On the average orders of the error term in the Dirichlet divisor problem

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Abstract

Let $A(x)$ be the error term in the Dirichlet divisor problem. The purpose of this paper is to study the difference between two kinds of mean value formulas of $A(x)$, that is, the mean value formulas $\int_1^x A(u)^k \, du$ and $\sum_{n \leq x} A(n)^k$ with a natural number $k$. In particular we study the case $k = 2$ and 3 in detail.

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1. Introduction and statement of results

Let $d(n)$ be the divisor function, and $A(x)$ the remainder term in the Dirichlet divisor problem defined by

$$A(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \quad (1.1)$$

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with the Euler constant $\gamma$. The Dirichlet divisor problem is to determine the best possible upper bound of $\Delta(x)$, and Dirichlet himself proved that $\Delta(x) = O(x^{1/2})$. Since then, many authors have sharpened Dirichlet’s bound of $\Delta(x)$, especially Huxley [7] proved the estimate $\Delta(x) = O(x^{23/73}(\log x)^{461/146})$. Recently, Huxley [8] himself improved this estimate to

$$\Delta(x) = O(x^{131/416}(\log x)^{26947/8320}).$$

(1.2)

As for mean value theorems for $\Delta(x)$, we know, for instance, that

$$\int_1^x \Delta(u) \, du = \frac{1}{4} \, x + \frac{1}{2\sqrt{2\pi^2}} \, x^{3/4} \sum_{n=1}^{\infty} \frac{d(n) n^{-5/4}}{n} \sin \left(4\pi \sqrt{n \, x} - \frac{\pi}{4}\right) + O(x^{1/4})$$

(1.3)

for $x \geq 1$, which was proved by Voronoï [18], and

$$\int_1^x \Delta(u)^2 \, du = \left(\frac{1}{6\pi^2} \sum_{n=1}^{\infty} d(n)^2 n^{-3/2}\right) x^{3/2} + F(x)$$

(1.4)

with $F(x) = O(x \log^4 x)$ and $x \geq 2$, which was proved by Preissmann [15]. Lau and Tsang [14, Theorem 1] proved an $\Omega_-$-estimate of $F(x)$, which is $F(x) = \Omega_-(x \log^2 x)$. Lau and Tsang [14, Eq. (1.4)] conjectured, inspired by the mean value formula for $F(x)$ obtained in [14, Theorem 2], that the asymptotic formula

$$F(x) = -\frac{1}{4\pi^2} x \log^2 x + A_1 x \log x + O(x)$$

(1.5)

would hold for $x \geq 2$ with a certain constant $A_1$, and Tsang [17, Corollary] proved that the asymptotic formula (1.5) holds true for almost all values of $x$. Ivić [10, Eq. (4.22)] conjectured the asymptotic formula for $F(x)$ with a stronger error term than (1.5), which is of the form

$$F(x) = -\frac{1}{4\pi^2} x \log^2 x + A_1 x \log x + A_2 x + O(x^\alpha),$$

(1.6)

where $\alpha$ is any constant satisfying $\frac{3}{4} < \alpha < 1$. Further, Ivić proved that formula (1.6) is not possible for $\alpha < \frac{3}{4}$, and that the conjectural bound $\Delta(x) = O(x^{1/4+\varepsilon})$ would follow from (1.6) if we could choose the constant $\alpha$ as $\alpha = \frac{3}{4} + \varepsilon$ with an arbitrarily small positive $\varepsilon$.

Our main object of this paper is to study another kind of mean values for $\Delta(x)$, that is, of the type

$$\sum_{n \leq x} \Delta(n)^k$$
for $x > 0$, where $k$ is any fixed natural number (Here and in what follows, we call this type the “discrete mean value” in order to distinguish from the average by integration. We call the latter type the “continuous mean value”.) For the case $k=1$, Voronoï [18] proved the asymptotic formula

$$\sum_{n \leq x} A(n) = \frac{1}{2} x \log x + \left( \gamma - \frac{1}{4} \right) x + O(x^{3/4}) \quad (1.7)$$

(cf. [1, p. 127], [16, p. 279]). In view of this formula, one can say that the behaviour of $\sum_{n \leq x} A(n)$ with respect to $x$ is the same as that of $\sum_{n \leq x} d(n)$ rather than $\int_{1}^{x} A(u) \, du$. (See [12, Lemma 1], [16, Lemma] and Lemma 1 below for this kind of average of the error term in general setting.) We also note that the error term in (1.7) can be replaced by $\Omega_{+}(x^{3/4})$ by Segal [16, Lemma] and formula (1.7) in [13].

For the case $k=2$, Hardy [5] investigated the relationship between the “discrete mean square” and the “continuous mean square”, and proved that

$$\sum_{n \leq x} A(n)^2 = \int_{1}^{x} A(u)^2 \, du + O(x^{1+\epsilon}) \quad (1.8)$$

with an arbitrarily small positive number $\epsilon$. From (1.4) and this formula, we can see that $\sum_{n \leq x} A(n)^2$ and $\int_{1}^{x} A(u)^2 \, du$ have the same order (which equals to $x^{3/2}$). Therefore in this case there is no difference of the order between $\sum_{n \leq x} A(n)^2$ and $\int_{1}^{x} A(u)^2 \, du$.

In this paper, we shall investigate the difference between the discrete mean value formula and the continuous mean value formula for general $k \geq 2$. Our first purpose is to refine the error term in Hardy’s formula (1.8). In fact, we can obtain the asymptotic formula with the best possible error term as follows.

**Theorem 1.** Let $A(x)$ be the function defined by (1.1). For $x \geq 2$, we have

$$\sum_{n \leq x} A(n)^2 = \int_{1}^{x} A(u)^2 \, du + \frac{1}{6} x \log^2 x + c_1 x \log x + c_2 x + \left\{ \frac{O}{\Omega_{+}} \right\} (x^{3/4} \log x)$$

with $c_1 = (8\gamma - 1)/12$ and $c_2 = (8\gamma^2 - 2\gamma + 1)/12$.

This theorem shows that the difference between the continuous and the discrete mean value formulas has the order $x \log^2 x$, which is not negligible. Hence we should say that the mean value behaviour of $\int_{1}^{x} A(u)^2 \, du$ and $\sum_{n \leq x} A(n)^2$ are different although the leading orders of these mean value formulas are the same. And also, this theorem implies that the studies of the mean value formulas $\sum_{n \leq x} A(n)^2$ and $\int_{1}^{x} A(u)^2 \, du$ can be reduced to each other.

One of the motivations of the study of $\sum_{n \leq x} A(n)^2$ is to study the detailed properties of the continuous mean square $\int_{1}^{x} A(u)^2 \, du$, especially the conjecture (1.5) or (1.6) by using the relation in Theorem 1. Actually, if we could derive the following asymptotic formula for $\sum_{n \leq x} A(n)^2$, then we could prove the conjecture (1.6):
Conjecture 1. The asymptotic formula

\[
\sum_{n \leq x} A(n)^2 = \left( \frac{1}{6\pi^2} \sum_{n=1}^{\infty} \frac{d(n) - 1}{n} \right) x^{3/2} + \left( \frac{1}{6} - \frac{1}{4\pi^2} \right) x \log^2 x + A_3 x \log x + A_4 x + O(x^z)
\]

would hold, where \(A_3\) and \(A_4\) are certain constants and \(z\) is a constant with \(\frac{3}{4} < z < 1\). More precisely, we can choose the constant \(z\) as \(z = \frac{3}{4} + \varepsilon\).

Obviously, we see that the first assertion of this conjecture is equivalent to formula (1.6) and this implies conjecture (1.5). And the second assertion is connected with Ivić’s lower bound of the exponent \(z\) in (1.6). We note that from Theorem 1 and the result of Tsang [17, Corollary] the asymptotic formula

\[
\sum_{n \leq x} A(n)^2 = \left( \frac{1}{6\pi^2} \sum_{n=1}^{\infty} \frac{d(n) - 1}{n} \right) x^{3/2} + \left( \frac{1}{6} - \frac{1}{4\pi^2} \right) x \log^2 x + A_3 x \log x + O(x)
\]

is valid for almost all \(x \geq 2\), namely the formula in Conjecture 1 is true for almost all \(x \geq 2\) with the choice \(z = 1\). Further, we immediately see that the second assertion of this conjecture implies the solution of the divisor problem \(A(x) = O(x^{1/4+\varepsilon})\), similarly to the continuous square case (1.6).

Next we study the difference between the continuous and the discrete mean value formulas for \(A(x)^k\) for general \(k \geq 3\). We can see that

\[
\sum_{n \leq x} A(n)^k = \int_1^x A(u)^k \, du + \begin{cases} O(x^{(k+3)/4+\varepsilon}) & \text{if } 3 \leq k \leq 10, \\
O(x^{(35k+3)/108+\varepsilon}) & \text{if } k \geq 11 \end{cases} \quad (1.9)
\]

for \(x \geq 1\). (Note that this formula is an immediate consequence of Lemma 1 below and the known result concerning the estimate of \(\int_1^x |A(u)|^A \, du\) with a positive number \(A\). In this sense, this formula can be said as a “trivial formula” (cf. Section 3 below).)

Now we discuss the error estimate in (1.9) for the case \(k = 3\) closely. In [2, Theorem 2], the author derived the asymptotic formula

\[
\sum_{n \leq x} A(n)^3 = \int_1^x A(u)^3 \, du + c_3 x^{3/2} \log x + O(x^{3/2})
\]

with \(c_3 = 3C/2\), where \(C\) denotes the coefficient of the main term in formula (1.4) (cf. [2, Section 3]). This formula gives the improvement of (1.9) for \(k = 3\). Here we derive more precise form of this asymptotic formula as follows:
Theorem 2. Let $A(x)$ be the function defined by (1.1). For $x \geq 2$, we have

$$\sum_{n \leq x} A(n)^3 = \int_1^x A(u)^3 \, du + c_3 x^{3/2} \log x + c_4 x^{3/2} + \left\{ O(x \log^5 x), \Omega(x \log^3 x) \right\}$$

with $c_3 = 3C/2$ and $c_4 = C(3\gamma - 1)$, where $C$ denotes the coefficient of the main term in (1.4). Furthermore, assuming conjecture (1.5) (or (1.6)), we have

$$\sum_{n \leq x} A(n)^3 = \int_1^x A(u)^3 \, du + c_3 x^{3/2} \log x + c_4 x^{3/2} + c_5 x \log^3 x$$

$$+ c_6 x \log^2 x + O(x \log x),$$

where $c_3$ and $c_4$ are as above, $c_5 = -3/8\pi^2$ and

$$c_6 = \frac{3}{8\pi^2} - \frac{3}{4\pi^2} \gamma + \frac{1}{8} + \frac{3}{2} A_1.$$

Here $A_1$ is the constant defined in (1.5).

Unfortunately, in the third power case we cannot derive the asymptotic formula with the best possible error term. One of the reasons is that we have only little information about the function $F(x)$ defined in (1.4). Indeed, formula (1.5) is still unsolved, especially we have not yet obtained the explicit value of the constant $A_1$. (However, we can show that the second formula in Theorem 2 is true for almost all $x \geq 2$ without using conjecture (1.5) (or (1.6)).)

Now, we will make a certain conjecture about the asymptotic behaviour of the third power case as follows:

Conjecture 2. The constant $c_6$ defined in Theorem 2 satisfies the condition $c_6 \neq 0$, namely, under assuming conjecture (1.5), the asymptotic formula

$$\sum_{n \leq x} A(n)^3 = \int_1^x A(u)^3 \, du + c_3 x^{3/2} \log x + c_4 x^{3/2} + c_5 x \log^3 x + \Omega(x \log^2 x)$$

would be true.

For $k \geq 4$, we can derive the asymptotic formula for $\sum_{n \leq x} A(n)^k$ for $4 \leq k \leq 9$ by applying formula (1.9) and the results about the asymptotic formulas for $\int_1^x A(u)^k \, du$ obtained in [19]. However, it seems to be quite difficult to derive a result analogous to Theorem 1, the second statement of Theorem 2 and the formula in Conjecture 2 (See Remark in Section 5 below.)
2. Two preliminary lemmas

Throughout this paper, $\varepsilon$ denotes an arbitrarily small positive number which need not be the same at each occurrence. Let $k$ be a fixed natural number. The symbols $O(\cdot)$, $\ll$, $\Omega$ and $\Omega_\varepsilon$ have their usual meaning, and the implied constants in those symbols will depend at most on $k$ and $\varepsilon$. For a real number $x$, $\psi(x)$ denotes the periodic Bernoulli function defined by $\psi(x) = x - \lfloor x \rfloor - 1/2$, where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$, and we define the function $\psi_1(x)$ as $\psi_1(x) = \int_1^x \psi(u) \, du$ for $x \geq 1$.

As a preparation for proving our theorems, we first study the asymptotic formula for the summatory function of number-theoretic error terms in general setting. Let $f(n)$ be an arithmetical function, and $E(x)$ the number-theoretic error term defined by

$$E(x) = \sum_{n \leq x} f(n) - g(x), \quad (2.1)$$

where $g(x)$ denotes a certain function (the “main term”), sometimes we additionally assume some conditions for the derivatives of $g(x)$. In [16, Lemma], Segal proved the asymptotic formula for the average order of $E(x)$, which is

$$\sum_{n \leq x} E(n) = \left( \frac{1}{2} - \psi(x) \right) E(x) + \frac{1}{2} g(x) + \int_1^x E(u) \, du + O(1 + |g'(x)|) \quad (2.2)$$

if $g(x)$ is twice continuously differentiable and $g''(x)$ has constant sign throughout the interval $[1, x]$. Kanemitsu and Sita Rama Chandra Rao [12, Lemma 1] proved that the error term in (2.2) can be replaced by

$$-\frac{1}{2} P_2(x) g'(x) + O(1 + |g''(x)|)$$

if $g(x)$ is thrice continuously differentiable and $g^{(3)}(x)$ has constant sign throughout the interval $[1, x]$. Here $P_2(x)$ is the periodic Bernoulli function of order 2. We note here that formula (2.2) looks like the Euler–Maclaurin summation formula. However, since the function $E(x)$ is discontinuous at the integer points, we cannot apply the Euler–Maclaurin summation formula directly to prove formula (2.2).

Now we shall generalize these formulas to more general power case $\sum_{n \leq x} E(n)^k$ with an arbitrarily fixed natural number $k$. In particular, we shall derive not an asymptotic formula but an identity between this function and the continuous mean value $\int_1^x E(u)^k \, du$. We prove the following lemma.

**Lemma 1.** Let $E(x)$ and $g(x)$ be the functions defined in (2.1), and assume that $g(x)$ is continuously differentiable. For a fixed natural number $k$, we have

$$\sum_{n \leq x} E(n)^k = \left( \frac{1}{2} - \psi(x) \right) E(x)^k + \int_1^x E(u)^k \, du + k \int_1^x \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^{k-1} \, du.$$
Proof. This lemma can be proved by induction on \( k \). If \( k = 1 \), we can obtain the assertion of the lemma by the method in [16, Lemma] as

\[
\sum_{n \leq x} E(n) = \sum_{m \leq x} f(m) \sum_{m \leq n \leq x} 1 - \sum_{n \leq x} g(n)
\]

\[
= \left( \frac{1}{2} - \psi(x) \right) \sum_{m \leq x} f(m) + \int_1^x (g(u) + E(u)) \, du - \sum_{n \leq x} g(n)
\]

\[
= \left( \frac{1}{2} - \psi(x) \right) E(x) + \int_1^x E(u) \, du + \frac{1}{2} (g(x) - g(1)) - \int_1^x g'(u) \psi(u) \, du.
\]

This completes the proof of the lemma for \( k = 1 \), since \( g(x) - g(1) = \int_1^x g'(u) \, du \).

Assume that the assertion of the lemma is valid for all \( k \leq K \) with some \( K \geq 1 \). We have

\[
\sum_{n \leq x} E(n)^{K+1} = \sum_{n \leq x} E(n)^K \left( \sum_{m \leq n} f(m) - g(n) \right)
\]

\[
= \sum_{m \leq x} f(m) \sum_{m \leq n \leq x} E(n)^K - \sum_{n \leq x} g(n) E(n)^K
\]

\[
= E(x) \sum_{n \leq x} E(n)^K - \sum_{m \leq x} f(m) \sum_{n \leq m} E(n)^K + \sum_{m \leq x} f(m) E(m)^K
\]

\[
+ \int_1^x g'(u) \sum_{n \leq u} E(n)^K \, du. \tag{2.3}
\]

Furthermore, by applying the assumption of induction, we have

\[
E(x) \sum_{n \leq x} E(n)^K = \left( \frac{1}{2} - \psi(x) \right) E(x)^{K+1} + E(x) \int_1^x E(u)^K \, du
\]

\[
+ KE(x) \int_1^x \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^{K-1} \, du, \tag{2.4}
\]

\[
- \sum_{m \leq x} f(m) \sum_{n \leq m} E(n)^K = - \sum_{m \leq x} f(m) E(m)^K - (E(x) + g(x)) \int_1^x E(u)^K \, du
\]

\[
- K (g(x) + E(x)) \int_1^x \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^{K-1} \, du
\]
\begin{equation}
+K \int_1^x (g(u) + E(u)) \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^K - 1 \, du
+ \int_1^x E(u)^K (E(u) + g(u)) \, du \tag{2.5}
\end{equation}

and

\begin{align*}
\int_1^x g'(u) \sum_{n \leq u} E(n)^K \, du
&= \int_1^x \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^K \, du + g(x) \int_1^x E(u)^K \, du \\
&\quad - \int_1^x g(u) E(u)^K \, du + Kg(x) \int_1^x \left( \frac{1}{2} - \psi(u) \right) g'(u) E(u)^K - 1 \, du \\
&\quad - Kg \int_1^x \left( \frac{1}{2} - \psi(u) \right) g(u) g'(u) E(u)^K - 1 \, du. \tag{2.6}
\end{align*}

By substituting (2.4)–(2.6) into (2.3), the assertion of the lemma can be proved for \( k = K + 1 \), and therefore for all natural number \( k \) by induction. \( \square \)

Next, we study the representation of the error function \( \Delta(x) \) as a sum of the \( \psi \)-function. It is well-known that the function \( \Delta(x) \) can be represented as

\begin{equation}
\Delta(x) = -2 \sum_{n \leq x^{1/2}} \psi \left( \frac{x}{n} \right) + O(1). \tag{2.7}
\end{equation}

However, the error term \( O(1) \) in (2.7) is too large for our purposes. Thus we need to improve this estimate to a sharper form than (2.7). Moreover, in order to prove Theorem 2, we also need to use an asymptotic formula for the derivative of this part. For that purpose, we use a formula for \( \int_1^y \psi_1(u) \, du \) for \( y \geq 1 \). From the definition of the \( \psi \)-function, we can see easily that \( \psi(u)^2 = 2\psi_1(u) + \frac{1}{4} \), and hence we have

\begin{equation}
\int_1^y \psi_1(u) \, du = -\frac{1}{12} y + \frac{1}{6} \psi(y)^3 - \frac{1}{24} \psi(y) + \frac{1}{12}. \tag{2.8}
\end{equation}

By using these facts, we prove the following lemma.

**Lemma 2.** Let \( R(x) \) be the error term defined by

\[ R(x) = \Delta(x) + 2 \sum_{n \leq x^{1/2}} \psi \left( \frac{x}{n} \right). \]
Then the function $R(x)$ can be written as

$$R(x) = -4\psi_1(x^{1/2}) - \frac{1}{6} + O(x^{-1/2}).$$  \hspace{1em} (2.9)

And if $x^{1/2}$ is not an integer, then we have that the derivative of $R(x)$ can be written down as

$$R'(x) = -2x^{-1/2}\psi(x^{1/2}) - 2x^{-1}\psi_1(x^{1/2}) - \frac{1}{6} x^{-1} + O(x^{-3/2}).$$  \hspace{1em} (2.10)

**Proof.** By using the method known as “Dirichlet’s hyperbola method” (cf. [3, Lemma 4.3]), the Euler–Maclaurin summation formula and the relation $\psi(u)^2 = 2\psi_1(u) + \frac{1}{4}$, we have

$$R(x) = -4\psi_1(x^{1/2}) + 4x \int_{x^{1/2}}^{\infty} u^{-3}\psi_1(u) \, du.$$  \hspace{1em} (2.11)

By applying integration by parts to the integral in (2.11) and by using formula (2.8), or its weaker version

$$\int_1^y \psi_1(u) \, du = -\frac{1}{12} y + O(1)$$  \hspace{1em} (2.12)

for $y \geq 1$, we can prove formula (2.9). (cf. [12, Lemma 7].)

Next we assume that $x^{1/2}$ is not an integer. Then $\psi_1(x^{1/2})$ is differentiable, and we have by differentiating the both sides of (2.11) with respect to $x$ that

$$R'(x) = -2x^{-1/2}\psi(x^{1/2}) - 2x^{-1}\psi_1(x^{1/2}) - \frac{1}{6} x^{-1} + O(x^{-3/2}).$$  \hspace{1em} (2.13)

This concludes the proof of the lemma. \hspace{1em} □

3. Integral formulas involving the $\psi$-functions

In our proofs, we need several formulas for the integrals involving the $\psi$-functions. First, we consider the integral

$$\int_1^y \psi(u)\psi\left(\frac{u}{n}\right) \, du$$

with a natural number $n$. We prove the following lemma.
Lemma 3. For a natural number \( n \), we have

\[
\int_1^y \psi(u) \psi \left( \frac{u}{n} \right) \, du = \frac{1}{12n} \, y - \frac{1}{6n} \, \psi(y)^3 + \frac{1}{2} \psi(y)^2 \psi \left( \frac{y}{n} \right) - \frac{1}{8} \psi \left( \frac{y}{n} \right) + \frac{1}{24n} \psi(y) - \frac{1}{12n}.
\]

Proof. Without loss of generality, we may assume that \( n > 1 \). At first, we assume that \( y \leq n \). It is easy to see that

\[
\psi \left( \frac{u}{n} \right) = \frac{u}{n} - \frac{1}{2} \left\{ \begin{array}{ll}
1 & \text{if } u = n, \\
0 & \text{otherwise}
\end{array} \right.
\]  

(3.1)

for \( u \leq y \). Then we have

\[
\int_1^y \psi(u) \psi \left( \frac{u}{n} \right) \, du = \frac{1}{n} \int_1^y u \psi(u) \, du - \frac{1}{2} \int_1^y \psi(u) \, du
\]

say. By the definition, we have immediately \( S_2 = -\psi(y)^2/4 + 1/16 \). As for the integral in \( S_1 \), we have by using integration by parts that

\[
\int_1^y u \psi(u) \, du = y \psi_1(y) - \int_1^y \psi_1(u) \, du
\]

\[
= \frac{1}{2} \, y \psi(y)^2 - \frac{1}{24} \, y - \frac{1}{6} \, \psi(y)^3 + \frac{1}{24} \, \psi(y) - \frac{1}{12}
\]  

(3.2)

(Note that this formula is valid for all \( y \geq 1 \).) Hence we obtain

\[
\int_1^y \psi(u) \psi \left( \frac{u}{n} \right) \, du = \frac{1}{2n} \, y \psi(y)^2 - \frac{1}{24n} \, y - \frac{1}{6n} \, \psi(y)^3
\]

\[
+ \frac{1}{24n} \psi(y) - \frac{1}{12n} - \frac{1}{4} \psi(y)^2 + \frac{1}{16}.
\]  

(3.3)

This with (3.1) completes the proof of the lemma for \( y \leq n \).
Next, we prove this lemma for $y > n$. We first note that $[n^{-1}y] = [n^{-1}y]$ for every $y \geq 1$. We divide the integral as

\[
\int_1^y \psi(u)\psi\left(\frac{u}{n}\right) \, du = \int_1^n \psi(u)\psi\left(\frac{u}{n}\right) \, du + \sum_{j=1}^{[n^{-1}y]-1} \int_{jn}^{(j+1)n} \psi(u)\left(\frac{u}{n} - j - \frac{1}{2}\right) \, du
\]

\[
+ \int_{[n^{-1}y]n}^y \psi(u)\psi\left(\frac{u}{n}\right) \, du
\]

\[= S_3 + S_4 + S_5,
\]
say, where we interpret the summation in $S_4$ as zero if the sum is empty. We have from (3.3) that $S_3 = (1 - n^{-1})/12$. From (3.2), we have $\int_1^N u\psi(u) \, du = (N - 1)/12$ for any natural number $N$, hence $S_4 = ([n^{-1}y] - 1)/12$.

We consider $S_5$. Since $[n^{-1}u] = [n^{-1}y]$ for $[n^{-1}y]n \leq u \leq y$, we have

\[
S_5 = \frac{1}{n} \int_{[n^{-1}y]n}^y u\psi(u) - \left([\frac{y}{n}] + \frac{1}{2}\right) \int_{[n^{-1}y]n}^y \psi(u) \, du
\]

\[= \frac{1}{12n} y - \frac{1}{6n} \psi(y)^3 + \frac{1}{2} \psi(y)^2\psi\left(\frac{y}{n}\right) - \frac{1}{8} \psi\left(\frac{y}{n}\right) + \frac{1}{24n} \psi(y) - \frac{1}{12} \left[\frac{y}{n}\right].
\]

Combining the formulas for $S_3$, $S_4$ and $S_5$, we can prove the assertion of the lemma for $n < y$. The proof of this lemma is complete. □

We note that, in order to prove Theorem 1, it is sufficient to apply

\[
\int_1^y \psi(u)\psi\left(\frac{u}{n}\right) \, du = \frac{1}{12n} y + O(1),
\]

(3.4)

which is weaker than Lemma 3. However, we will use the explicit formula in Lemma 3 to prove Theorem 2, especially to prove the next lemma.

To prove Theorem 2, we need not only Lemma 3 (or (3.4)) but also the formula for the integral

\[
\int_1^y \psi(u)\psi\left(\frac{u}{n_1}\right) \psi\left(\frac{u}{n_2}\right) \, du,
\]

where $n_1$ and $n_2$ are natural numbers with $n_1 \leq n_2$. Next we shall derive the asymptotic formula for this integral. As a preparation for this purpose, we prove the following lemma.
Lemma 4. For a natural number $n$ and a real number $y$ with $y \geq 1$, we have
\[
\int_{1}^{y} u \psi(u) \psi\left(\frac{u}{n}\right) \, du = \frac{1}{24n} y^2 - \frac{1}{6n} y \psi(y)^3 + \frac{1}{2} y \psi(y)^2 \psi\left(\frac{y}{n}\right) - \frac{1}{8} y \psi\left(\frac{y}{n}\right) + \frac{1}{24n} y \psi(y) \\
+ \frac{1}{12n} \psi(y)^4 + \frac{1}{24} n \psi\left(\frac{y}{n}\right)^2 - \frac{1}{96} n + \frac{1}{24} - \frac{1}{6} \psi(y)^3 \psi\left(\frac{y}{n}\right) \\
+ \frac{1}{24} \psi(y) \psi\left(\frac{y}{n}\right) - \frac{1}{24n} \psi(y)^2 - \frac{5}{64n}.
\]

Proof. We first note that
\[
\int_{1}^{y} u^2 \psi(u) \, du = y^2 \psi_1(y) - 2 \int_{1}^{y} u \psi_1(u) \, du \\
= \frac{1}{2} y^2 \psi(y)^2 - \frac{1}{24} y^2 - \frac{1}{3} y \psi(y)^3 + \frac{1}{12} y \psi(y) - \frac{1}{24} \psi(y)^2 \\
+ \frac{1}{12} \psi(y)^4 - \frac{5}{64} \tag{3.5}
\]
for every $y \geq 1$, since $\psi(u)^2 = 2\psi_1(u) + \frac{1}{4}$ and
\[
\int_{1}^{y} \psi(u)^3 \, du = \frac{1}{4} \left( \psi(y)^4 - \frac{1}{16} \right).
\]
First consider the case $y \leq n$. From (3.2) and (3.5), we have
\[
\int_{1}^{y} u \psi(u) \psi\left(\frac{u}{n}\right) \, du \\
= \frac{1}{n} \int_{1}^{y} u^2 \psi(u) \, du - \frac{1}{2} \int_{1}^{y} u \psi(u) \, du \\
= \frac{1}{2n} y^2 \psi(y)^2 - \frac{1}{24n} y^2 - \frac{1}{3n} y \psi(y)^3 - \frac{1}{4} y \psi(y)^2 + \frac{1}{12n} y \psi(y) + \frac{1}{48} y \\
+ \frac{1}{12n} \psi(y)^4 + \frac{1}{12} \psi(y)^3 - \frac{1}{24n} \psi(y)^2 - \frac{1}{48} \psi(y) + \frac{1}{24} - \frac{5}{64n} \tag{3.6}
\]
This with (3.1) completes the proof of the lemma for $y \leq n$. 
Next we assume that \( y > n \). Dividing the integral into three parts and using the same method as that used in the proof of Lemma 3, we have

\[
\int_{1}^{y} u\psi(u)\psi\left(\frac{u}{n}\right) \, du = \int_{1}^{n} u\psi(u)\psi\left(\frac{u}{n}\right) \, du + \sum_{j=1}^{[n^{-1}y]-1} \int_{jn}^{(j+1)n} u\psi(u)\psi\left(\frac{u}{n}\right) \, du \\
+ \int_{[n^{-1}y]n}^{y} u\psi(u)\psi\left(\frac{u}{n}\right) \, du \\
= S_6 + S_7 + S_8,
\]
say. As for \( S_6 \), it follows from formula (3.6) with \( y = n \) that

\[
S_6 = \frac{1}{24} n + \frac{1}{12} - \frac{1}{12n}.
\]

Further, since formulas (3.2) and (3.5) are reduced to

\[
\int_{1}^{N} u\psi(u) \, du = \frac{1}{12} N - \frac{1}{12} \quad \text{and} \quad \int_{1}^{N} u^2\psi(u) \, du = \frac{1}{12} N^2 - \frac{1}{12}
\]
for any natural number \( N \), we have

\[
S_7 = \frac{1}{n} \sum_{j=1}^{[n^{-1}y]-1} \int_{jn}^{(j+1)n} u^2\psi(u) \, du - \sum_{j=1}^{[n^{-1}y]-1} \left( j + \frac{1}{2} \right) \int_{jn}^{(j+1)n} u\psi(u) \, du \\
= \frac{1}{24n} y^2 - \frac{1}{12} \psi\left(\frac{y}{n}\right) - \frac{1}{24} y + \frac{1}{24} n\psi\left(\frac{y}{n}\right)^2 + \frac{1}{24} n\psi\left(\frac{y}{n}\right) - \frac{1}{32} n.
\]

Consider \( S_8 \). We again note that \([n^{-1}u] = [n^{-1}y]\) for \([n^{-1}y]n \leq u \leq y\). We have

\[
S_8 = \frac{1}{n} \int_{[n^{-1}y]n}^{y} u^2\psi(u) \, du - \left( \left[\frac{y}{n}\right] + \frac{1}{2} \right) \int_{[n^{-1}y]n}^{y} u\psi(u) \, du \\
= -\frac{1}{24} \psi\left(\frac{y}{n}\right) + \frac{1}{24} y - \frac{1}{6n} y\psi(y)^3 + \frac{1}{24n} y\psi(y) + \frac{1}{2} y\psi(y)^2\psi\left(\frac{y}{n}\right) \\
- \frac{1}{6} \psi(y)^3\psi\left(\frac{y}{n}\right) + \frac{1}{24} \psi(y)\psi\left(\frac{y}{n}\right) - \frac{1}{24} n\psi\left(\frac{y}{n}\right) - \frac{1}{48} n + \frac{1}{12n} \psi(y)^4 \\
- \frac{1}{24n} \psi(y)^2 + \frac{1}{192n}.
\]

Therefore combining the formulas for \( S_6, S_7 \) and \( S_8 \), we obtain the assertion of the lemma for \( y > n \). This completes the proof of this lemma. \( \square \)
Lemma 5. Let $n_1$ and $n_2$ be natural numbers with $n_1 \leq n_2$. We have, for $y \geq 1$,
\[
\int_1^y \psi(u) \psi \left( \frac{u}{n_1} \right) \psi \left( \frac{u}{n_2} \right) \, du = \frac{n_2}{24n_1} \psi \left( \frac{y}{n_2} \right)^2 - \frac{n_2}{96n_1} + O(1)
\]
uniformly in $y, n_1$ and $n_2$.

Proof. If $y \leq n_2$, we easily see that
\[
\int_1^y \psi(u) \psi \left( \frac{u}{n_1} \right) \psi \left( \frac{u}{n_2} \right) \, du = \frac{1}{n_2} \int_1^y u \psi(u) \psi \left( \frac{u}{n_1} \right) \, du - \frac{1}{2} \int_1^y \psi(u) \psi \left( \frac{u}{n_1} \right) \, du
\]
\[
= \frac{1}{24n_1n_2} y^2 - \frac{1}{24n_1} y + O(1)
\]
bym Lemmas 3 and 4. Therefore we obtain the assertion of this lemma for $y \leq n_2$ by using this formula and the definition of the $\psi$-function.

Next we assume that $y > n_2$. Then by the definition of the $\psi$-function and using Lemmas 3 and 4, we have
\[
\int_1^y \psi(u) \psi \left( \frac{u}{n_1} \right) \psi \left( \frac{u}{n_2} \right) \, du = \frac{1}{24n_1n_2} y^2 - \frac{1}{6n_1n_2} y \psi(y)^3 + \frac{1}{2n_2} y \psi(y)^2 \psi \left( \frac{y}{n_1} \right)
\]
\[
- \frac{1}{8n_2} y \psi \left( \frac{y}{n_1} \right) + \frac{1}{24n_1n_2} y \psi(y) - \frac{1}{24n_1} y
\]
\[
- \int_{n_2}^y \psi(u) \psi \left( \frac{u}{n_1} \right) \left[ \frac{u}{n_2} \right] \, du + O(1). \quad (3.7)
\]
We divide the integral on the right-hand side of the above formula as
\[
\int_{n_2}^y \psi(u) \psi \left( \frac{u}{n_1} \right) \left[ \frac{u}{n_2} \right] \, du = \sum_{j=1}^{[n_2^{-1}y] - 1} j \int_{jn_2}^{(j+1)n_2} \psi(u) \psi \left( \frac{u}{n_1} \right) \, du
\]
\[
+ \left[ \frac{y}{n_2} \right] \int_{[n_2^{-1}y]n_2}^y \psi(u) \psi \left( \frac{u}{n_1} \right) \, du
\]
\[
= S_9 + S_{10},
\]
say. Then we have by applying the methods used in the proofs of the previous lemmas that
\[
S_9 = \frac{1}{24n_1n_2} y^2 - \frac{1}{12n_1} y \psi \left( \frac{y}{n_2} \right) - \frac{1}{12n_1} y + \frac{n_2}{24n_1} \psi \left( \frac{y}{n_2} \right)^2
\]
\[
+ \frac{n_2}{12n_1} \psi \left( \frac{y}{n_2} \right) + \frac{n_2}{32n_1}
\]
and
\[ S_{10} = \frac{1}{12n_1} y\psi\left(\frac{y}{n_2}\right) + \frac{1}{24n_1} y - \frac{n_2}{12n_1} \psi\left(\frac{y}{n_2}\right)^2 - \frac{n_2}{12n_1} \psi\left(\frac{y}{n_2}\right) - \frac{n_2}{48n_1} \]
\[ + \frac{1}{24n_1n_2} y\psi(y) - \frac{1}{6n_1n_2} y\psi(y)^3 + \frac{1}{2n_2} y\psi(y)^2\psi\left(\frac{y}{n_1}\right) \]
\[ - \frac{1}{8n_2} y\psi\left(\frac{y}{n_1}\right) + O(1). \]

Hence we obtain
\[ \int_{n_2}^{y} \psi(u)\psi\left(\frac{u}{n_1}\right) \left[ \frac{u}{n_2} \right] du = \frac{1}{24n_1n_2} y^2 - \frac{1}{6n_1n_2} y\psi(y)^3 + \frac{1}{2n_2} y\psi(y)^2\psi\left(\frac{y}{n_1}\right) \]
\[ - \frac{1}{8n_2} y\psi\left(\frac{y}{n_1}\right) - \frac{1}{24n_1} y + \frac{1}{24n_1n_2} y\psi(y) \]
\[ - \frac{n_2}{24n_1} \psi\left(\frac{y}{n_2}\right)^2 + \frac{n_2}{96n_1} + O(1). \]

By substituting this formula into (3.7), the proof of this lemma is complete. □

Finally, we shall consider the explicit value of the integral
\[ \int_{1}^{\infty} u^{-2}\psi(u) du \]  \hspace{2cm} (3.8)
and derive an asymptotic formula for finite truncation of (3.8):

**Lemma 6.** We have
\[ \int_{1}^{\infty} u^{-2}\psi(u) du = \frac{1}{2} - \gamma \]
and
\[ \int_{1}^{y} u^{-2}\psi(u) du = \frac{1}{2} - \gamma + (\psi_1(y) + \frac{1}{12})y^{-2} + O(y^{-3}). \]

**Proof.** It is well-known that the Euler constant \( \gamma \) can be written as
\[ \gamma = \lim_{N \to \infty} \left( \sum_{n \leq N} n^{-1} - \log N \right). \]
Hence we can get the first assertion of the lemma by applying the Euler–Maclaurin summation formula to the right-hand side of this formula. And the second assertion of the lemma can be proved by integration by parts, (2.12) and the first assertion of this lemma. □

We note that the formulas in Lemma 6 are equivalent to

\[ \int_{1}^{\infty} u^{-3} \psi_1(u) \, du = \frac{1}{2} \left( \frac{1}{2} - \gamma \right) \]  

(3.9)

and

\[ \int_{1}^{y} u^{-3} \psi_1(u) \, du = \frac{1}{2} \left( \frac{1}{2} - \gamma \right) + \frac{1}{24} y^{-2} + O(y^{-3}), \]  

(3.10)

respectively. We also use these two formulas in order to prove Theorem 2.

4. Proof of Theorem 1

We now consider the difference between the discrete mean value \( \sum_{n \leq x} A(n)^k \) and the continuous mean value \( \int_{1}^{x} A(u)^k \, du \) for general power moments. In Lemma 1, we specify \( f(n) = d(n) \), hence \( E(u) = A(u) \) and \( g(u) = u(\log u + 2\gamma - 1) \). Then by using (1.2) and the estimate

\[
\int_{1}^{x} |A(u)|^A \, du \ll \begin{cases} 
A^{(A+4)/4+\varepsilon} & \text{if } 0 \leq A \leq \frac{28}{3}, \\
A^{(35A+38)/108+\varepsilon} & \text{if } A > \frac{28}{3}
\end{cases}
\]

(cf. [9, Theorem 2], [6, pp. 402–403]), we get

\[
\sum_{n \leq x} A(n)^k = \int_{1}^{x} A(u)^k \, du + O(|A(x)|^k) + O\left( \log x \int_{1}^{x} |A(u)|^{k-1} \, du \right) \\
= \int_{1}^{x} A(u)^k \, du + \begin{cases} 
O(x^{(k+3)/4+\varepsilon}) & \text{if } 1 \leq k \leq 10, \\
O(x^{(35k+3)/108+\varepsilon}) & \text{if } k \geq 11.
\end{cases}
\]

This formula implies (1.9) (We note that this formula does not give a refinement of the previous results for \( k = 1 \) and 2, and also for \( k = 3 \).)

Next we study the case \( k = 2 \) more closely to prove Theorem 1. From Lemma 1, we have

\[
\sum_{n \leq x} A(n)^2 = \int_{1}^{x} A(u)^2 \, du - 2 \int_{1}^{x} \psi(u)(\log u + 2\gamma)A(u) \, du \\
+ \int_{1}^{x} (\log u + 2\gamma)A(u) \, du + O(x^{2/3}) \\
= \int_{1}^{x} A(u)^2 \, du + T_1 + T_2 + O(x^{2/3}),
\]

where
say. As for \( T_2 \), it follows from (1.3) and partial summation that

\[
T_2 = \frac{1}{4} x (\log x + 2\gamma - 1) + \frac{1}{2\sqrt{2\pi}} x^{3/4} \log x \sum_{n=1}^{\infty} d(n) n^{-5/4} \sin \left( 4\pi \sqrt{nx} - \frac{\pi}{4} \right) + O(x^{3/4}).
\]

Next, substituting the formula in Lemma 2 into \( T_1 \), we have

\[
T_1 = -2 \int_1^{x/2} \psi(u)(\log u + 2\gamma) \left\{ -2 \sum_{n \leq u^{1/2}} \psi \left( \frac{u}{n} \right) + R(u) \right\} du
\]

\[
= 4 \sum_{n \leq x^{1/2}} \int_1^{x/n^2} \psi(u) \psi \left( \frac{u}{n} \right) (\log u + 2\gamma) du - 2 \int_1^{x/n^2} \psi(u)(\log u + 2\gamma) R(u) du
\]

\[
= T_{11} + T_{12},
\]

say. We consider \( T_{12} \). We note that formula (2.10) in Lemma 2 may be true if \( x^{1/2} \) is an integer, however we can write the function \( T_{12} \) by using the right-hand side in (2.10) as

\[
T_{12} = 2 \int_1^{x/2} \psi_1(u)(\log u + 2\gamma)(-2u^{-1/2} \psi(u^{1/2}) - 2u^{-1} \psi_1(u^{1/2}) - \frac{1}{6} u^{-1}) du + O(\log x).
\]

Hence by using this formula we have \( T_{12} = O(x^{1/2} \log x) \). (In other words, we can prove this estimate by dividing the integral as, for example,

\[
T_{12} = -2 \sum_{j=1}^{[x^{1/2}]-1} \int_{j^{2+1/4}}^{(j+1)^2-1/4} \psi(u)(\log u + 2\gamma) R(u) du + O(x^{1/2} \log x)
\]

and by applying integration by parts, (2.9) and (2.10).)

We consider \( T_{11} \). By applying integration by parts and Lemma 3, we have

\[
\int_1^{x/n^2} \psi(u) \psi \left( \frac{u}{n} \right) (\log u + 2\gamma) du = \frac{1}{12n} x \log x + \frac{1}{12n} (2\gamma - 1)x - \frac{1}{6} n \log n - \frac{1}{12} (2\gamma - 1)n + O(\log x).
\]
Hence we obtain

\[
T_{11} = \frac{1}{3} x (\log x + 2\gamma - 1) \sum_{n \leq x^{1/2}} n^{-1} - \frac{2}{3} \sum_{n \leq x^{1/2}} n \log n
- \frac{1}{3} (2\gamma - 1) \sum_{n \leq x^{1/2}} n + O(x^{1/2} \log x).
\]

By applying the Euler–Maclaurin summation formula to the sums in the above, we obtain

\[
T_{11} = \frac{1}{6} x \log^2 x + \frac{1}{3} (2\gamma - 1) x \log x + \frac{1}{3} (2\gamma^2 - 2\gamma + 1) x + O(x^{1/2} \log x),
\]

(4.1)

and hence the asymptotic formula for \( T_1 \) is also given by the right-hand side of (4.1).

By collecting the asymptotic formulas for \( T_1 \) and \( T_2 \), we obtain

\[
\sum_{n \leq x} \Delta(n)^2 = \int_1^x \Delta(u)^2 \, du + \frac{1}{6} x \log^2 x + c_1 x \log x + c_2 x
+ \frac{1}{2} \sqrt{\frac{2}{\pi}} x^{3/4} \log x \sum_{n=1}^\infty d(n)n^{-5/4} \sin \left( 4\pi \sqrt{nx} - \frac{\pi}{4} \right) + O(x^{3/4}),
\]

(4.2)

where the constants \( c_1 \) and \( c_2 \) are defined in the statement of Theorem 1. The \( O \)-estimate in Theorem 1 is easily proved, since the infinite series in (4.2) is convergent absolutely. And the \( \Omega_{\pm} \)-estimates in this theorem are equivalent to formula (1.7) in [13]. This completes the proof of Theorem 1.

5. Proof of Theorem 2

In Lemma 1, we put \( k = 3 \). By specifying \( f(n) = d(n) \) as was done in Section 4, we have

\[
\sum_{n \leq x} \Delta(n)^3 = \int_1^x \Delta(u)^3 \, du + \frac{3}{2} \int_1^x (\log u + 2\gamma) \Delta(u)^2 \, du
- 3 \int_1^x \psi(u)(\log u + 2\gamma) \Delta(u)^2 \, du + O(x)
= \int_1^x \Delta(u)^3 \, du + U_1 + U_2 + O(x),
\]

where...
say. As for $U_1$, we have by applying integration by parts, (1.4) and the mean value formula for $F(x)$ (cf. [14, Theorem 2]) that

\[
U_1 = \frac{3}{2} C x^{3/2} \left( \log x + 2\gamma - \frac{2}{3} \right) + \frac{3}{2} (\log x + 2\gamma) F(x) + \frac{3}{8\pi^2} x \log^2 x + O(x \log x),
\]

where $F(x)$ is the error term in formula (1.4), and $C$ is the coefficient of the main term in (1.4).

We consider $U_2$. By using Lemma 2, we have

\[
U_2 = -12 \int_1^x \psi(u)(\log u + 2\gamma) \sum_{m_1, m_2 \leq u^{1/2}} \psi \left( \frac{u}{m_1} \right) \psi \left( \frac{u}{m_2} \right) du \\
+ 12 \int_1^x \psi(u) R(u)(\log u + 2\gamma) \sum_{m \leq u^{1/2}} \psi \left( \frac{u}{m} \right) du - 3 \int_1^x \psi(u) R(u)^2 (\log u + 2\gamma) du
\]

say. It is obviously seen that $U_{23} = O(x \log x)$, since $\psi(u) = O(1)$ and $R(u) = O(1)$. Concerning $U_{22}$, we have by using the arguments used in the estimation of the function $T_{12}$, Lemmas 2 and 3 (and also the first formula of (2.13)) that

\[
U_{22} = \frac{1}{2} R(x) x \log^2 x + \sum_{m \leq x^{1/2}} m^{-1} \int_1^x u(\log u + 2\gamma)(2u^{-1/2}\psi(u^{1/2}) \\
+ 2u^{-1}\psi_1(u^{1/2}) + \frac{1}{6} u^{-1}) du + O(x \log x)
\]

say. We have by applying integration by parts and by using Lemma 5 that $U_{212} = O(x^{1/2} \log x)$. 

Consider $U_{21}$. We see that

\[
U_{21} = -24 \sum_{m_2 \leq x^{1/2}} \sum_{m_1 \leq m_2} \int_{m_2}^x \psi(u) \psi \left( \frac{u}{m_1} \right) \psi \left( \frac{u}{m_2} \right) (\log u + 2\gamma) du \\
+ 12 \sum_{m \leq x^{1/2}} \int_{m^2}^x \psi(u) \psi \left( \frac{u}{m} \right)^2 (\log u + 2\gamma) du
\]

say. We have by applying integration by parts and by using Lemma 5 that $U_{212} = O(x^{1/2} \log x)$. 

As for $U_{211}$, by using Lemma 5, we have

$$U_{211} = -(\log x + 2\gamma) \sum_{m_2 \leq \sqrt{x}} \sum_{m_1 \leq m_2} \left( \frac{m_2}{m_1} \psi \left( \frac{x}{m_2} \right)^2 - \frac{m_2}{4m_1} \right)$$

$$+ \sum_{m_2 \leq \sqrt{x}} \sum_{m_1 \leq m_2} \int_{m_2^2}^{x} u^{-1} \left( \frac{m_2}{m_1} \psi \left( \frac{u}{m_2} \right)^2 - \frac{m_2}{4m_1} \right) du + O(x \log x)$$

$$= \frac{1}{16} x \log^2 x - \log x \sum_{m \leq \sqrt{x}} \psi \left( \frac{x}{m} \right)^2 m \log m$$

$$+ \sum_{m \leq \sqrt{x}} m \log m \int_{m^2}^{x} u^{-1} \psi \left( \frac{u}{m} \right)^2 du + O(x \log x),$$

since

$$\sum_{m \leq \sqrt{x}} m \log^2 m = \frac{1}{8} x \log^2 x + O(x \log x).$$

Furthermore, by using Lemma 3, we have

$$\sum_{m \leq \sqrt{x}} m \log m \int_{m^2}^{x} u^{-1} \psi \left( \frac{u}{m} \right)^2 du = \sum_{m \leq \sqrt{x}} m \log \int_{m^2}^{x} u^{-1} \psi (u)^2 du$$

$$= \frac{1}{12} \sum_{m \leq \sqrt{x}} m \log m \int_{m}^{x/m} u^{-1} du + O(x \log x)$$

$$= O(x \log x).$$

On the other hand, by applying the formula $\psi(u)^2 = 2\psi_1(u) + \frac{1}{4}$, we have

$$\sum_{m \leq \sqrt{x}} \psi \left( \frac{x}{m} \right)^2 m \log m = 2 \sum_{m \leq \sqrt{x}} \psi_1 \left( \frac{x}{m} \right) m \log m + \frac{1}{16} x \log x + O(x)$$

$$= \frac{1}{16} \left( 8\psi_1 (x^{1/2}) + 1 \right) x \log x$$

$$+ 2x \int_{1}^{x^{1/2}} u^{-2} \psi \left( \frac{x}{u} \right) \sum_{m \leq u} m \log m du + O(x).$$
We apply the formula
\[
\sum_{m \leq u} m \log m = \frac{1}{2} u^2 \log u - \frac{1}{4} u^2 - \psi(u)u \log u + O(\log u)
\]
to get
\[
\int_1^{x^{1/2}} u^{-2} \psi \left( \frac{x}{u} \right) \sum_{m \leq u} m \log m \ du = \frac{1}{2} \int_1^{x^{1/2}} \psi \left( \frac{x}{u} \right) \log u \ du - \frac{1}{4} \int_1^{x^{1/2}} \psi \left( \frac{x}{u} \right) \ du \\
- \int_1^{x^{1/2}} u^{-1} \psi \left( \frac{x}{u} \right) \psi(u) \log u \ du + O(1).
\]
Furthermore, since
\[
\int_1^{x^{1/2}} \psi \left( \frac{x}{u} \right) \ du = x \int_{x^{1/2}}^{x} u^{-2} \psi(u) \ du = O(1)
\]
by integration by parts, and
\[
\int_1^{x^{1/2}} \psi \left( \frac{x}{u} \right) \log u \ du = x \log x \int_{x^{1/2}}^{x} u^{-2} \psi(u) \ du - x \int_{x^{1/2}}^{x} u^{-2} \psi(u) \log u \ du \\
= - \frac{1}{2} \left( \psi_1(x^{1/2}) + \frac{1}{12} \right) \log x + O(1),
\]
which follows from integration by parts and Lemma 6, we have
\[
\int_1^{x^{1/2}} u^{-2} \psi \left( \frac{x}{u} \right) \sum_{m \leq u} m \log m \ du \\
= - \frac{1}{4} \left( \psi_1(x^{1/2}) + \frac{1}{12} \right) \log x - \frac{1}{2} x^{-1/2} \log x \int_1^{x^{1/2}} \psi(u) \psi \left( \frac{x}{u} \right) \ du \\
+ \int_1^{x^{1/2}} u^{-2} (1 - \log u) \int_1^{u} \psi(t) \psi \left( \frac{x}{t} \right) \ dt \ du + O(1).
\]
Hence we obtain
\[
U_{211} = \frac{1}{24} x \log^2 x + x^{1/2} \log^2 x \int_1^{x^{1/2}} \psi(u) \psi \left( \frac{x}{u} \right) \ du \\
- 2x \log x \int_1^{x^{1/2}} u^{-2} (1 - \log u) \int_1^{u} \psi(t) \psi \left( \frac{x}{t} \right) \ dt + O(x \log x).
\]
We treat the integral \( \int_1^y \psi(u) \psi(x/u) \, du \) for \( y \leq x^{1/2} \). From the definition of the \( \psi \)-function, we get

\[
\int_1^y \psi(u) \psi\left(\frac{x}{u}\right) \, du = \int_1^y \left( u - \frac{1}{2} \right) \psi\left(\frac{x}{u}\right) \, du - \int_1^y \left[ u \right] \psi\left(\frac{x}{u}\right) \, du
\]

\[
= x \int_{xy-1}^x \left( xu^{-3} - \frac{1}{2} u^{-2} \right) \psi(u) \, du - \sum_{j=1}^{[y]-1} \int_j^{j+1} \psi\left(\frac{x}{u}\right) \, du
\]

\[
- [y] \int_{[y]}^y \psi\left(\frac{x}{u}\right) \, du
\]

\[
= V_1 + V_2 + V_3,
\]
say. Consider the function \( V_2 \). We divide this function as

\[
V_2 = -x^{-1} \sum_{j=1}^{[y]-1} j \left\{ j^2 \psi_1\left(\frac{x}{j}\right) - (j + 1)^2 \psi_1\left(\frac{x}{j+1}\right) \right\}
\]

\[
-2x \sum_{j=1}^{[y]-1} j \int_{x/(j+1)}^{x/j} u^{-3} \psi_1(u) \, du
\]

\[
= V_{21} + V_{22},
\]
say. Consider \( V_{21} \). We first note that

\[
\sum_{j=1}^N j^3 (a_j - a_{j+1}) = \sum_{j=1}^N (3j^2 - 3j + 1)a_j - N^3 a_{N+1}
\]

for any sequence \( \{a_j\}_{j \in \mathbb{N}} \) and any natural number \( N \). Then by using this formula, the formula

\[
\sum_{n \leq y} j^2 B_2 \left( \frac{x}{j} - \left[ \frac{x}{j} \right] \right) = O(x^{1/2} y^{3/2})
\]
for $x^{2/5} \ll y \leq x^{1/2}$ [11, Theorem 2], and the relation $B_2(y - \lfloor y \rfloor) = 2\psi_1(y) + \frac{1}{6}$ we have

$$V_{21} = \frac{y^3}{x} \psi_1 \left( \frac{x}{\lfloor y \rfloor} \right) - x^{-1} \sum_{j \leq y} j^2 \psi_1 \left( \frac{x}{j} \right) + O(1)$$

$$= \frac{y^3}{x} \psi_1 \left( \frac{x}{\lfloor y \rfloor} \right) + \frac{1}{36x} y^3 + O(1) + \begin{cases} O(x^{-1/2}y^{3/2}) & \text{if } x^{2/5} \ll y \leq x^{1/2}, \\ O(x^{-1}y^3) & \text{if } 1 \leq y \ll x^{2/5}. \end{cases}$$

(5.1)

Here, $B_2(y)$ denotes the Bernoulli function of order 2 (Note that in the case $1 \leq y \ll x^{2/5}$, the leading terms in (5.1) are absorbed into the error term.)

Now we derive asymptotic formulas for the functions $V_1, V_22$ and $V_3$. We first treat $V_{22}$. Since

$$\sum_{j=1}^{N} j(a_j - a_{j+1}) = \sum_{j=1}^{N} a_j - Na_{N+1}$$

for any sequence $\{a_j\}_{j \in \mathbb{N}}$ and any natural number $N$, using (3.10) we obtain

$$V_{22} = -2x \sum_{j=1}^{\lfloor y \rfloor-1} \int_{x/j}^{x/j+1} u^{-3} \psi_1(u) du + 2x(\lfloor y \rfloor - 1) \int_{1}^{\lfloor y \rfloor} u^{-3} \psi_1(u) du$$

$$= \frac{1}{18x} y^3 + O(1).$$

Finally, we evaluate $V_1$ and $V_3$. These can be treated by using integration by parts. Actually, it is easy to see that

$$V_1 = -\frac{y^3}{x} \psi_1 \left( \frac{x}{y} \right) - \frac{y^3}{12x} + O(1)$$

by (2.12), and also

$$V_3 = \frac{y^3}{x} \psi_1 \left( \frac{x}{y} \right) - \frac{y^3}{x} \psi_1 \left( \frac{x}{\lfloor y \rfloor} \right) - 2x [y] \int_{x/y}^{x/[y]} u^{-3} \psi_1(u) du + O(1)$$

$$= \frac{y^3}{x} \psi_1 \left( \frac{x}{y} \right) - \frac{y^3}{x} \psi_1 \left( \frac{x}{\lfloor y \rfloor} \right) + O(1),$$

since

$$\int_{x/y}^{x/[y]} u^{-3} \psi_1(u) du = \frac{[y]^2}{24x^2} - \frac{y^2}{24x^2} + O(x^{-3/2}) = O(x^{-3/2})$$
for $y \leq x^{1/2}$. Therefore we obtain

$$\int_1^y \psi(u)\psi\left(\frac{x}{u}\right) \, du \ll 1 + \begin{cases} x^{-1/2}y^{3/2} & \text{if } x^{2/5} \ll y \leq x^{1/2}, \\ x^{-1}y^3 & \text{if } 1 \leq y \ll x^{2/5}, \end{cases}$$

and thus we obtain

$$U_{211} = \frac{1}{24} x \log^2 x + O(x \log x).$$

By combining all the above formulas we obtain

$$\sum_{n \leq x} A(n)^3 = \int_1^x A(u)^3 \, du + \frac{3}{2} Cx^{3/2} \log x + (3\gamma - 1)Cx^{3/2} + \frac{3}{2}(\log x + 2\gamma)F(x)$$

$$+ \left(\frac{3}{8\pi^2} + \frac{1}{8}\right)x \log^2 x + O(x \log x).$$

Hence applying the estimates $F(x) = O(x \log^4 x)$ and $F(x) = \Omega_-(x \log^2 x)$ to this formula, we can prove the first assertion of Theorem 2. And also applying formula (1.5), we can obtain the second assertion of this theorem. This completes the proof of Theorem 2. $\square$

**Remark.** For $k \geq 4$, to derive the asymptotic formula for the difference between $\sum_{n \leq x} A(n)^k$ and $\int_1^x A(u)^k \, du$ with sharper error term than (1.9) by using the method in this paper, we should consider the asymptotic formula and the identity for the integrals of the type

$$\int_1^x \psi(u)\psi\left(\frac{u}{n_1}\right) \psi\left(\frac{u}{n_2}\right) \cdots \psi\left(\frac{u}{n_j}\right) \, du, \quad (5.2)$$

where $n_1, n_2, \ldots, n_j$ are natural numbers with $n_1 \leq n_2 \leq \cdots \leq n_j$ and $j$ is an arbitrary natural number with $j \leq k$. In particular, we need to use these kinds of formulas for all $j \leq k$. For example, in the case $k = 4$, we should first derive the identity for the integral $\int_1^x \psi(u)\psi(u/n_1)\psi(u/n_2) \, du$ instead of Lemma 5 in this paper, and then we consider the asymptotic formula for the integral

$$\int_1^x \psi(u)\psi\left(\frac{u}{n_1}\right) \psi\left(\frac{u}{n_2}\right) \psi\left(\frac{u}{n_3}\right) \, du.$$
Moreover, for that purpose, we should also know the detailed information about the function $F_k(x)$ defined by

$$F_k(x) = \int_1^x A(u)^k \, du - C_k x^{1+k/4},$$

where $C_k$ is a certain constant. In particular, we should apply the results of $F_{k-1}(x)$ for $k \geq 4$ when we study the mean value formula for $\sum_{n \leq x} A(n)^k$. Actually, if we consider the analogue of Theorem 1 or the second assertion of Theorem 2 to general $k$th moment, it would be necessary to derive an $\Omega$-estimate of $F_{k-1}(x)$, or, an asymptotic formula for $F_{k-1}(x)$ analogous to (1.5). It seems to be quite difficult to obtain such an $\Omega$-estimate and an asymptotic formula for general $k \geq 3$.

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