} \alpha p \tau^{p-1} &= a\alpha^2\tau^{2p} + \alpha\beta\tau^{p+q} \\ \beta q \tau^{q-1} &= -(1+a)(1+2a)\alpha^2\tau^{2p} - 3(a+1)\alpha\beta\tau^{p+q} - \beta^2\tau^{2q} \end{aligned} \tag{4.2}$$

so that the powers to be compared are

$$\begin{matrix} -1 & p & q & \\ q-1 & 2p & p+q & 2q, \end{matrix} \tag{4.3}$$

in which we have removed a common p from the first line. If the first term on the right side of (4.2) were zero, we would also be able to subtract a common q from the second line. Evidently there are three possible systems which should be analysed separately.

4.1. *The General Case*

In general the product $(1+a)(1+2a) \neq 0$. In this case we can have two possible types of leading order behaviour. The first is the generic one for which $p = -1 = q$. The second is $q = -1$, $p > -1$.

We commence our analysis with the first case. The two coefficients α and β are found from the solution of

$$\begin{aligned} -\alpha &= a\alpha^2 + \alpha\beta \\ -\beta &= -(1+a)(1+2a)\alpha^2 - 3(1+a)\alpha\beta - \beta^2 \end{aligned} \tag{4.4}$$

and we obtain $\alpha = 1, 2$ and $\beta = -(1+a), -(1+2a)$ respectively.

To determine the resonances we make the substitution

$$\begin{aligned} x &= \alpha\tau^{-1} + \mu\tau^{r-1} \\ y &= \beta\tau^{-1} + \nu\tau^{r-1}, \end{aligned} \tag{4.5}$$

which allows both sets of coefficients to be treated at once. The condition for the coefficients μ and ν to be arbitrary is given by

$$\begin{vmatrix} r - a\alpha & -\alpha \\ (1+a)(a\alpha + 2\alpha - 3) & r - (3 - 3\alpha - a\alpha) \end{vmatrix} = 0 \tag{4.6}$$

with the solution $r = -1, 4 - 3\alpha$ so that in the case $\alpha = 1$ the resonances are at $r = -1, 1$, indicating a right Painlevé series, and in the case $\alpha = 2$ the resonances are of $r = -1, -2$, indicating a left Painlevé series. Note that the resonances are independent of the value of a. The coefficients at the resonances are given by

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - a \end{pmatrix} \sigma \qquad \text{and} \qquad \begin{pmatrix} -1 \\ 1 + a \end{pmatrix} \sigma, \tag{4.7}$$

where σ is a parameter, for the cases $\alpha = 1$ and $\alpha = 2$, respectively.

The other possible leading order behaviour is given by $q = -1$ and $p > -1$ which means that the series for x starts at the constant term and so the standard Painlevé analysis cannot be used. We write

$$x = a_i \tau^i \qquad y = b_i \tau^{i-1} \tag{4.8}$$

in which the summation convention is adopted and the counter i commences at zero. We substitute (4.8) into the system (1.19) and (1.20) and obtain

$$i a_i \tau^{i-1} = a a_i a_j \tau^{i+j} + a_i b_j \tau^{i+j-1} \tag{4.9}$$

$$\begin{aligned} (i-1) b_i \tau^{i-2} = {} & -(1+a)(1+2a) a_i a_j \tau^{i+j} \\ & - 3(1+a) a_i b_j \tau^{i+j-1} - b_i b_j \tau^{i+j-2}. \end{aligned} \tag{4.10}$$

We equate the coefficients of like powers of τ to zero and obtain

$$\begin{aligned} a_0 &= 0 & b_0 &= 1 \\ a_1 &= a_1 & b_1 &= 0 \\ a_2 &= 0 & b_2 &= 3(1+a)a_1 \\ a_3 &= \frac{1}{2}(3+4a)a_1^2 & b_3 &= 0 \\ a_4 &= 0 & b_4 &= -\frac{1}{5}(1+a)(19+20a)a_1^2. \end{aligned} \tag{4.11}$$

It is a simple matter to verify that the first few terms of the series for x give the first few terms of the series for y when substituted into (1.21).

4.2. *The Case* $(1 + a)(1 + 2a) = 0$

As there are two roots to this quadratic equation, we have to consider two subcases.

4.2.1. $1 + 2a = 0$

The two-dimensional system, (1.19) and (1.20), is now

$$\begin{aligned} \dot{x} &= -\frac{1}{2}x^2 + xy \\ \dot{y} &= -\frac{3}{2}xy - y^2. \end{aligned} \tag{4.12}$$

For the case when all terms are dominant, viz, $p = q = -1$, the coefficients of the leading order terms are found from the solution of the system

$$\begin{aligned} -\alpha &= -\frac{1}{2}\alpha^2 + \alpha\beta \\ -\beta &= -\frac{3}{2}\alpha\beta - \beta^2 \end{aligned} \tag{4.13}$$

and are $\alpha = 1$ and $\beta = -\frac{1}{2}$. The resonances are given by $r = \pm 1$ and so we obtain a right Painlevé series. The arbitrary vector at the resonance $r = 1$ is

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \sigma. \tag{4.14}$$

In addition to all the terms contributing to the dominant behaviour there is the possibility that just some of the terms can be dominant. Such terms are termed subdominant. These terms must have the same symmetry, which will be more restrictive than the rescaling symmetry of the system as a whole, and this property is termed subselfsimilarity. The first possible subdominant behaviour is given by $p = -1$, $q > -1$. As the series for y does not contain a singularity, we must, as we did above, substitute a full series into the system (4.12). We assume that

$$x = a_i \tau^{i-1}, \qquad y = b_i \tau^i \tag{4.15}$$

with the same conventions as before. We obtain the coefficients

$$\begin{aligned} a_0 &= 2 & b_0 &= 0 \\ a_1 &= 0 & b_1 &= 0 \\ a_2 &= 0 & b_2 &= 0 \\ a_3 &= 0 & b_3 &= 0 \\ a_4 &= 0 & b_4 &= 0, \end{aligned} \tag{4.16}$$

so that we only obtain an isolated solution.

There is another possibility and that is that the subdominant behaviour indicates the existence of an asymptotic series; i.e., we have $p = -1$, $q < -1$. We write

$$x = a_i t^{-i-1} \qquad y = b_i t^{-i-2} \tag{4.17}$$

with the counter beginning at zero. Note that we do not expand in terms of τ since the expansion is asymptotic. We obtain the set of coefficients

$$\begin{aligned} a_0 &= 2 & b_0 &= 0 \\ a_1 &= a_1 & b_1 &= b_1 \\ a_2 &= \frac{1}{2}a_1^2 - 2b_1 & b_2 &= \frac{3}{2}a_1 b_1 \\ a_3 &= \frac{1}{4}(a_1^2 - 8b_1)a_1 & b_3 &= \frac{3}{2}\left(a_1^2 - b_1\right) b_1 \end{aligned} \tag{4.18}$$

and, if the first few terms of the series for x are substituted into (1.21) with this value of a, we obtain verification of the first few terms of the series for y.

4.2.2. $1 + a = 0$

The system is now

$$\begin{aligned} \dot{x} &= -x^2 + xy \\ \dot{y} &= -y^2. \end{aligned} \tag{4.19}$$

In the case where all terms are dominant the coefficients of the leading order terms are given by

$$\begin{aligned} -\alpha &= -\alpha^2 + \alpha\beta \\ -\beta &= -\beta^2 \end{aligned} \tag{4.20}$$

and we find that $\alpha = 2$ and $\beta = 1$. The resonances occur at $r = -1, -2$ and consequently we have a left Painlevé series.

One of the possible patterns of subdominant behaviour is when $q = -1$, $p > -1$. We write

$$x = a_i \tau^i, \qquad y = b_i \tau^{i-1} \tag{4.21}$$

with the usual convention and find that we obtain the coefficients

$$\begin{aligned} a_0 &= 0 & b_0 &= 1 \\ a_1 &= 0 & b_1 &= 0 \\ a_2 &= 0 & b_2 &= 0 \\ a_3 &= 0 & b_3 &= b_1 \\ a_4 &= 0 & b_4 &= 0 \\ a_5 &= 0 & b_5 &= 0 \\ a_6 &= 0 & b_6 &= \frac{1}{3}b_1^2 \end{aligned} \tag{4.22}$$

which is very strange since we are generating a solution for y although it is defined in terms of a zero function.

If we try the pattern $q = -1$, $p < -1$, we find that the first few coefficients are

$$\begin{aligned} a_0 &= 0 & b_0 &= 1 \\ a_1 &= 0 & b_1 &= b_1 \\ a_2 &= 0 & b_2 &= b_1^2 \\ a_3 &= 0 & b_3 &= b_1^3 \\ a_4 &= 0 & b_4 &= b_1^4 \end{aligned} \tag{4.23}$$

from which it is evident that the solution to the system is

$$\begin{aligned} x &= 0 \\ y &= \frac{1}{t} + \frac{b_1}{t^2} + \frac{b_1^2}{t^3} + \frac{b_1^3}{t^4} + \cdots \\ &= \frac{1}{t - b_1}, \end{aligned} \tag{4.24}$$

which is a one-parameter solution of more than slightly suspect appearance.

5. COMMENT

We can make several observations on the results of these calculations.

1. There is a close relationship between the possession of the Painlevé property and the ease of explicit integration of this class of equations. Since they are all second-order equations with two symmetries, they are reducible to quadratures and so are integrable in the sense of Lie. However, as we have seen in Section 2, integrability in the sense of Painlevé is a much rarer occurrence. We recall that integrability in the sense of Painlevé imposes stronger requirements upon the solution since the function must be meromorphic.

2. The properties of the two-dimensional system are independent of the value of the parameter a in the definition of the new variable y, except for the two particular values of $a = -1, -\frac{1}{2}$. It is interesting to note that the quadratic equation which determines these values of a is the adjoint of the equation which determines the value of the coefficient, α, of the leading order term in the analysis of the original second-order equation (3.1).

3. In addition to the possibility of the existence of the left Painlevé series and the right Painlevé series we have seen the possibility of the existence of an asymptotic expansion which really has nothing to do with the Painlevé analysis even though it indicates singular behaviour.

4. What is most disturbing is the existence of solutions which are apparently spurious. The correctness of the proper interpretation of these solutions remains an open question.

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