J. Math. Anal. Appl. 348 (2008) 589–606

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Bounded sets of Lagrange multipliers for vector optimization problems in infinite dimension

M. Durea [a], J. Dutta [b], Chr. Tammer [c,*]

[a] Faculty of Mathematics, "Al. I. Cuza" University, Bd. Carol I, nr. 11, 700506 Iaşi, Romania
[b] Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kanpur 208016, India
[c] Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, Institut für Optimierung und Stochastik, D-06099 Halle (Saale), Germany

ARTICLE INFO

*Article history:*
Received 19 December 2007
Available online 23 July 2008
Submitted by B.S. Mordukhovich

*Keywords:*
Lagrange multipliers
Vector optimization
Mangasarian–Fromovitz conditions

ABSTRACT

The aim of this paper is to point out some sufficient constraint qualification conditions ensuring the boundedness of a set of Lagrange multipliers for vectorial optimization problems in infinite dimension. In some (smooth) cases these conditions turn out to be necessary for the existence of multipliers as well.



## 1. Introduction

Lagrange multipliers are one of the central notions of optimization theory. The method of Lagrange multipliers is known to be the classical approach to constrained optimization problems. Its roots can be traced back to Lagrange's work in 18th century and the use of this method emerged in 20th century in F. John's paper [14], in 1948. Basically, the method of Lagrange multipliers consists of transforming a constrained problem into an unconstrained one. In the 18th century most optimization problems were related to physical sciences and had equalities as constraints. Lagrange essentially applied his multiplier method to study problems arising in the Calculus of Variations. For more details see the entertaining book titled *Stories about Maxima and Minima* by Tikhomorov [21]. Though the problems of Calculus of Variations are actually infinite dimensional problems of optimization, very different techniques were developed in those days to study them, since there was no clear idea about infinite dimensional spaces.

The real study of optimization in finite dimensions in fact began with the advent of *Linear Programming* during the second World–War. Later on, it was realized that in many applications nonlinear functions had to be tackled and that gave rise to *Nonlinear Programming*. It was understood that in many applications, for example in engineering, constraints in the form of inequalities play a predominant role. John's seminal paper of 1948, that has been mentioned above, first brought the Lagrangian multiplier rule to the domain of inequality constraint. Another key contribution in this direction is due to Kuhn and Tucker [15]. They corrected the shortcomings of the optimality conditions as laid down by John. This lead to the now famous Kuhn–Tucker conditions which play a very fundamental role both in the theory of optimization and the analysis of optimization algorithms. It is also important to note that Kuhn and Tucker [15] were most probably the first to have a detailed study of necessary optimality conditions for multiobjective programming. Thus Kuhn and Tucker extended the Lagrangian multiplier rule to the domain of vector optimization.

* Corresponding author.
*E-mail addresses:* durea@uaic.ro (M. Durea), jdutta@iitk.ac.in (J. Dutta), christiane.tammer@mathematik.uni-halle.de (Chr. Tammer).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.07.039

The main draw back of the necessary optimality conditions laid down by John was that a multiplier gets associated with the gradient of the objective function and this multiplier can also become zero, thus making the objective function play no role in the optimization process. However, in order to put the necessary optimality conditions for problems with inequality constraints in the form of Lagrange multiplier rule one needs a positive multiplier associated with the gradient of the objective function. Kuhn and Tucker [15] realized that in order to achieve this one needs to impose certain conditions on the constraints. These constraints, known as constraint qualifications, play a vital role in optimization. One of such constraint qualification, due to Mangasarian and Fromovitz [16], was introduced in 1969 and it still plays a very central role in optimization theory. In order to motivate our study, it will be useful to briefly state the Mangasarian–Fromovitz constraint qualification (MFCQ).

Consider the following mathematical programming problem in finite dimensions,

$$\begin{aligned}&\min f(x),\\&\text{subject to}\\&\quad g_i(x) \leqslant 0, \quad i=1,\dots,m,\\&\quad h_j(x)=0, \quad j=1,\dots,k,\end{aligned}$$

where $f:\mathbb{R}^n \to \mathbb{R}$, $g_i:\mathbb{R}^n \to \mathbb{R}$ for all $i=1,\dots,m$ and $h_j:\mathbb{R}^n \to \mathbb{R}$ for all $j=1,\dots,k$. Assume that all the functions are continuously Fréchet differentiable. Let $\bar{x}$ be a feasible point of the above problem. Then the Mangasarian–Fromovitz constraint qualification holds at $\bar{x}$ if the set

$$\{\nabla h_1(\bar{x}),\dots,\nabla h_k(\bar{x})\}$$

is a linearly independent set and there exists $d \in \mathbb{R}^n$ such that

$$\begin{aligned}&\langle \nabla h_j(\bar{x}), d\rangle = 0, \quad \forall j=1,\dots,k,\\&\langle \nabla g_i(\bar{x}), d\rangle < 0, \quad \forall i \in I(\bar{x}),\end{aligned}$$

where $I(\bar{x}) = \{i \in \{1,\dots,m\} \mid g_i(\bar{x})=0\}$ denotes the set of active indices at the point $\bar{x}$.

In 1977, J. Gauvin [12] observed that the Mangasarian–Fromovitz constraint qualification condition is in fact equivalent (for finite dimensional scalar problems with smooth data) to the boundedness of the set of Lagrange multipliers associated with the mathematical programming problem stated above. More precisely, Gauvin [12] showed that the boundedness of the set

$$M(\bar{x}) = \left\{(\lambda,\mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k \;\middle|\; \nabla f(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^k \mu_j \nabla h_j(\bar{x}) = 0,\ \lambda_i g_i(\bar{x}) = 0,\ \forall i=1,\dots,m\right\},$$

is equivalent to the Mangasarian–Fromovitz constraint qualification. This remark gives more importance to the Mangasarian–Fromovitz constraint qualification condition, because the fact that the set of multipliers is bounded has many implications in the computational aspects concerning the Lagrangian function (see [1] and the references therein).

Let us now turn our attention to vector optimization problems in finite dimensions. Consider the following vector optimization problem

$$\begin{aligned}&\min f(x) = \bigl(f_1(x),\dots,f_p(x)\bigr)\\&\text{subject to}\\&\quad g_i(x) \leqslant 0, \quad i=1,\dots,m,\\&\quad h_j(x)=0, \quad j=1,\dots,k,\end{aligned}$$

where $f:\mathbb{R}^n \to \mathbb{R}$, $g_i:\mathbb{R}^n \to \mathbb{R}$ for all $i=1,\dots,m$ and $h_j:\mathbb{R}^n \to \mathbb{R}$ for all $j=1,\dots,k$. Again, assume that all the functions are continuously differentiable. It is well known that a feasible point $\bar{x}$ of the above problem is called Pareto minimum if there exists no feasible $x$ such that $f(x) - f(\bar{x}) \in -(\mathbb{R}^p_+ \setminus \{0\})$ and the point $\bar{x}$ is called a weak minimum if there exists no feasible $x$ such that $f(x) - f(\bar{x}) \in -\operatorname{int}\mathbb{R}^p_+$. The set of Lagrangian multipliers for the vector optimization problem stated above is given as

$$\begin{aligned}E(\bar{x}) = \Biggl\{&(\tau,\lambda,\mu) \in \bigl(\mathbb{R}^p_+ \setminus \{0\}\bigr) \times \mathbb{R}^m_+ \times \mathbb{R}^k \;\Bigm|\; \sum_{r=1}^p \tau_r \nabla f_r(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^k \mu_j \nabla h_j(\bar{x}) = 0,\\&\lambda_i g_i(\bar{x}) = 0,\ \forall i=1,\dots,m\Biggr\}.\end{aligned}$$

For example, if $\bar{x}$ is a weak minimum and some constraint qualification is satisfied, then $E(\bar{x})$ is nonempty. However, observe that $E(\bar{x})$ forms a cone and thus is unbounded. In Dutta and Lalitha [7] certain subsets of $E(\bar{x})$ were identified and necessary and sufficient conditions for those sets to be bounded were studied. In the quoted paper the authors considered the case for both Pareto minimum and the weak Pareto minimum and they studied the smooth case and also the nonsmooth case with locally Lipschitz data.

In the recent past, there has been a renewed interest in the study of non-convex vector optimization in infinite dimensional spaces. These approaches are based on a very different scalarization approach than the ones used in the literature (see, for example, [10] and [9] and the references therein). Further, there are few works which also concentrate on the study of set-valued vector optimization. Very recently, the coderivative of a set-valued map defined by Mordukhovich (see [17]) was used as an important tool in the study of set-valued optimization (see [3,8,19]).

In this paper, we focus our attention on vector optimization problems with cone constraints in normed spaces. In particular, our main aim is to devise sufficient conditions under which the set of Lagrange multipliers to the cone constrained vector optimization problem becomes a bounded set. We study both the smooth and nonsmooth case. While studying the nonsmooth case we take a more unified view by studying set-valued optimization problems in terms of the coderivative of Mordukhovich. We also show what happens when we consider the map to be single-valued. It is natural that the conditions needed to guarantee the boundedness of the set of Lagrange multipliers would be more complicated than the one in the finite dimensional case. We would like to stress that we use specific methods for each of the different cases that we are going to study.

The paper is organized as follows. Section 2 contains some basic facts required in the sequel. Our main focus in this section is the study of dually compact cones which play a pivotal role in our study. Section 3 is devoted to the case of cone-constrained vector optimization problem with single-valued and smooth data on general normed vector spaces, while Section 4 is devoted to the case of set-valued (nonsmooth) optimization problems on reflexive Banach spaces. The aim of Sections 3 and 4 is to develop some Mangasarian–Fromovitz type conditions in order to ensure that a certain set of Lagrange multipliers is bounded. In both cases we shall see that the conditions we shall find are similar to those in the classical theory despite the fact we use different techniques: In Section 3 we mainly follow the lines of the classical case while in Section 4 we consider the tools of generalized differentiation theory developed by B.S. Mordukhovich [17].

## 2. Preliminaries

In the first part of this paper we will be concerned with the following vector optimization problem with functional constraints

$$(\mathcal{P}) \quad \min f(x) \quad \text{such that } g(x) \in -Q,$$

where $X, Y$ and $Z$ are normed vector spaces, $f : X \to Y$, $g : X \to Z$ are functions and $Q$ is a closed convex cone in $Z$. We shall consider that the space $Y$ has a partial order induced by the closed convex cone $K$, as usual, by $y_1 \leqslant_K y_2$ if $y_2 - y_1 \in K$. We denote by $M$ the set of feasible points of the problem $(\mathcal{P})$, i.e., $M := \{x \in X \mid g(x) \in -Q\}$.

In the following, in some situations, we assume:

$(A_K)$ $\operatorname{int} K \neq \emptyset$,
$(A_Q)$ $\operatorname{int} Q \neq \emptyset$.

It is important to realize that in many normed spaces the natural ordering cone has an empty interior. For example, this fact is true in most of the Asplund spaces. However, there are some Banach spaces where the natural ordering cone has a nonempty interior. An important example of such a space is the space $C(\Omega)$ of all continuous real-valued functions defined on the compact Hausdorff set $\Omega$.

A feasible point $\bar{x}$ is said to be a Pareto minimum of $(\mathcal{P})$ if there exists no feasible $x$ to $(\mathcal{P})$ such that $f(x) - f(\bar{x}) \in -(K \setminus \{0\})$. In the event that the cone $K$ is also having a nonempty interior then $\bar{x}$ is called a weak minimum of $(\mathcal{P})$ if there exists no feasible $x$ such that $f(x) - f(\bar{x}) \in -\operatorname{int} K$.

As we mentioned in the previous section of this paper we also focus our attention on set-valued optimization problem. Hence, we mention the types of set-valued optimization problems that we shall consider in this study. Let us begin with the some preliminaries on set-valued maps.

Let us consider set-valued maps $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, where $X$, $Y$ and $Z$ are normed spaces. As usual, the graph of $F$ is $\operatorname{Gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, $F(A) := \bigcup_{x \in A} F(x)$ and the inverse set-valued map of $F$ is $F^{-1} : Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} F^{-1}$ if and only if $(x, y) \in \operatorname{Gr} F$.

We shall first consider the following set-valued problem where we will have no functional constraints. This problem is given as follows:

$$(\mathcal{P}_1) \quad \min F(x), \quad \text{subject to } x \in S \subset X,$$

where $F : X \rightrightarrows Y$ and the space $Y$ is partially-ordered by a closed convex cone $K$. Now, with respect to the ordering cone $K$ we introduce the definition of Pareto minimum and weak Pareto minimum for the above problem.

**Definition 2.1.** A point $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called Pareto minimum point for problem $(\mathcal{P}_1)$ if

$$\big(F(S) - \bar{y}\big) \cap -K = \{0\}.$$

For weak solutions (if $\operatorname{int} K \neq \emptyset$) one uses the following definition.

**Definition 2.2.** A point $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called weak minimum point for problem $(\mathcal{P}_1)$ if

$$\big(F(S) - \bar{y}\big) \cap -\operatorname{int} K = \emptyset.$$

We shall also consider the case when the feasible set of the problem $(\mathcal{P}_1)$ will be explicitly described via set-valued maps. This is given in form of the following problem

$$(\mathcal{P}_2) \quad \min F(x), \quad \text{subject to } x \in X,\ 0 \in G(x) + Q.$$

In fact, we shall also consider a simplified version of problem $(\mathcal{P}_2)$ where $G$ is not a set-valued map but a single-valued one denoted by $g$. Then the problem above is represented as

$$\big(\mathcal{P}_2'\big) \quad \min F(x), \quad \text{subject to } x \in X,\ g(x) \leqslant_Q 0.$$

The definitions for both types of solutions for $(\mathcal{P}_2)$ is obtained by replacing $S$ in the above definitions by $\{x \in X \mid 0 \in G(x) + Q\} = G^{-1}(-Q)$.

As mentioned in the previous section, our main aim here is to work on infinite dimensional normed vector spaces, not only with single-valued vector functions, but also with set-valued maps. We shall consider the smooth and nonsmooth data and the solutions will be understood in the sense of weak minimality and/or Pareto minimality.

We will now turn our attention to a specific class of cones called dually compact cones which play a central role in our analysis. This notion was introduced in [19] where the authors used it in order to develop Fermat type rules for set-valued maps.

Let $Y$ be a normed vector space and $K \subset Y$ be a closed, convex and pointed cone which introduces a partial order relation on $Y$. The notation $Y^*$ stands for the topological dual of $Y$. Then, the dual cone of $K$ is

$$K^* := \big\{y^* \in Y^* \bigm| y^*(k) \geqslant 0,\ \forall k \in K\big\}.$$

In several cases, we shall work with cones with nonempty interior, but we shall also consider more general cones, namely dually compact cones introduced in [19]. One says that $K$ is dually compact if there exists a compact set $C \subset Y$ s.t.

$$K^* \subset \Big\{y^* \in Y^* \Bigm| \|y^*\| \leqslant \sup_{y \in C} y^*(y)\Big\}.$$

In fact, this property is equivalent with the similar condition written for finite sets instead of compact sets: The cone is dually compact if and only if there exists a finite subset $P := \{y_1, y_2, \ldots, y_p\} \subset Y$ s.t.

$$K^* \subset \Big\{y^* \in Y^* \Bigm| \|y^*\| \leqslant \max_{y_i \in P} y^*(y_i)\Big\}. \tag{1}$$

Indeed, since every finite set is compact it is clear that the latter property implies the definition. Conversely (see also [19]), for the compact set $C$ there exists a finite set $\{y_1, y_2, \ldots, y_p\}$ s.t.

$$C \subset \bigcup_{i=1}^{p} \big(y_i + 2^{-1} U_Y\big),$$

where $U_Y$ stands for the closed unit ball in $Y$. Then for every $y^* \in K^*$,

$$\|y^*\| \leqslant \sup_{y \in \bigcup_{i=1}^{p}(y_i + 2^{-1}U_Y)} y^*(y) = \max_{i=\overline{1,p}} y^*(y_i) + 2^{-1}\|y^*\|.$$

We deduce that

$$\|y^*\| \leqslant \max_{i=\overline{1,p}} y^*(2y_i),$$

so we can take $P$ as $\{2y_1, 2y_2, \ldots, 2y_p\}$.

Using (1) as the basis of further considerations one can easily obtain two conclusions already observed in [19]:

- every closed convex cone in a Euclidian space $\mathbb{R}^n$ ($n \geqslant 1$) is dually compact: Take $P = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{-1, 1\}, \forall i = \overline{1, p}\}$;
- a cone with nonempty interior (in a general normed vector space) is dually compact.

Concerning this last assertion it is proved in [19] that for a cone with nonempty interior the set $P$ can be taken as a singleton. We can prove that this property characterizes the cones with nonempty interior among dually compact cones. This is shown in the following proposition.

**Proposition 2.1.** *Let $K$ be a cone with nonempty interior. Then it is dually compact and the set $P$ in (1) is a singleton. Conversely, assume that $K$ is dually compact and the set $P$ is a singleton. Then $K$ has a nonempty interior.*

**Proof.** Let us first observe that in the case $P := \{\bar{u}\}$ the right-hand side of (1) becomes $\{y^* \in Y^* \mid \|y^*\| \leqslant y^*(\bar{u})\}$ and this set is a closed convex cone. Moreover,

$$\begin{aligned}\left\{y^* \in Y^* \,\middle|\, \|y^*\| \leqslant y^*(\bar{u})\right\} &= \left\{y^* \in Y^* \,\middle|\, y^*(u) \leqslant y^*(\bar{u}),\ \forall u \in U_Y\right\} \\ &= \left\{y^* \in Y^* \,\middle|\, y^*(\bar{u} - u) \geqslant 0,\ \forall u \in U_Y\right\} \\ &= (\bar{u} - U_Y)^*,\end{aligned}$$

where $A^*$ stands, as usual, for the dual cone of a set $A$.

Suppose now that $\operatorname{int} K \neq \emptyset$. Then, there exists $\bar{u} \in K$ and a positive $a$ s.t. $\bar{u} - aU_Y \subset K$, i.e. $a^{-1}\bar{u} - U_Y \subset K$. Then $(a^{-1}\bar{u} - U_Y)^* \supset K^*$ so $P$ can be taken as $\{a^{-1}\bar{u}\}$.

Conversely, suppose that $K$ is a dually compact cone with the set $P = \{\bar{u}\}$. Then, from the discussion at the beginning of the proof we conclude that $K^* \subset (\bar{u} - U_Y)^*$. Then

$$K^{**} = K \supset (\bar{u} - U_Y)^{**} = \operatorname{cl}\operatorname{cone}(\bar{u} - U_Y),$$

where $\operatorname{cl}\operatorname{cone}(\bar{u} - U_Y)$ stands for the closed cone generated by $\bar{u} - U_Y$. Since this cone obviously contains balls, $\operatorname{int} K \neq \emptyset$. This completes the proof. □

Therefore, we can conclude that a closed convex cone $K$ is dually compact and has empty interior if the set $P$ contains at least two distinct elements $\bar{u}_1, \bar{u}_2 \in Y$ s.t. $(\bar{u}_1 - U_Y)^* \neq (\bar{u}_2 - U_Y)^*$, $K^* \cap (\bar{u}_i - U_Y)^* \setminus (\bar{u}_j - U_Y)^* \neq \emptyset$, $i \neq j$, $i, j \in \overline{1,2}$. Thus, from the point of view of vector optimization, it is important to have an example of a cone which is dually compact and yet has an empty interior. We now present here the example very kindly provided to us by Professor K.F. Ng.

**Example 2.1.** Let $X = l_2$ and let the cone $C$ be given as

$$C = \left\{(0, t_1, t_2, \ldots) \,\middle|\, \sum_{n=2}^{\infty} t_n^2 < +\infty\right\}.$$

It is clear that $X$ is an Asplund space and $C$ has an empty interior. Further, it is simple to see that the dual cone of $C$ is given as

$$C^* = \left\{(t, 0, 0, \ldots) \,\middle|\, t \in \mathbb{R}\right\}.$$

It is simple to see that $C^*$ is weakly locally compact. Hence, using Proposition 3.1 in [19], we conclude that $C$ is dually compact. In fact, one can see that the corresponding set P consists of two elements: $(1, 0, 0, \ldots)$ and $(-1, 0, 0, \ldots)$.

However, we would like to stress that the natural ordering cone of many normed spaces needs not be dually compact. In the sequel, we use a remarkable property of dually compact cones [19, Relation 3.9] which follows from relation (1): If $K$ is dually compact then every (generalized) sequence $(y_n^*) \subset K^*$ with $y_n^* \xrightarrow{w^*} 0$ converges strongly to 0. Here $w^*$ denotes the weak star topology in $Y^*$ generated by the natural duality between $Y$ and $Y^*$. We interpret this relation in the following way: if a sequence $(y_n^*) \subset K^*$ with $\|y_n^*\| = 1$ for every $n$, converges weakly star to some $y^*$, then $y^* \neq 0$ (contrary, $y_n^* \to 0$ in norm and this is not possible since $\|y_n^*\| = 1$).

## 3. The case of smooth data

This section is devoted to development of main ideas in the simplest case: We consider weak solutions for vectorial problems under inequality constraints with smooth data. For this section, we suppose

$(A_f)$ The function $f : X \to Y$ is continuously Fréchet differentiable on $X$.
$(A_g)$ The function $g : X \to Z$ is continuously Fréchet differentiable on $X$.

In what follows we denote by $\nabla f(x)$ the Fréchet derivative of $f$ at the point $x \in X$ and by $T_B(M, z)$ the Bouligand tangent cone to $M \subset X$ at $z \in M$. We recall that

$$T_B(M, z) := \left\{u \in X \,\middle|\, \exists (t_n) \subset (0, \infty),\ t_n \downarrow 0,\ \exists (u_n) \subset X,\ u_n \to u,\ \forall n \in \mathbb{N},\ z + t_n u_n \in M\right\}.$$

The next results present a method for obtaining Lagrange multipliers for problem $(\mathcal{P})$ along the lines of the classical scalar case. Corresponding results are proven by Jahn [13, Theorem 7.3], using the Lyusternik theorem. For reader's convenience, we give the following necessary condition for weakly minimal points of $(\mathcal{P})$ with the feasible set $M$.

**Proposition 3.1.** *Assume $(A_K)$ and $(A_f)$. If $\bar{x} \in M$ is a weakly minimal point for the problem $(\mathcal{P})$, then $\nabla f(\bar{x})(u) \notin -\operatorname{int} K$ for every $u \in T_B(M, \bar{x})$.*

**Proof.** We proceed by contradiction: Suppose that there is an element $u \in T_B(M, \bar{x})$ s.t. $\nabla f(\bar{x})(u) \in -\operatorname{int} K$. Since $f$ is Fréchet differentiable at $\bar{x}$, one has that there exists $\alpha : X \to Y$ with $\lim_{h \to 0} \alpha(h) = \alpha(0) = 0$ s.t.

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \|x - \bar{x}\| \alpha(x - \bar{x}) \tag{2}$$

for every $x \in X$. Since $u \in T_B(M, \bar{x})$, following the definition, there exist $(t_n) \subset [0, \infty)$, $t_n \downarrow 0$ and $(u_n) \subset X$, $u_n \to u$, s.t. $\bar{x} + t_n u_n \in M$, for every $n \in \mathbb{N}$. Using (2) one gets:

$$f(\bar{x} + t_n u_n) - f(\bar{x}) = t_n \nabla f(\bar{x})(u_n) + t_n \|u_n\| \alpha(t_n u_n) = t_n \big( \nabla f(\bar{x})(u_n) + \|u_n\| \alpha(t_n u_n) \big)$$

for every $n \in \mathbb{N}$. Therefore, for $n$ large enough, $\nabla f(\bar{x})(u_n) + \|u_n\| \alpha(t_n u_n) \in -\operatorname{int} K$, whence $t_n(\nabla f(\bar{x})(u_n) + \|u_n\| \alpha(t_n u_n)) \in -\operatorname{int} K$, and taking into account that $\bar{x} + t_n u_n \in M$, this contradicts the weak minimality of $\bar{x}$. The proposition is proved. $\square$

Now, we present a calculation of the tangent cone which appears in the preceding proposition taking into account the specific form of our feasible set $M = \{x \in X \mid g(x) \in -Q\}$. This is done in the next proposition where one needs to employ a condition concerning the existence of an element $\bar{u} \in X$ with $\nabla g(\bar{x})(\bar{u}) \in -\operatorname{int} Q$. This is a *Mangasarian–Fromovitz type condition* (denoted in short as (*MF*) condition).

**Proposition 3.2.** *Assume $(A_Q)$ and $(A_g)$. Furthermore, suppose that $g(\bar{x}) \in -Q$ and (MF) holds at $\bar{x}$. Then,*

$$\big\{ u \in X \bigm| \nabla g(\bar{x})(u) \in -Q \big\} \subset T_B(M, \bar{x}).$$

*Moreover, if $g(\bar{x}) = 0$ then*

$$T_B(M, \bar{x}) = \big\{ u \in X \bigm| \nabla g(\bar{x})(u) \in -Q \big\}.$$

**Proof.** First, observe that the set $M$ coincides with $g^{-1}(-Q)$ and the announced form of the tangent cone to $M$ at $\bar{x}$ coincides with $\nabla g(\bar{x})^{-1}(-Q)$. In order to prove the first conclusion, consider an element $u \in X$ with $\nabla g(\bar{x})(u) \in -\operatorname{int} Q$ and a sequence $(t_n) \subset (0, \infty)$, $t_n \downarrow 0$. We use that $g$ is differentiable at $\bar{x}$ and we get the existence of a sequence $(\alpha_n) \subset Z$, $\alpha_n \to 0$ with

$$g(\bar{x} + t_n u) = g(\bar{x}) + t_n \nabla g(\bar{x})(u) + t_n \|u\| \alpha_n = g(\bar{x}) + t_n \big( \nabla g(\bar{x})(u) + \|u\| \alpha_n \big) \in -Q - \operatorname{int} Q \subset -Q,$$

for $n$ large enough. This shows that $g(\bar{x} + t_n u) \in -Q$ for large $n$, whence $u \in T_B(M, \bar{x})$.

Let us now take $u \in X$ with $\nabla g(\bar{x})(u) \in -Q$ and for every $n \in \mathbb{N} \setminus \{0\}$ consider $u_n := n^{-1} \bar{u} + (1 - n^{-1}) u$ where $\bar{u}$ is given by condition (*MF*). It is clear that the sequence $(u_n)$ converges to $u$ and since

$$\nabla g(\bar{x})(u_n) = n^{-1} \nabla g(\bar{x})(\bar{u}) + \big(1 - n^{-1}\big) \nabla g(\bar{x})(u) \in -\operatorname{int} Q - Q \subset -\operatorname{int} Q,$$

we have that $\nabla g(\bar{x})(u_n) \in -\operatorname{int} Q$ for every $n$. Then $u_n \in T_B(M, \bar{x})$ for every $n$ and since $T_B(M, \bar{x})$ is a closed cone, we get $u \in T_B(M, \bar{x})$. This ends the proof of the first part.

Now, we study the case where $g(\bar{x}) = 0$. Let us consider $u \in T_B(M, \bar{x})$. Hence there exists a sequence $\{u_n\}$ such that $u_n \to u$ and a sequence $t_n \downarrow 0$ such that $\bar{x} + t_n u_n \in M$. Hence $g(\bar{x} + t_n u_n) \in -Q$. Since $g(\bar{x}) = 0$ we have

$$g(\bar{x} + t_n u_n) - g(\bar{x}) \in -Q.$$

Since $t_n > 0$ for all $n \in \mathbb{N}$ and $Q$ is a cone we have that

$$\frac{g(\bar{x} + t_n u_n) - g(\bar{x})}{t_n} \in -Q.$$

Since $g$ is smooth and $Q$ is a closed cone we have as $n \to \infty$

$$\nabla g(\bar{x})(u) \in -Q.$$

This yields

$$T_B(M, \bar{x}) \subset \big\{ u \in X \bigm| \nabla g(\bar{x})(u) \in -Q \big\},$$

i.e., the assertions of the proposition are shown. $\square$

From the above proof it is clear that for the equality to hold in the above result the fact that $g(\bar{x}) = 0$ plays a very crucial role. However, we want to stress that such a requirement is not mandatory to calculate the Bouligand tangent cone to the set $M$ at $\bar{x}$. The result would be more generalized and more involved than that of the above proposition. We show such a calculation through the following proposition.

**Proposition 3.3.** *Assume $(A_Q)$ and $(A_g)$. Let $\bar{x} \in M$ and let us assume that there exists $\bar{u} \in X$ such that*

$$\nabla g(\bar{x})(\bar{u}) \in \operatorname{int} T_B\big(-Q, g(\bar{x})\big). \tag{3}$$

*Then*

$$T_B(M, \bar{x}) = \big\{u \in X \;\big|\; \nabla g(\bar{x})(u) \in T_B(-Q, \bar{x})\big\}.$$

**Proof.** Let us first prove that one has

$$\big\{u \in X \;\big|\; \nabla g(\bar{x})(u) \in T_B(-Q, \bar{x})\big\} \subset T_B(M, \bar{x})$$

under the qualification condition (3). First, let us consider an element $u \in X$ such that

$$\nabla g(\bar{x})(u) \in \operatorname{int} T_B\big(-Q, g(\bar{x})\big).$$

Since $-Q$ is convex and $\operatorname{int}(-Q) \neq \emptyset$ we have from Proposition 4.2.3 of Aubin and Frankowska [2]

$$\operatorname{int} T_B\big(-Q, g(\bar{x})\big) = \bigcup_{\mu > 0} \left(\frac{\operatorname{int}(-Q) - g(\bar{x})}{\mu}\right).$$

This shows that there exists $\bar{\mu} > 0$ such that

$$g(\bar{x}) + \bar{\mu} \nabla g(\bar{x})(u) \in \operatorname{int}(-Q).$$

Now consider a sequence $\lambda_n > 0$ such that $\lambda_n \downarrow 0$. Thus we have

$$g(\bar{x} + \lambda_n \bar{\mu} u) = g(\bar{x}) + \lambda_n \bar{\mu} \nabla g(\bar{x})(u) + \lambda_n \bar{\mu} \|u\| v_n,$$

where $v_n \in Z$ and $v_n \to 0$ when $n \to \infty$. Therefore, we have

$$g(\bar{x} + \lambda_n \bar{\mu} u) = (1 - \lambda_n) g(\bar{x}) + \lambda_n \big(g(\bar{x}) + \bar{\mu} \nabla g(\bar{x})(u) + \bar{\mu} \|u\| v_n\big).$$

For $n$ sufficiently large one can show that

$$g(\bar{x} + t_n u) \in \operatorname{int}(-Q) \subset -Q,$$

where $t_n = \lambda_n \bar{\mu}$. Thus, it is clear that $t_n \downarrow 0$ and $\bar{x} + t_n u \in M$ and hence $u \in T_B(M, \bar{x})$.

Now consider the case

$$\nabla g(\bar{x})(u) \in T_B\big(-Q, g(\bar{x})\big).$$

We take a point $\bar{u}$ which satisfies (3). Consider now the sequence (for $n \geqslant 2$)

$$u_n = \frac{1}{n}\bar{u} + \left(1 - \frac{1}{n}\right) u.$$

It is clear that $u_n \to u$. Now observe that

$$\nabla g(\bar{x})(u_n) = \frac{1}{n} \nabla g(\bar{x})(\bar{u}) + \left(1 - \frac{1}{n}\right) \nabla g(\bar{x})(u).$$

This shows that

$$\nabla g(\bar{x})(u_n) \in \frac{1}{n} \operatorname{int} T_B\big(-Q, g(\bar{x})\big) + \left(1 - \frac{1}{n}\right) T_B\big(-Q, g(\bar{x})\big).$$

Since $Q$ is a convex cone we have that $T_B(-Q, g(\bar{x}))$ is a convex cone and hence

$$\nabla g(\bar{x})(u_n) \in \operatorname{int} T_B(-Q, \bar{x}).$$

Therefore, $u_n \in T_B(M, \bar{x})$ for all $n \geqslant 2$. Since $T_B(M, \bar{x})$ is closed we have $u \in T_B(M, \bar{x})$. This proves the required inclusion.

For the opposite inclusion we proceed as follows. From [2, p. 124, Table 4.1, item (4)] we have

$$\nabla g(\bar{x})\big(T_B\big(g^{-1}(-Q), \bar{x}\big)\big) \subset T_B\big(-Q, g(\bar{x})\big).$$

Since $g^{-1}(-Q) = M$, we have

$$T_B(M, \bar{x}) \subset \big\{u \in X \;\big|\; \nabla g(\bar{x})(u) \in T_B\big(-Q, g(\bar{x})\big)\big\}$$

and this yields the result. $\square$

**Remark 3.1.** It is simple to observe that if $g(\bar{x}) = 0$ in the above proposition, the conclusion of Proposition 3.2 follows immediately by noting the fact that

$$T_B(-Q, 0) = -Q.$$

The conclusion of the above proposition also holds under the *Robinson constraint qualification*. The Robinson constraint qualification requires that

$$0 \in -\operatorname{int}\{g(\bar{x}) + \nabla g(\bar{x})(X) + Q\},$$

where $\nabla g(\bar{x})(X)$ denotes the image of $\nabla g(\bar{x})$. This has been demonstrated for example in Bonnans and Shapiro [4]. Observe that (3) reduces to the (MF) condition when $g(\bar{x}) = 0$.

Putting together Propositions 3.1 and 3.2 one gets the next corollary.

**Corollary 3.1.** *Assume $(A_K)$, $(A_Q)$, $(A_f)$ and $(A_g)$. If $\bar{x} \in M$ is a weakly minimal point for the problem $(\mathcal{P})$ and there exists $\bar{u} \in X$ s.t. $\nabla g(\bar{x})(\bar{u}) \in -\operatorname{int} Q$, then $\nabla f(\bar{x})(u) \notin -\operatorname{int} K$ for every $u \in X$ with $\nabla g(\bar{x})(u) \in -Q$.*

We present now an existence result for the Lagrange multipliers in our case. A corresponding result, in a finite dimensional setting, is shown by Jahn [13, Theorem 7.8].

**Proposition 3.4.** *Assume $(A_K)$, $(A_Q)$, $(A_f)$ and $(A_g)$. Let $\bar{x} \in M$ be a weakly minimal point for the problem $(\mathcal{P})$ s.t. the condition $(MF)$ holds at $\bar{x}$. Then there exists $y^* \in K^* \setminus \{0\}$ and $z^* \in Q^*$ s.t.*

$$y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) = 0. \tag{4}$$

**Proof.** From Corollary 3.1 one has that

$$\big(\nabla f(\bar{x})(u), \nabla g(\bar{x})(u)\big) \notin -\operatorname{int} K \times -Q$$

for every $u \in X$. Since the set $\{(\nabla f(\bar{x})(u), \nabla g(\bar{x})(u)) \mid u \in X\} \subset Y \times Z$ is convex one can use a standard convex separation theorem to obtain a pair $(y^*, z^*) \in (Y^* \times Z^*) \setminus \{(0,0)\}$ s.t.

$$\big(y^* \circ \nabla f(\bar{x})\big)(u) + \big(z^* \circ \nabla g(\bar{x})\big)(u) \geqslant y^*(k) + z^*(q)$$

for every $u \in X$, $k \in -K$ and $q \in -Q$. Now, by some standard arguments, we have that $y^* \in K^*$, $z^* \in Q^*$ and

$$\big(y^* \circ \nabla f(\bar{x})\big)(u) + \big(z^* \circ \nabla g(\bar{x})\big)(u) \geqslant 0$$

for every $u \in X$. Then, by the linearity of the involved functionals, one gets

$$\big(y^* \circ \nabla f(\bar{x})\big)(u) + \big(z^* \circ \nabla g(\bar{x})\big)(u) = 0$$

for every $u \in X$. Suppose by contradiction that $y^* = 0$; then $z^* \neq 0$ and $(z^* \circ \nabla g(\bar{x}))(\bar{u}) = 0$. But this is a contradiction, because $z^* \in Q^*$ and $\nabla g(\bar{x})(\bar{u}) \in -\operatorname{int} Q$, so $(z^* \circ \nabla g(\bar{x}))(\bar{u}) < 0$. The proof is complete. □

In the above result, fix $y^* \in K^* \setminus \{0\}$ and consider the set

$$L_{y^*} := \{z^* \in Q^* \mid (4) \text{ holds}\}.$$

We have seen in Proposition 3.4 that condition $(MF)$ ensures the nonemptiness of $L_{y^*}$ for at least one $y^* \in K^* \setminus \{0\}$. We ask when this set is bounded. In order to give an appropriate (positive) answer to this question, let us observe that condition $(MF)$ is equivalent to the following *regularity condition* $(RC)$:

$$z^* \in Q^*,\ z^* \circ \nabla g(\bar{x}) = 0 \quad \Rightarrow \quad z^* = 0.$$

Indeed, suppose first that $(MF)$ holds and $(RC)$ does not. Then there exists $\bar{u} \in X$ s.t. $\nabla g(\bar{x})(\bar{u}) \in -\operatorname{int} Q$, and hence $z^* \in Q^* \setminus \{0\}$ with $z^* \circ \nabla g(\bar{x}) = 0$. This implies $z^*(\nabla g(\bar{x})(\bar{u})) = 0$, but, on the other hand $z^*(\nabla g(\bar{x})(\bar{u})) < 0$, hence a contradiction. Suppose now that $(RC)$ holds and $(MF)$ does not. This is $\nabla g(\bar{x})(X) \cap -\operatorname{int} Q = \emptyset$. It is clear that the set $\nabla g(\bar{x})(X)$ is convex, so from a separation theorem, there exists $z^* \in Q^* \setminus \{0\}$ with $z^*(\nabla g(\bar{x})(u)) \geqslant 0$ for every $u \in X$. The linearity of $z^* \circ \nabla g(\bar{x})$ implies that $z^*(\nabla g(\bar{x})(u)) = 0$ for every $u \in X$ and this yields (by (ii)) $z^* = 0$, a contradiction.

In order to prove our result concerning the boundedness of $L_{y^*}$ one needs the following assumption on the space $Z$:

$(A_Z)$ The closed unit ball of $Z^*$ is $w^*$-sequentially compact.

Note that this assumption holds if $Z$ is a subset of a weakly compactly generated space (see [6,11]). In particular, $(A_Z)$ holds if $Z$ is a separable Banach space or a reflexive Banach space.

**Remark 3.2.** We show the following theorem assuming (among other things) $(A_Z)$ and that $Q$ is dually compact. In Example 2.1 a dually compact cone (with empty interior) in $X = l_2$ is given such that $(A_Z)$ is fulfilled. However, it would be possible to replace in Theorem 3.1 the assumption that $Q$ is dually compact by $(A_Q)$.

**Theorem 3.1.** *Assume $(A_Z)$, $(A_f)$ and $(A_g)$. Furthermore, suppose that $Q$ is dually compact. Let $y^* \in K^* \setminus \{0\}$. If the regularity condition (RC) holds, then the set $L_{y^*}$ is (norm) bounded. Conversely, if $L_{y^*}$ is nonempty and bounded then (RC) holds.*

**Proof.** Suppose, again by contradiction, that $L_{y^*}$ is unbounded. Then, one can find a sequence $(z_n^*) \subset L_{y^*}$ with $\|z_n^*\| \to \infty$. By normalization, one has

$$\big(\|z_n^*\|^{-1} y^* \circ \nabla f(\bar{x})\big)(u) + \big(\|z_n^*\|^{-1} z_n^* \circ \nabla g(\bar{x})\big)(u) = 0 \tag{5}$$

for every $u \in X$. It is clear that the sequence $(\|z_n^*\|^{-1} z_n^*) \subset Q^*$ is bounded, whence, by $(A_Z)$, it contains a $w^*$-convergent subsequence to some $z^* \in Q^*$. Without relabeling, one can write $\|z_n^*\|^{-1} z_n^* \xrightarrow{w^*} z^*$. Since $Q$ is dually compact, one gets that $z^* \neq 0$ (contrary, $\|z_n^*\|^{-1} z_n^* \to 0$ in norm and this is not possible because $\|\|z_n^*\|^{-1} z_n^*\| = 1$ for every $n$). On the other hand, $\|z_n^*\|^{-1} y^* \to 0$. For every fixed $u \in X$ one passes to the weak* limit in (5) and, consequently,

$$\big(z^* \circ \nabla g(\bar{x})\big)(u) = 0$$

for every $u \in X$, i.e., $z^* \circ \nabla g(\bar{x}) = 0$. Since $z^* \neq 0$, this contradicts the regularity condition (RC), hence the conclusion follows. For the opposite, let us consider $\bar{z}^* \in L_{y^*}$ and suppose that there exists $z^* \in Q^* \setminus \{0\}$ with $z^* \circ \nabla g(\bar{x}) = 0$. Then is clear that $\bar{z}^* + nz^* \in L_{y^*}$ for every $n \in \mathbb{N}$, so $L_{y^*}$ is unbounded. This is again a contradiction and the result is proved. □

**Remark 3.3.** Observe that the set $L_{y^*}$ is always convex, so under (MF), it is weakly* compact.

We can summarize the results of this section in a corollary.

**Corollary 3.2.** *Assume $(A_Z)$, $(A_K)$, $(A_Q)$, $(A_f)$ and $(A_g)$. Suppose that $\bar{x} \in M$ is a weakly minimal point for the problem $(\mathcal{P})$. The following assumptions are equivalent*:

(i) (MF) *holds*;
(ii) (RC) *holds*;
(iii) *there exists $y^* \in K^* \setminus \{0\}$ s.t. $L_{y^*}$ is nonempty and bounded.*

Similar conditions can be obtained (along the lines of scalar theory) for problems with mixed constraints, i.e., problems having some additional constraints given as $h(x) = 0$, where $h$ is a smooth function acting between $X$ and another normed vector space $W$. Of course, in this case one needs a constraint qualification condition with respect to $h$ as the surjectivity of its derivative and, moreover (for the boundedness of the set of Lagrange multipliers) one needs the space $W$ to be finite dimensional. We briefly consider this situation.

Let $f : X \to Y$ and $g : X \to Z$, $h : X \to W$ be functions and consider the problem:

$$(\mathcal{P}') \quad \text{minimize } f(x) \quad \text{s.t. } g(x) \in -Q, \ h(x) = 0.$$

We denote by $M'$ the set of feasible points of the problem $(\mathcal{P}')$, i.e.,

$$M' := \big\{x \in X \mid g(x) \in -Q, \ h(x) = 0\big\}.$$

The Proposition 3.1 must be considered with $M'$ instead of $M$.

Suppose

$(A_h)$ The function $h$ is continuously Fréchet differentiable on $X$.

The (MF) condition for this case (referred as $(MF)'$) is that there exists an $\bar{u} \in X$ with $\nabla g(\bar{x})(\bar{u}) \in -\operatorname{int} Q$ and $\nabla h(\bar{x})(\bar{u}) = 0$. We follow a correspondence with the results concerning problem $(\mathcal{P})$ and we only point out the differences in the proofs.

**Proposition 3.5.** *Assume $(A_Q)$, $(A_g)$ and $(A_h)$. Suppose that $g(\bar{x}) \in -Q$ and $(MF)'$ holds at $\bar{x}$ and the linear operator $\nabla h(\bar{x})$ is surjective. Then,*

$$\big\{u \in X \mid \nabla g(\bar{x})(u) \in -Q, \ \nabla h(\bar{x})(u) = 0\big\} \subset T_B(M', \bar{x}).$$

**Proof.** First, let $u \in X$ s.t. $\nabla g(\bar{x})(u) \in -\operatorname{int} Q$, $\nabla h(\bar{x})(u) = 0$. Since $\nabla h(\bar{x})$ is surjective there exist $(t_n) \downarrow 0$ and $(u_n) \to 0$ s.t. $h(\bar{x} + t_n u_n) = 0$ (apply [2, Theorem 4.3.3]). The differentiability of $g$ ensures the existence of a sequence $(\alpha_n) \subset Z$, $\alpha_n \to 0$ with

$$g(\bar{x}+t_nu_n) = g(\bar{x}) + t_n\nabla g(\bar{x})(u_n) + t_n\|u_n\|\alpha_n$$
$$= g(\bar{x}) + t_n\big(\nabla g(\bar{x})(u_n) + \|u_n\|\alpha_n\big) \in -Q - \operatorname{int} Q \subset -Q$$

for $n$ large enough. This shows that $g(\bar{x}+t_nu_n) \in -Q$ for large $n$, hence $u \in T_B(M', \bar{x})$.

For the general case of $u \in X$ with $\nabla g(\bar{x})(u) \in -Q$, $\nabla h(\bar{x})(u) = 0$ we proceed as in the proof of Proposition 3.1 using $(MF)'$ instead of $(MF)$. □

**Corollary 3.3.** *Assume $(A_K)$, $(A_Q)$, $(A_f)$, $(A_g)$ and $(A_h)$. If $\bar{x} \in M$ is a weakly minimal point for the problem $(\mathcal{P}')$ such that $(MF)'$ holds at $\bar{x}$ and $\nabla h(\bar{x})$ is surjective then $\nabla f(\bar{x})(u) \notin -\operatorname{int} K$ for every $u \in X$ with $\nabla g(\bar{x})(u) \in -Q$, $\nabla h(\bar{x})(u) = 0$.*

The Lagrange multipliers condition is as below, cf. Jahn [13, Theorem 7.8], for the finite dimensional setting.

**Proposition 3.6.** *Assume $(A_K)$, $(A_Q)$, $(A_f)$, $(A_g)$ and $(A_h)$. Let $\bar{x} \in M$ be a weakly minimal point for the problem $(\mathcal{P}')$ such that the condition $(MF)'$ holds at $\bar{x}$ and $\nabla h(\bar{x})$ is surjective. Then there exists $y^* \in K^* \setminus \{0\}$, $z^* \in Q^*$, $p^* \in W^*$ s.t.*

$$y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x}) = 0. \tag{6}$$

**Proof.** One can proceed as above because

$$\big(\nabla f(\bar{x})(u), \nabla g(\bar{x})(u), \nabla h(\bar{x})(u)\big) \notin -\operatorname{int} K \times -Q \times \{0\}$$

for every $u \in X$. □

Again, in the above result fix $y^* \in K^* \setminus \{0\}$ and consider the set

$$L'_{y^*} := \big\{(z^*, p^*) \in Q^* \times W^* \;\big|\; (6) \text{ holds}\big\}.$$

It is easy to observe that, under the surjectivity of $\nabla h(\bar{x})$, condition $(MF)'$ is equivalent to the following *regularity condition $(RC)'$*:

$$z^* \in Q^*,\ p^* \in W^*,\ z^* \circ \nabla g(\bar{x}) + p^* \circ \nabla h(\bar{x}) = 0 \quad \Rightarrow \quad z^* = 0,\ p^* = 0.$$

**Theorem 3.2.** *Assume $(A_Z)$, $(A_f)$, $(A_g)$ and $(A_h)$. Furthermore, suppose that $Q$ is dually compact. Let $y^* \in K^* \setminus \{0\}$. Suppose that $W$ is finite dimensional and $\nabla h(\bar{x})$ is surjective. If the regularity condition $(RC)'$ holds at $\bar{x}$ then the set $L'_{y^*}$ is (norm) bounded. Conversely, if $L'_{y^*}$ is nonempty and bounded, then $(RC)'$ holds.*

**Proof.** The proof is as in Theorem 3.1 because the cone $Q \times \{0\} \subset Z \times W$ is obviously dually compact. □

## 4. The case of nonsmooth data

In this section we pass to the case where the data are not necessarily smooth and even not necessarily single-valued. We shall discuss the problems $(\mathcal{P}_1)$, $(\mathcal{P}_2')$ and $(\mathcal{P}_2)$. Further, we will be only concerned with Pareto minima of these classes of problems. Moreover, we will not make any assumption on the interior of the ordering cone $K$. In fact, the interior of the ordering cone can be empty. The details of the solution concepts associated with these problems are given in Section 2. The optimality conditions and the results associated with the boundedness of the set of Lagrange multipliers depend on an important construction associated with the set-valued map—the coderivative of Mordukhovich. In the infinite dimensional setting there are two notions of coderivative, namely the *mixed coderivative* and the *normal coderivative* whose definition we present below. However, before defining the coderivatives it is important to state the various notions of normal cones that we use in this section, since the coderivatives in turn depend on the notion of normal cones to non-convex sets. The definitions that appear below are give in the comprehensive monograph of Mordukhovich [17] but we present them here mainly for the sake of completeness.

**Definition 4.1.** Let $X$ be an Asplund space and $S \subset X$ be a nonempty closed subset of $X$ and let $x \in S$. The limiting normal cone to $S$ at $x$ is

$$N(S,x) := \big\{x^* \in X^* \;\big|\; \exists x_n \xrightarrow{S} x,\ x_n^* \xrightarrow{w^*} x^*,\ x_n^* \in N_F(S,x_n)\big\},$$

where $N_F(S,z)$ denotes the Fréchet normal cone to $S$ at a point $z \in S$, given as

$$N_F(S,z) := \left\{x \in X^* \;\middle|\; \limsup_{u \in S,\, u \to z} \frac{x^*(u-z)}{\|u-z\|} \leqslant 0\right\}.$$

It is important to note that the Fréchet normal cone is always convex and closed in the strong topology in $X^*$ and hence it is *weak*$^*$-closed in $X^*$ if $X$ is reflexive. Further, the limiting normal cone needs not to be closed in the norm topology even in the setting of a Hilbert space. See [17] for details.

In order to define the two coderivative notions of Mordukhovich (see, for example [17]) one needs to define another fundamental construction, namely the Fréchet coderivative. We would like to stress that all the spaces involved in this section are Asplund spaces.

**Definition 4.2.** Let $F : X \rightrightarrows Y$ be a set-valued map. Then the Fréchet coderivative of $F$ at $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is defined as the multifunction $\hat{D}^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ such that

$$\hat{D}^* F(\bar{x}, \bar{y})(y^*) = \bigl\{x^* \in X^* \bigm| (x^*, -y^*) \in N_F\bigl(\operatorname{Gr} F, (\bar{x}, \bar{y})\bigr)\bigr\}.$$

**Definition 4.3.** Let $F : X \rightrightarrows Y$ be a set-valued map. Then the mixed coderivative of $F$ at $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is a set-valued map $D^*_M : Y^* \rightrightarrows X^*$ given as

$$D^*_M F(\bar{x}, \bar{y})\bigl(\bar{y}^*\bigr) = \limsup_{(x,y)\to(\bar{x},\bar{y}),\ y^*\to\bar{y}^*} \hat{D}^* F(x, y)(y^*).$$

This means that the mixed coderivative is a collection of $\bar{x}^* \in X^*$ such that there are sequences $(x_k, y_k, y^*_k) \to (\bar{x}, \bar{y}, \bar{y}^*)$ and $x^*_k \xrightarrow{w^*} \bar{x}^*$ with $(x_k, y_k) \in \operatorname{Gr} F$ and $x^*_k \in \hat{D}^* F(x, y)(y^*_k)$.

**Definition 4.4.** Let $F : X \rightrightarrows Y$ be a set-valued map. Then the normal coderivative of $F$ at $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is a set-valued map $D^*_N : Y^* \rightrightarrows X^*$ given as

$$D^*_N F(\bar{x}, \bar{y})(\bar{y}^*) = \limsup_{(x,y)\to(\bar{x},\bar{y}),\ y^* \xrightarrow{w^*} \bar{y}^*} \hat{D}^* F(x, y)(y^*).$$

This means that the normal coderivative is a collection of $\bar{x}^* \in X^*$ such that there are sequences $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $(x^*_k, y^*_k) \xrightarrow{w^*} (\bar{x}^*, \bar{y}^*)$ with $(x_k, y_k) \in \operatorname{Gr} F$ and $x^*_k \in \hat{D}^* F(x, y)(y^*_k)$. The normal coderivative can also be equivalently represented as

$$D^*_N F(x, y)(y^*) := \bigl\{x^* \in X^* \bigm| (x^*, -y^*) \in N\bigl(\operatorname{Gr} F, (x, y)\bigr)\bigr\}.$$

When $F : X \to Y$ is a single-valued mapping then we denote the mixed and normal coderivatives as $D^*_M F(\bar{x})(y^*)$ and $D^*_N F(\bar{x})(y^*)$, respectively. Observe that the major difference in the definition of the mixed coderivative and normal coderivative is in the manner in which $y^*_k$ converges to $\bar{y}^*$. In case of the mixed derivative the convergence is in the norm topology, while in the case of the normal coderivative the convergence is in the *weak*$^*$-star topology. It is clear that if $X$ and $Y$ are finite dimensional spaces both notions of coderivative coincide. Moreover, all the three coderivatives mentioned above are positively homogenous set-valued maps. A set-valued map $F : X \rightrightarrows Y$ is said to be positively homogeneous if $\alpha F(x) \subset F(\alpha x)$, with $\alpha > 0$. Further, it is clear that for any $y^* \in Y^*$ one has

$$\hat{D} F(\bar{x}, \bar{y})(y^*) \subset D^*_M F(\bar{x}, \bar{y})(y^*) \subset D^*_N F(\bar{x}, \bar{y})(y^*). \tag{7}$$

If $f : X \to Y$ is continuously Fréchet differentiable at $\bar{x}$ then one has from Theorem 1.38 in Mordukhovich [17] the following fact:

$$D^*_M f(\bar{x})(y^*) = D^*_N f(\bar{x})(y^*) = \bigl\{\nabla f(\bar{x})^* y^*\bigr\}, \quad \forall y^* \in Y^*, \tag{8}$$

where $\nabla f(\bar{x})^*$ denotes the adjoint operator associated with $\nabla f(\bar{x})$. However, keeping in view of the notations already used in Section 3 we can express $\nabla f(\bar{x})^* y^*$ equivalently as $y^* \circ \nabla f(\bar{x})$, i.e., $\nabla f(\bar{x})^* y^* = y^* \circ \nabla f(\bar{x})$.

There are two-regularity notions which will play an important role in the sequel. A set-valued map $F : X \rightrightarrows Y$ is called $N$-regular at $(\bar{x}, \bar{y})$ if

$$D^*_N F(\bar{x}, \bar{y}) = \hat{D}^* F(\bar{x}, \bar{y}),$$

and $F$ is called $M$-regular at $(\bar{x}, \bar{y})$ if

$$D^*_M F(\bar{x}, \bar{y}) = \hat{D}^* F(\bar{x}, \bar{y}).$$

It is clear from (7) that if $F$ is $N$-regular then it is also $M$-regular though the converse may not hold. If $F$ is $N$-regular, then one has

$$\hat{D} F(\bar{x}, \bar{y})(y^*) = D^*_M F(\bar{x}, \bar{y})(y^*) = D^*_N F(\bar{x}, \bar{y})(y^*). \tag{9}$$

Further, $N$-regularity can also be equivalently stated as

$$N\bigl(\operatorname{Gr} F, (\bar{x}, \bar{y})\bigr) = \hat{N}\bigl(\operatorname{Gr} F, (\bar{x}, \bar{y})\bigr).$$

It is well known that locally Lipschitz functions play a major role in optimization theory, see for example Clarke [5]. Thus it is natural expect a generalization of the notion of locally Lipschitz functions in the setting of set-valued mappings. In fact there are two concepts, namely, the pseudo-Lipschitz set-valued maps and Lipschitz set-valued maps. We present the definition of these two well-known concepts.

**Definition 4.5.** A set-valued map $F : X \rightrightarrows Y$ is called pseudo-Lipschitz at a point $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ if there exist a positive number $L > 0$ and some neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ s.t. for every $x, u \in U$ one has:

$$F(x) \cap V \subset F(u) + L\|x - u\| U_Y.$$

When $V = Y$ then $F$ is said to be a Lipschitz set-valued map relative to the set $U$. When $F$ is single-valued map and Lipschitz in the above sense then we say that $f$ is locally Lipschitz at $\bar{x}$ or Lipschitz around $\bar{x}$.

We shall now provide an interesting characterization of the pseudo-Lipschitz property of a set-valued map $F$ in terms of its normal coderivative through the following lemma which play a pivotal role in the sequel. The simple proof of this lemma is a direct application of Theorem 1.38 in Mordukhovich [17] and $N$-regularity.

**Lemma 4.1.** *Let $F : X \rightrightarrows Y$ be a set-valued map and let $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. Let $F$ be pseudo-Lipschitz and $N$-regular, then*

$$D_N^* F(\bar{x}, \bar{y})(0) = \{0\}.$$

Now, we present some robust optimality conditions in terms of the normal coderivative for $(\mathcal{P}_1)$ and $(\mathcal{P}_2')$.

**Theorem 4.1.** *(See [19, Corollary 4.1] and see also [8, Corollary 3.4].) Suppose that $X, Y$ are Asplund spaces, $(\bar{x}, \bar{y})$ is a Pareto minimum point for $(\mathcal{P}_1)$ and $F$ is pseudo-Lipschitz at $(\bar{x}, \bar{y})$. If $K$ is dually compact, then there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t.*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + N(S, \bar{x}).$$

Since in finite dimensions every closed convex cone is dually compact, one has the following result.

**Corollary 4.1.** *Suppose that $X$ and $Y$ are finite dimensional spaces, $(\bar{x}, \bar{y})$ is a Pareto minimum point for $(\mathcal{P}_1)$ and $F$ is pseudo-Lipschitz at $(\bar{x}, \bar{y})$. Then there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t.*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + N(S, \bar{x}).$$

Now, we shall focus our attention on the problem $(\mathcal{P}_2')$.

**Theorem 4.2.** *(See [19, Theorem 4.3].) Suppose that $X, Y, Z$ are Asplund spaces and $(\bar{x}, \bar{y})$ is a Pareto minimum point for $(\mathcal{P}_2')$. If $F$ is pseudo-Lipschitz at $(\bar{x}, \bar{y})$ and $g$ is locally Lipschitz, and both $K$ and $Q$ are dually compact, then there exist $y^* \in K^*$ and $z^* \in Q^*$ with $\|y^*\| + \|z^*\| = 1$ s.t.*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + D_N^* g(\bar{x})(z^*). \tag{10}$$

Now, if $g$ is continuously Fréchet differentiable then using (8) and Theorem 4.2 we get the following proposition.

**Proposition 4.1.** *Suppose that $X, Y$ are Asplund spaces, $(\bar{x}, \bar{y})$ is a Pareto minimum point for $(\mathcal{P}_2')$, $F$ is pseudo-Lipschitz at $(\bar{x}, \bar{y})$ and $g$ is continuously Fréchet differentiable at $\bar{x}$. Let $K$ and $Q$ be dually compact. Then there exists $y^* \in K^*$ and $z^* \in Q^*$ with $\|y^*\| + \|z^*\| = 1$ s.t.*

$$-z^* \circ \nabla g(\bar{x}) \in D_N^* F(\bar{x}, \bar{y})(y^*). \tag{11}$$

The elements $y^*$and $z^*$ are the Lagrange multipliers for our problem. The question which arise in the context of our discussion is when the set

$$L_{y^*} = \{z^* \in Q^* \mid (11) \text{ holds}\}$$

is nonempty and bounded for an $y^* \in K^* \setminus \{0\}$. The answer is provided in the next result under some regularity assumptions on $F$.

**Theorem 4.3.** *Let $X, Y, Z$ be reflexive Banach spaces and $(\bar{x}, \bar{y})$ be a Pareto minimum point for $(\mathcal{P}_2')$. Suppose that $F$ is pseudo-Lipschitz and $N$-regular at $(\bar{x}, \bar{y})$, $g$ is continuously Fréchet differentiable at $\bar{x}$, $K$ and $Q$ are dually compact and the following constraint qualification condition holds*:

$$z^* \in Q^*,\ z^* \circ \nabla g(\bar{x}) = 0 \quad \Rightarrow \quad z^* = 0. \tag{12}$$

*Then there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t. $L_{y^*}$ is nonempty and bounded.*

**Proof.** Proposition 4.1 ensures the existence of some $y^* \in K^*$, $z^* \in Q^*$ with $\|y^*\| + \|z^*\| = 1$ s.t. $-z^* \circ \nabla g(\bar{x}) \in D_N^* F(\bar{x}, \bar{y})(y^*)$. If $y^* = 0$, since $F$ is pseudo-Lipschitz and $N$-regular at $(\bar{x}, \bar{y})$ from Lemma 4.1 we have

$$D_N^* F(\bar{x}, \bar{y})(0) = \{0\},$$

whence $-z^* \circ \nabla g(\bar{x}) = 0$. The constraint qualification condition (12) implies $z^* = 0$, a contradiction. Therefore, $y^* \neq 0$ and $L_{y^*}$ is nonempty. Suppose, by contradiction, that there exists a sequence $(z_n^*) \subset L_{y^*}$ with $\|z_n^*\| \to +\infty$. Therefore, for every natural $n$,

$$-z_n^* \circ \nabla g(\bar{x}) \in D_N^* F(\bar{x}, \bar{y})(y^*),$$

hence

$$\left(-z_n^* \circ \nabla g(\bar{x}), -y^*\right) \in N\left(\operatorname{Gr} F, (\bar{x}, \bar{y})\right).$$

Consequently,

$$\left(\left\|z_n^*\right\|^{-1}\left(-z_n^* \circ \nabla g(\bar{x})\right), -\left\|z_n^*\right\|^{-1} y^*\right) \in N\left(\operatorname{Gr} F, (\bar{x}, \bar{y})\right).$$

It is clear that $-\|z_n^*\|^{-1} y^* \to 0$ while, using the dually compactness of $Q$,

$$\left\|z_n^*\right\|^{-1}\left(-z_n^* \circ \nabla g(\bar{x})\right) \xrightarrow{w^*} z^* \circ \nabla g(\bar{x})$$

with $z^* \in Q^* \setminus \{0\}$ (note that $(A_Z)$ holds, because $Z$ is a reflexive Banach space). On the other hand, the $N$-regularity of $F$ at $(\bar{x}, \bar{y})$ ensures the closedness (in the product topology of strong topologies) and the convexity of $N(\operatorname{Gr} F, (\bar{x}, \bar{y}))$. Since $X, Y$ are reflexive Banach spaces, $N(\operatorname{Gr} F, (\bar{x}, \bar{y}))$ is closed in $w^* \times w^*$ topology. So,

$$\left(-z^* \circ \nabla g(\bar{x}), 0\right) \in N\left(\operatorname{Gr} F, (\bar{x}, \bar{y})\right),$$

i.e.

$$-z^* \circ \nabla g(\bar{x}) \in D_N^* F(\bar{x}, \bar{y})(0).$$

Again, since $F$ is pseudo-Lipschitz and $N$-regular at $(\bar{x}, \bar{y})$, Lemma 4.1 implies

$$D_N^* F(\bar{x}, \bar{y})(0) = \{0\},$$

whence

$$-z^* \circ \nabla g(\bar{x}) = 0.$$

In our assumptions, this would imply $z^* = 0$, which is a contradiction. The proposition is proved. □

We make use of Theorem 4.2 in order to study the problem of the nonemptiness and boundedness of the set $L'_{y^*} := \{z^* \in Q^* \mid (10) \text{ holds}\}$ for an element $y^* \in K^*$, $\|y^*\| = 1$.

**Theorem 4.4.** *Let $X, Y, Z$ be reflexive Banach spaces and $(\bar{x}, \bar{y})$ be a Pareto minimum point for $(\mathcal{P}_2')$. Suppose that $F$ is pseudo-Lipschitz and $N$-regular at $(\bar{x}, \bar{y})$ and $g$ is locally Lipschitz at $\bar{x}$ and $N$ - regular at $(\bar{x}, g(\bar{x}))$. If both $K$ and $Q$ are dually compact and the following constraint qualification condition holds*

$$z^* \in Q^*,\ 0 \in D_N^* g(\bar{x})(z^*) \quad \Rightarrow \quad z^* = 0 \tag{13}$$

*then there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t. $L'_{y^*}$ is nonempty and bounded.*

**Proof.** Theorem 4.2 ensures the existence of some $y^* \in K^*$ and $z^* \in Q^*$ with $\|y^*\| + \|z^*\| = 1$ s.t.

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + D_N^* g(\bar{x})(z^*).$$

Now suppose that $y^* = 0$, hence $z^* \neq 0$. Then since $F$ is pseudo-Lipschitz and $N$-regular at $(\bar{x}, \bar{y})$, one has that $D_N^* F(\bar{x}, \bar{y})(y^*) = \{0\}$. Accordingly, relation (10) can be written down as

$$0 \in D_N^* g(\bar{x})(z^*)$$

and in our constraint qualification condition we get $z^* = 0$, a contradiction. We conclude that there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t. $L'_{y^*}$ is nonempty. Suppose, by contradiction, that there exists a sequence $(z_n^*) \subset L'_{y^*}$ with $\|z_n^*\| \to +\infty$. Therefore, for every natural $n$ one has:

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + D_N^* g(\bar{x})(z_n^*).$$

Since the normal coderivative is a positively homogenous set-valued map we have

$$0 \in D_N^* F(\bar{x}, \bar{y})\left(\left\|z_n^*\right\|^{-1} y^*\right) + D_N^* g(\bar{x})\left(\left\|z_n^*\right\|^{-1} z_n^*\right).$$

Therefore, there exists $(u_n^*) \in X^*$ s.t., for every $n$,

$$u_n^* \in D_N^* F(\bar{x}, \bar{y})\big(\|z_n^*\|^{-1} y^*\big); \qquad -u_n^* \in D_N^* g(\bar{x})\big(\|z_n^*\|^{-1} z_n^*\big).$$

This means,

$$\big(u_n^*, -\|z_n^*\|^{-1} y^*\big) \in N\big(\operatorname{Gr} F, (\bar{x}, \bar{y})\big); \qquad \big(-u_n^*, \|z_n^*\|^{-1} z_n^*\big) \in N\big(\operatorname{Gr} g, \big(\bar{x}, g(\bar{x})\big)\big).$$

Of course, the sequences $(\|z_n^*\|^{-1} y^*)$ and $(\|z_n^*\|^{-1} z_n^*)$ are bounded and pseudo-Lipschitzianity of $F$ ensure that $(u_n^*)$ is bounded too. So, we can suppose that $u_n^* \xrightarrow{w^*} u \in X^*$. Dual compactness of $Q$ allows us to write that $\|z_n^*\|^{-1} z_n^* \xrightarrow{w^*} z^* \in Q^* \setminus \{0\}$. Clearly, $(\|z_n^*\|^{-1} y^*) \to 0$. The same argument as in the proof of the preceding result gives

$$(u^*, 0) \in N\big(\operatorname{Gr} F, (\bar{x}, \bar{y})\big); \qquad (-u^*, z^*) \in N\big(\operatorname{Gr} g, \big(\bar{x}, g(\bar{x})\big)\big),$$

i.e.

$$0 \in D_N^* F(\bar{x}, \bar{y})(0) + D_N^* g(\bar{x})(z^*).$$

The final argument is as above: $D_N^* F(\bar{x}, \bar{y})(0)$ consists only of 0 and then $0 \in D_N^* g(\bar{x})(z^*)$ which implies $z^* = 0$. This is a contradiction and the thesis is proved. □

We will now focus on the problem $(\mathcal{P}_2)$. Let us note that the problem $(\mathcal{P}_2)$ is the most studied problem in the literature on set-valued optimization. In order to make the exposition simple and in order to reduce the technical complication that may arise in the setting of an Asplund space we shall present our results for the problem $(\mathcal{P}_2)$ in the finite dimensional setting. In particular, we shall assume that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and $Z = \mathbb{R}^k$. Thus in this setting in the problem $(\mathcal{P}_2)$ we have $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$. In Ng and Zheng [19] the main focus was the problem $(\mathcal{P}_2')$. Thus, the following result will be an improvement of the results presented in Ng and Zheng [19] at least in the finite dimensional setting. Let us begin by presenting the notion of inner-semicompactness of a set-valued map around a reference point. A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be inner-semicompact at $\bar{x}$ if for every sequence $x_k \to \bar{x}$ there is a sequence $y_k \in S(x_k)$ which has a convergent subsequence. For example if $S$ is uniformly bounded (local boundedness) at $\bar{x}$ then $S$ is inner-semicompact at $\bar{x}$. By uniform boundedness or local boundedness at $\bar{x}$ we mean that there exists a neighborhood $U$ of $\bar{x}$ and a positive number $M$ such that for all $x \in U$ we have $\|y\| \leqslant M$ for all $y \in S(x)$. Also we use in the following result the notion of the kernel of a set-valued map $S$ denoted by $\ker S$ which simply is the collection of those elements $x \in \mathbb{R}^n$ such that $0 \in S(x)$.

**Theorem 4.5.** *Consider the problem $(\mathcal{P}_2)$ and let $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ be a Pareto minimum of $(\mathcal{P}_2)$. Assume that $F$ is pseudo-Lipschitz at $(\bar{x}, \bar{y})$ and $G(\bar{x})$ is compact. Further assume that the mapping $x \mapsto G(x) \cap (-Q)$ is inner-semicompact at $\bar{x}$. Moreover, assume that the following qualification condition hold true: for any $y \in G(\bar{x}) \cap (-Q)$ one has*

$$N(-Q, y) \cap \ker D_N^* G(\bar{x}, y) = \{0\}. \tag{14}$$

*Then there exist $y^* \in K^*$ with $\|y^*\| = 1$, $\hat{y} \in G(\bar{x}) \cap (-Q)$ and $z^* \in N(-Q, \hat{y})$ such that*

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + D_N^* G(\bar{x}, \hat{y})(z^*). \tag{15}$$

**Proof.** Since $(\bar{x}, \bar{y})$ is a Pareto minimum and we are in a finite dimensional setting, from Corollary 4.1, we have that there exists $y^* \in K^*$ with $\|y^*\| = 1$ such that

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + N\big(G^{-1}(-Q), \bar{x}\big).$$

Thus, the proof would be complete if we compute $N(G^{-1}(-Q), \bar{x})$. This will be done by applying Theorem 3.8 in Mordukhovich [17]. The qualification conditions in the statement of the theorem imply the qualification conditions in Theorem 3.8 of [17]. Hence, from Theorem 3.8 in [17] we have

$$N\big(G^{-1}(-Q), \bar{x}\big) \subset \bigcup \big\{ D_N^* G(\bar{x}, y')(w^*) \;\big|\; w^* \in N(-Q, y'),\ \ y' \in G(\bar{x}) \cap (-Q) \big\}.$$

Hence there exist $\hat{y} \in F(\bar{x}) \cap (-Q)$ and $z^* \in N(-Q, \hat{y})$ such that

$$0 \in D_N^* F(\bar{x}, \bar{y})(y^*) + D_N^* G(\bar{x}, \hat{y})(z^*).$$

This proves the result. □

The vectors $y^*$ and $z^*$ are the Lagrange multipliers associated with the problem $(\mathcal{P}_2)$. Our aim is to see under what conditions the set of Lagrange multipliers for the problem $(\mathcal{P}_2)$ remains bounded. It suffices to study the following set for $y^* \in K^*$ with $\|y^*\| = 1$:

$$L_{y^*}^{\circ} = \big\{ z^* \in \mathbb{R}^k \;\big|\; \exists \hat{y} \in G(\bar{x}) \cap (-Q),\ \text{with } z^* \in N(-Q, \hat{y}) \text{ such that (15) holds} \big\}.$$

Of course the set $L^{\circ}_{y^*}$ do depend on the point $(\bar{x},\bar{y})$. It is easy to observe that the qualification condition in the above theorem can also be equivalently written as follows: for any $\hat{y}\in G(\bar{x})\cap(-Q)$ and $z^*\in N(-Q,\hat{y})$

$$0\in D^*_N G(\bar{x},\hat{y})(z^*) \quad\Rightarrow\quad z^*=0.$$

In the following result we state the condition under which the set $L^{\circ}_{y^*}$ is bounded.

**Theorem 4.6.** *Let us consider the problem $(\mathcal{P}_2)$. Let $(\bar{x},\bar{y})\in \operatorname{Gr} F$ be Pareto minimum for $(\mathcal{P}_2)$. Further assume that $F$ is pseudo-Lipschitz at $(\bar{x},\bar{y})$ and $G$ is pseudo-Lipschitz at each $(\bar{x},y)$ with $y\in G(\bar{x})$. Assume further that $G(\bar{x})$ is compact. Also assume that $x\mapsto G(x)\cap(-Q)$ is inner-semicompact and for any $y\in G(\bar{x})\cap(-Q)$ one has*

$$N(-Q,y)\cap\ker D^*_N G(\bar{x},y)=\{0\}. \tag{16}$$

*Then there exists $y^*\in K^*$ with $\|y^*\|=1$ s.t. $L^{\circ}_{y^*}$ is a nonempty and bounded set.*

**Proof.** By use of Theorem 4.5 there exists $y^*\in K^*$ with $\|y^*\|=1$ s.t. $L^{\circ}_{y^*}$ is a nonempty set. Further, let us assume that $L^{\circ}_{y^*}$ is not bounded. Thus there exists a sequence $\{z^*_n\}\subset L^{\circ}_{y^*}$ such that $\|z^*_n\|\to\infty$ as $n\to\infty$. Now from the definition of $L^{\circ}_{y^*}$ for each $n$ there exists $\hat{y}_n\in G(\bar{x})\cap(-Q)$ with $z^*_n\in N(-Q,\hat{y}_n)$ such that

$$0\in D^*_N F(\bar{x},\bar{y})(y^*)+D^*_N G(\bar{x},\hat{y}_n)\big(z^*_n\big).$$

Now consider the sequence $\{\frac{z^*_n}{\|z^*_n\|}\}$. It is clear that this sequence is bounded and has a cluster point, say $z^*$, with $\|z^*\|=1$. Now, noting the fact that the normal coderivative is a positively homogenous set-valued map, we have for each $n$

$$0\in D^*_N F(\bar{x},\bar{y})\left(\frac{y^*_n}{\|z^*_n\|}\right)+D^*_N G(\bar{x},\hat{y}_n)\left(\frac{z^*_n}{\|z^*_n\|}\right).$$

Hence, there exists an element $u^*\in\mathbb{R}^n$ such that

$$u^*_n\in D^*_N F(\bar{x},\bar{y})\left(\frac{y^*_n}{\|z^*_n\|}\right)$$

and

$$-u^*_n\in D^*_N G(\bar{x},\hat{y}_n)\left(\frac{z^*_n}{\|z^*\|}\right).$$

From the definition of the normal coderivative we have

$$\left(u^*_n,-\frac{y^*}{\|z^*_n\|}\right)\in N\big(\operatorname{Gr} F,(\bar{x},\bar{y})\big)$$

and

$$\left(-u^*_n,-\frac{z^*_n}{\|z^*_n\|}\right)\in N\big(\operatorname{Gr} G,(\bar{x},\hat{y}_n)\big).$$

It has been mentioned in Rockafellar and Wets [20] that in finite dimensions the pseudo-Lipschitz property is equivalent to the fact that the normal coderivative is locally bounded. Since $F$ is pseudo-Lipschitz, it is clear that $D^*_N F(\bar{x},\bar{y})$ is locally bounded. The sequence $\{\frac{y^*}{\|z^*_n\|}\}$ converges to zero and thus, using the local boundedness of $D^*_N F(\bar{x},\bar{y})$, we conclude that the sequence $\{u^*_n\}$ is bounded. Let $u^*$ be a cluster point of the sequence. Further, as $G(\bar{x})$ is compact and $-Q$ is closed it is clear that $G(\bar{x})\cap(-Q)$ is compact and hence the sequence $\{\hat{y}_n\}$ is bounded and thus have a cluster point, say, $\hat{y}$. Hence, as $n\to\infty$, we have

$$(u^*,0)\in N\big(\operatorname{Gr} F,(\bar{x},\bar{y})\big)$$

since the limiting normal cone is closed and we have

$$(-u^*,z^*)\in N\big(\operatorname{Gr} G,(\bar{x},\hat{y})\big)$$

since the limiting normal cone has a closed graph in finite dimensions. Thus, using the definition of the normal coderivative, we have

$$0\in D^*_N F(\bar{x},\bar{y})(0)+D^*_N G(\bar{x},\hat{y})(z^*). \tag{17}$$

Since $F$ is pseudo-Lipschitz and since the mixed coderivative and the normal coderivative coincide in finite dimensions we have (for example from Theorem 1.38 in [17] or Theorem 9.40 [20]) that

$$D^*_N F(\bar{x},\bar{y})(0)=\{0\}.$$

Hence, from (17) we get

$$0 \in D_N^* G(\bar{x}, \hat{y})(z^*),$$

thus we have

$$z^* \in \ker D_N^* G(\bar{x}, \hat{y}).$$

Further, since $z_n^* \in N(-Q, \hat{y}_n)$ we have that

$$\frac{z_n^*}{\|z_n^*\|} \in N(-Q, \hat{y}_n).$$

Again, since the limiting normal has a closed graph in finite dimensions we have that

$$z^* \in N(-Q, \hat{y}).$$

Moreover, we know that $\|z^*\| = 1$ and hence $z^* \neq 0$ and this contradicts (16). Hence the assertion is shown. □

Till now in this section we have concentrated on the fact that at least the objective function in our analysis above is always a set-valued map. However, we can consider the map $F$ above as single-valued map between the Asplund spaces $X$ and $Y$. In the single-valued scenario let us represent $F$ by $f$. Thus, for the usual nonsmooth-vector optimization problem we consider the problem $(\mathcal{P})$ (see Section 2), now with locally Lipschitz data. The necessary optimality conditions for the single-valued nonsmooth vector optimization problem $(\mathcal{P})$ can be easily derived from the results given in this section via the well-known scalarization formula (see Mordukhovich [17] for details). However, we shall state below the definition of the limiting subdifferential and scalarization rule. Since we are essentially interested in locally Lipschitz functions we shall mention the definition of the limiting subdifferential for a locally Lipschitz function. Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function where as before $X$ is an Asplund space. Then the limiting subdifferential or the Mordukhovich subdifferential of $f$ at $\bar{x} \in X$ is given as

$$\partial\phi(\bar{x}) = \{\xi \in X^* \mid (x^*, -1) \in N_{\operatorname{epi}\phi}(\bar{x}, f(\bar{x})\},$$

where $\operatorname{epi}\phi$ denotes the epigraph of $\phi$. Note that this subdifferential was firstly introduced in [18]. Let us now consider the function $f : X \to Y$, where $X$ and $Y$ are Asplund spaces. For the scalarization formula to be compatible with the normal coderivative one needs to assume that the function $f : X \to Y$ is strictly Lipschitzian at each point of $X$. We say that $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ if it is locally Lipschitz there and if there is a neighborhood $V$ of the origin in $X$ such that the sequence of the differential quotient

$$\frac{f(x_k + t_k v) - f(x_k)}{t_k}$$

contains a norm convergent subsequence whenever one has $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$. Further, it is important to note that if $Y$ is finite dimensional, then the notion of strict Lipschitzianess at $\bar{x}$ coincides with the notion of locally Lipschitzianess at $\bar{x}$. It has been shown in [17, Chapter 3] that if $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ then one has

$$D_N^* f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}), \quad \forall y^* \in Y^*.$$

It is clear now that by the application of the above scalarization formula one can deduce necessary optimality conditions and sufficient conditions for the boundedness of the set of Lagrange multipliers for the problem $(\mathcal{P})$ using the results from this section. Note that, since $f$ is single-valued, the notion of pseudo-Lipschitz continuity coincides with the usual notion of Lipschitz continuity. If $f$ is just locally Lipschitz one has

$$D_M^* f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}), \quad \forall y^* \in Y^*.$$

So, if $f$ is locally Lipschitz and $N$-regular, then

$$D_N^* f(\bar{x})(y^*) = D_M^* f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}), \quad \forall y^* \in Y^*. \tag{18}$$

## 5. Application

As an application we formulate a vector control approximation problem in general spaces and show the boundedness of the set of Lagrange multipliers of this problem applying Theorem 4.3. Suppose that $X, Y$ and $W$ are real Banach spaces; as in the preceding, $K \subset Y$ is a closed convex cone. We introduce a vector-valued norm (see [13]) $|||\cdot||| : W \to K$ that for all $w, w_1, w_2 \in W$ and for all $\lambda \in \mathbb{R}$ satisfies:

1. $|||w||| = 0 \iff w = 0$;
2. $|||\lambda w||| = |\lambda|\,|||w|||$;
3. $|||w_1 + w_2||| \in |||w_1||| + |||w_2||| - K$.

A subdifferential (denoted by $\partial^{\leqslant}$) for vector-valued functions was proposed by Jahn in [13] and for the particular case of above vector-valued norm $|||\cdot|||$ it has the following form:

$$\partial^{\leqslant}|||\cdot|||(w_0) = \left\{x^* \in L(W,Y) \,\middle|\, x^*(w_0) = |||w_0|||,\ |||w||| - x^*(w) \in K\ \forall w \in W\right\}, \tag{19}$$

where $L(W,Y)$ denotes the space of linear continuous operators from $W$ into $Y$.

Sufficient conditions for $\partial^{\leqslant}|||\cdot||| \neq \emptyset$ are given by Jahn [13] (for instance that $|||\cdot|||$ is continuous and $K$ has the Daniell property which means that every decreasing net (i.e., $i \leqslant j$ implies $x_j \leqslant x_i$) having a lower bound converges to its infimum).

Suppose now that $f : X \to Y$ is a locally Lipschitz function, $A_i \in L(X,W)$ and $\alpha_i \geqslant 0$ $(i = 1,\dots,n)$. Then we consider for $x \in X$ and $a^i \in W$ $(i = 1,\dots,n)$ the vector-valued function

$$h(x) := f(x) + \sum_{i=1}^{n} \alpha_i \left|\left|\left|A_i(x) - a^i\right|\right|\right|$$

and $g : X \to Z$, where $Z$ is a normed space and $Q$ is a closed convex cone in $Z$.

Now, we study the vector control approximation problem

$$(\mathcal{P}_{app}) \quad \min h(x), \quad \text{subject to } x \in X,\ g(x) \leqslant_Q 0.$$

**Theorem 5.1.** *Let $W, X, Y, Z$ be reflexive Banach spaces and let $\bar{x}$ be a Pareto minimum point for $(\mathcal{P}_{app})$. Suppose that $K$ and $Q$ are dually compact, $g$ is continuously Fréchet differentiable on $X$, $|||\cdot|||$ is locally Lipschitz, $h$ is $N$-regular and the following constraint qualification condition holds*:

$$z^* \in Q^*,\ z^* \circ \nabla g(\bar{x}) = 0 \quad \Rightarrow \quad z^* = 0. \tag{20}$$

*Then there exists $y^* \in K^*$, $\|y^*\| = 1$ s.t. $L_{y^*} = \{z^* \in Q^* \mid -z^* \circ \nabla g(\bar{x}) \in \partial\langle y^*, h\rangle(\bar{x})\}$ is nonempty and bounded.*

**Proof.** We apply Proposition 4.1. Taking into account that $-z^* \circ \nabla g(\bar{x}) \in D_N^* h(\bar{x})(y^*)$ and so because of (18), under the assumption that $h$ is locally Lipschitz and $N$-regular, we have

$$-z^* \circ \nabla g(\bar{x}) \in D_M^* h(\bar{x})(y^*) = \partial\langle y^*, h\rangle(\bar{x}).$$

Consequently, we get the result with Theorem 4.3. □

## 6. Conclusions

The aim of our paper was to show the boundedness of the set of Lagrange multipliers for vector optimization problems in infinite dimensional spaces. Corresponding results for the finite dimensional case were already published by Dutta and Lalitha [7]. In order to extend the results to the infinite dimensional case (cf. Theorem 3.1) we need additional assumptions concerning the ordering cone $Q$, namely that this cone has a nonempty interior. This nonemptyness of the interior of the cone implies that $Q$ is dually compact. This is an important property for cones in infinite dimensional spaces in order to show the existence of nontrivial multipliers (cf. Ng and Zheng [19]).

## Acknowledgments

The authors would like to thank Prof. K.F. Ng for providing an example of a dually compact cone and the anonymous referee for helpful comments.

## References

[1] M. Anitescu, Degenerate nonlinear programming with quadratic growth condition, SIAM J. Optim. 10 (2000) 1116–1135.
[2] J.P. Aubin, H. Frankowska, Set-valued Analysis, Birkhäuser, Basel, 1990.
[3] T.Q. Bao, B.S. Mordukhovich, Existence of minimizers and necessary conditions in multiobjective optimization with equilibrium constraints, Appl. Math. 26 (2007) 453–472.
[4] J.F. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, 2000.
[5] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley–Interscience, 1983.
[6] J. Diestel, Geometry of Banach Spaces—Selected Topics, Springer, Berlin, 1975.
[7] J. Dutta, C.S. Lalitha, Bounded sets of KKT multipliers in vector optimization, J. Global Optim. 36 (2006) 425–437.
[8] M. Durea, Estimations of the Lagrange multipliers' norms in set-valued optimization, Pacific J. Optim. 2 (2006) 487–499.
[9] M. Durea, C. Tammer, Fuzzy necessary optimality conditions for vector optimization problems, Optimization, in press.
[10] J. Dutta, C. Tammer, Lagrangian conditions for vector optimization in Banach spaces, Math. Methods Oper. Res. 64 (2006) 521–541.
[11] M. Fabian, Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces, Wiley, New York, 1997.
[12] J. Gauvin, A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming, Math. Program. 12 (1977) 136–138.
[13] J. Jahn, Vector Optimization. Theory, Applications, and Extensions, Springer-Verlag, Berlin, 2004.
[14] F. John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays Presented to R. Courant on His 60th Birthday, January 8, 1948, Interscience Publishers Inc., New York, 1948, pp. 187–204.
[15] H.W. Kuhn, A.W. Tucker, Nonlinear programming, in: Proceedings of the Second Berkley Symposium on Mathematical Statistics and Probability, 1951, pp. 481–492.

[16] O.L. Mangasarian, S. Fromovitz, The Fritz John necessary optimality conditions in the presence of equality and inequality constraints, J. Math. Anal. Appl. 17 (1967) 37–47.
[17] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, vol. I: Basic Theory, vol. II: Applications, Grundlehren Math. Wiss. (A Series of Comprehensive Stud. in Math.), vols. 330 and 331, Springer, Berlin, 2006.
[18] B.S. Mordukhovich, Maximum principle in problems of time optimal control with nonsmooth constraints, J. Appl. Math. Mech. 40 (1976) 960–969.
[19] K.F. Ng, X.Y. Zheng, The Fermat rule for multifunctions on Banach spaces, Math. Program. Ser. A 104 (2005) 69–90.
[20] R.T. Rockafellar, R.J.B. Wets, Variational Analysis, Springer, 1998.
[21] V.M. Tikhomirov, Stories about Maxima and Minima, Math. Assoc. Amer., Amer. Math. Soc., 1990.