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Singular Value Decompositions and Inversion Methods for the Exterior Radon Transform and a Spherical Transform*

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The classical Radon transform, R, maps an integrable function in R^n to its integrals over all $n-1$ dimensional hyperplanes, and the exterior Radon transform is the transform R restricted to hyperplanes that do not intersect a given disc. A singular value decomposition for the exterior transform is given for spaces of square integrable functions on the exterior of the disc. This decomposition in orthogonal functions explicitly produces the null space and range of the exterior transform and gives a new method for inverting the transform modulo the null space. A modification of this method is given that will exactly invert functions of compact support. These results generalize theorems of R. M. Perry and the author. A singular value decomposition for the Radon transform that integrates over spheres in R^n containing the origin is also given. This follows from the singular value decomposition for R and yields the null space and a new inversion method for this transform.

1. INTRODUCTION

In this article we investigate the problem of reconstructing a function defined outside of a disc in R^n from integrals over hyperplanes not intersecting that disc. The problem can be solved, at least for functions of compact support [12, 15, 18]; however, there are "null functions" not of compact support for which this problem does not have a solution [13, 21]. The existence of these functions complicates the reconstruction problem, even for functions of compact support [26, p. 426].

Our solution to this problem, the singular value decomposition of Theorem 3.2, provides a method to reconstruct a function defined outside the disc up to a null function, and in Remark 3.6 we outline a refinement of this method that will exactly reconstruct a compactly supported function. The singular value decomposition proves other useful results, including a simple characterization of the entire set of null functions (see also [21]).

Special cases of the reconstruction problem are important for astronomy and computerized tomography, and this theoretical research is motivated, in

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part, by these practical problems. This problem for lines in R^2 comes up in the analysis of astronomical data about the corona of the sun [2]. The mathematical problem in X-ray tomography is to recover a function in the plane—the density of a planar cross section of the body—from values of its integrals over lines in that plane [26, 27]. If the planar cross section intersects the heart, movement of the heart will create blurring in the reconstruction of that section. So, if one only wants to reconstruct the organs around the heart, a reconstruction method is needed that does not use integrals over lines passing through the heart. For similar reasons, it is possible that a solution to the proposed problem for integrals over planes in R^3 might become useful for a type of nuclear magnetic resonance zeugmatography [25].

Our singular value decomposition is the extension to R^n of R. Michael Perry's decomposition for this problem over lines in R^2 [20]. A singular value decomposition provides theoretical information about the problem (Corollaries 3.3–3.5), and, in this case, an inversion method (Remark 3.6). The solutions to this problem in [3] and [8] use integral equations and do not give good numerical results [11], but Lewitt and Bates [16] and Natterer [19] give numerically better solutions for the discretized problem in R^2.

We also prove a singular value decomposition for the problem of recovering a function on R^n from integrals over spheres containing the origin. Besides being of intrinsic interest, this second problem has applications to partial differential equations [6].

2. Definitions

First let $\cdot$ denote the standard inner product on R^n; let $|\ |$ be the induced norm, and let dx be Lebesgue measure on R^n. At the same time let S^{n-1} be the unit sphere in R^n; let $\omega \in S^{n-1}$, and let $s \in R$. Now let $d\omega$ and ds denote the standard measures on S^{n-1} and R, respectively. Then we denote the volume of S^{n-1} in its measure by ω_n. In order to define the Radon transform, let $H(\omega, s)$ be $\{x \in R^n \mid x \cdot \omega = s\}$; this is the $n-1$ dimensional hyperplane with normal vector ω and directed distance s from the origin. The points (ω, s) and $(-\omega, -s)$ parametrize the same hyperplane $H(\omega, s)$. Therefore we will always assume $s \geqslant 0$ in this article. Let dx_H be the measure on $H(\omega, s)$ induced from Lebesgue measure on R^n.

The classical Radon transform is defined for an integrable function f on R^n by

$$Rf(\omega, s) = \int_{H(\omega,s)} f(x)\, dx_H. \tag{2.1}$$

$Rf(\omega, s)$ is just the integral of f over the hyperplane $H(\omega, s)$.

Let E be the exterior of the unit disc in R^n, $E = \{x \in R^n \mid |x| \geq 1\}$, and let $E' = S^{n-1} \times [1, \infty)$. E' corresponds to the set of hyperplanes $H(\omega, s)$ that are contained in E.

The *exterior Radon transform* is the transform R as a map from integrable functions on E to integrable functions on E'. The problem posed in the first sentence of this article, recovering a function defined outside a disc from integrals over hyperplanes not intersecting that disc, is solved by inverting the exterior Radon transform.

The following spaces of functions will be used in Section 3. Let $L^2_p(E)$ be the Hilbert space of functions on E defined by the inner product

$$\langle f_1, f_2 \rangle_{E,p} = \int_E f_1(x) \overline{f_2(x)}\ 2|x|^{n-1}\,(1 - |x|^{-2})^p\, dx \tag{2.2}$$

and let $L^2_p(E')$ be the Hilbert space of functions on E' defined by the inner product

$$\langle g_1, g_2 \rangle_{E',p} = \int_{E'} g_1(\omega, s)\, \overline{g_2(\omega, s)}\ 2s^{-n+1}(1 - s^{-2})^{p-(n-1)/2}\, d\omega\, ds. \tag{2.3}$$

A spherical harmonic is the restriction to S^{n-1} of a harmonic polynomial on R^n. The spherical harmonics on S^1 are just trigonometric polynomials. Let $\{Y_{kl}(\omega) \mid l = 0, 1, 2,\ldots,\ k = 1, 2,\ldots, N(l)\}$ be an orthonormal basis of $L^2(S^{n-1}, d\omega)$ of spherical harmonics, where Y_{kl} is the restriction to S^{n-1} of a homogeneous polynomial of degree l, and $N(l)$ is the dimension of the subspace of spherical harmonics of degree l [24]. Any function $f \in L^2_p(E)$ can be decomposed uniquely

$$f(x) = \sum_{kl} f_{kl}(|x|)\, Y_{kl}(x/|x|), \tag{2.4}$$

where

$$f_{kl}(r) = \int_{S^{n-1}} f(r\omega)\, \overline{Y_{kl}(\omega)}\, d\omega \tag{2.5}$$

and

$$f_{kl} \in L^2([1, \infty), 2r^{2n-2}(1 - r^{-2})^p\, dr). \tag{2.6}$$

For similar reasons each $g \in L^2_p(E')$ can be written uniquely

$$g(\omega, s) = \sum_{kl} g_{kl}(s)\, Y_{kl}(\omega), \tag{2.7}$$

where g_{kl} is defined in a manner analogous to (2.5) and

$$g_{kl} \in L^2([1, \infty), 2s^{-n+1}(1-s^{-2})^{p-(n-1)/2}\, ds). \tag{2.8}$$

We now define the concept *singular value decomposition* (see Theorem VI.17 of [23]). Let $R: S \to T$ be a continuous map between separable Hilbert spaces. Let N be the null space of R (the "null functions" that have zero transforms, $N = R^{-1}\{0\}$). Orthonormal bases $\{f_m^R, f_n^N, m, n = 1, 2,...\}$ of S and $\{g_m\}$ of T constitute a singular value decomposition of R if $\{f_n^N\}$ is a basis of the null space N and there are constants $R_m \neq 0$ such that

$$Rf_m^R = R_m g_m. \tag{2.9}$$

In this situation R can be inverted modulo the null space; the method is outlined in Remark 3.6 and specifically (3.18). From the behavior of the constants R_m one can predict the stability of the inverse operator.

3. A Singular Value Decomposition for the Exterior Radon Transform

In this section we give a singular value decomposition for the exterior transform on the Hilbert spaces $L_p^2(E)$ and $L_p^2(E')$. First the bases are given (Proposition 3.1) and then the calculation (2.9) is made (Theorem 3.2), and, from this, the continuity (Corollary 3.3), null space (Corollary 3.4) and range (Corollary 3.5) of the exterior transform are proved. Finally, in Remark 3.6, we describe a method to exactly invert functions of compact support using the singular value decomposition.

The bases will be expressed in terms of the following shifted Jacobi polynomials. We define $Q_m(\alpha, \beta, t)$ to be a real-valued polynomial of degree m in t for $\alpha > -1$, $\beta > -1$, $m = 0, 1, 2,...$ such that $\{Q_m(\alpha, \beta, t)\}$ is an orthonormal basis of $L^2([0, 1], t^\alpha(1-t)^\beta\, dt)$; specifically we require

$$\int_0^1 t^m Q_m(\alpha, \beta, t)\, t^\alpha (1-t)^\beta\, dt > 0 \tag{3.1}$$

and

$$\int_0^1 Q_m(\alpha, \beta, t)\, Q_M(\alpha, \beta, t)\, t^\alpha (1-t)^\beta\, dt = \delta_{mM}, \tag{3.2}$$

where the Kronecker delta, δ_{mM}, is equal to one if $m = M$ and is zero otherwise.

PROPOSITION 3.1. *Assume $p > (n-3)/2$, and let $l, m = 0, 1, 2,\ldots$ and $k = 1,\ldots, N(l)$. The functions f^p_{klm} defined for $x \in E$ by*

$$f^p_{klm}(x) = |x|^{-n}\, Q_m(-\tfrac{1}{2}, p, |x|^{-2})\, Y_{kl}(x/|x|) \tag{3.3}$$

for l even and

$$f^p_{klm}(x) = |x|^{-n-1}\, Q_m(\tfrac{1}{2}, p, |x|^{-2})\, Y_{kl}(x/|x|) \tag{3.4}$$

for l odd form an orthonormal basis of $L^2_p(E)$. The functions g^p_{klm} defined for $(\omega, s) \in E'$ by

$$g^p_{klm}(\omega, s) = s^{-1-l} Q_m(l + (n-2)/2, p - (n-1)/2, s^{-2})\, Y_{kl}(\omega) \tag{3.5}$$

form an orthonormal basis of $L^2_p(E')$. The orthonormal polynomials $Q_m(\alpha, \beta, t)$ are defined by (3.1) *and* (3.2).

Proof. We prove that $\{g^p_{klm}\}$ is an orthonormal basis of $L^2_p(E')$. The other proof is similar. Because $p > (n-3)/2$, the parameter $\beta = p-(n-1)/2$ in the expression for Q_m in (3.5) is greater than -1. Because the Y_{kl} and the Q_m are orthonormal bases of their respective L^2 spaces, $\{g^p_{klm}\}$ is an orthonormal basis of the space $L^2_p(E')$ (see (2.3), (2.7) and (2.8)).

The gamma function $\Gamma(z)$ is a meromorphic function for $z \in C$ that satisfies several useful identities [1]:

$$\begin{gathered} \Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(n) = (n-1)! \qquad \text{for} \quad n = 1, 2,\ldots, \\ \Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\,\Gamma(z+1/2). \end{gathered} \tag{3.6}$$

For $t \in R$ we let $\lfloor t \rfloor$ denote the greatest integer that is less than or equal to t.

We can now state our main theorem.

THEOREM 3.2. *For $p > (n-3)/2$ the exterior Radon transform on domain $L^2_p(E)$ in R^n satisfies*

$$Rf^p_{klm}(\omega, s) = 0 \qquad \text{for} \quad m < \lfloor l/2 \rfloor \tag{3.7}$$

and, for $m \geqslant \lfloor l/2 \rfloor$,

$$Rf^p_{klm}(\omega, s) = R^p_{lm'}\, g^p_{klm'}(\omega, s), \tag{3.8}$$

where $m' = m - \lfloor l/2 \rfloor$, and

$$R^p_{lm'} = 2^p \left(\frac{\pi^{n-1}\Gamma(l+2m'+1)\,\Gamma(p+m'+(3-n)/2)\,\Gamma(p+l+m'+\frac{1}{2})}{\Gamma(2p+l+2m'+1)\,\Gamma(m'+1)\,\Gamma(l+m'+n/2)} \right)^{1/2}. \tag{3.9}$$

Because of the length of the proof we will first make some observations and then prove the theorem. This theorem is the generalization to R^n of R. M. Perry's decomposition for the exterior Radon transform on R^2 and is proved using identities in [20] about the Q_m. Allan Cormack [4], Mark Davison [7] and Alfred Louis [17] have given nice singular value decompositions for the classical Radon transform and have applied them to interesting practical problems.

In the case $p = (n-1)/2$, $R^p_{lm'}$ takes the simple form $R^{(n-1)/2}_{lm'} = (2\pi)^{(n-1)/2} \prod_{j=1}^{n-1} (l + 2m' + j)^{-1/2}$ as can be seen using (3.6).

COROLLARY 3.3. *For each* $p > (n-3)/2$, $R: L^2_p(E) \to L^2_p(E')$ *is a compact operator.*

Proof. It is sufficient to show that $R^p_{lm'} \to 0$ as $l + m' \to \infty$ (see Theorem VI.12a and the example on p. 199 in [23]). Stirling's approximation of the gamma function gives constants T, C_1, and C_2 such that for all $u > T$, $C_1 \leqslant (\Gamma(u)\, e^u)/u^{u-1/2} \leqslant C_2$ [1, p. 204]. Using this and the expression (3.9) it is straightforward to show $(R^p_{lm'})^2 \leqslant C(m')^{p-(n-1)/2} \times (l + 2m')^{-p-(n-1)/2}$ for some constant C. This shows that $R^p_{lm'} \to 0$ as $l + m' \to \infty$.

By the nature of the singular value decomposition, (3.7), and the continuity of the exterior transform we have:

COROLLARY 3.4. *The null space of* $R: L^2_p(E) \to L^2_p(E')$ *is the closure of the span of functions* $|x|^{-n-k}\, Y_l(x/|x|)$, *where* Y_l *is a spherical harmonic of degree* l *and* $0 \leqslant k < l$ *and* $k - l$ *is even.*

The author proved this result in [21] using different techniques.

Theorem 3.2 gives a simple characterization of the range of the exterior transform.

COROLLARY 3.5. *A function* $g \in L^2_p(E')$ *is in the range of the exterior transform if and only if*

$$\sum_{l,m',k} ((\langle g, g^p_{klm'} \rangle_{E',p})/R^p_{lm'})^2 < \infty.$$

Proof of Theorem 3.2. We first calculate the Radon transform of functions of the form $|x|^{-n-l-2j}\, Y_{kl}(x/|x|)$ and then use this information to calculate $R(f^p_{klm})$. This is straightforward because the coefficients of the polynomials Q_m are known.

Let $C^\lambda_l(t)$ denote the classical Gegenbauer polynomial (depending on $\lambda > -1/2$) of degree l. These polynomials are orthogonal on $[-1, 1]$ with weight $(1-t^2)^{\lambda-1/2}$. If $\lambda = 0$ these polynomials are replaced by the

Chebyshev polynomials, $T_l(t)$ [9]. If $w(|x|)$ is an integrable radial function then [18, Lemma 5.2]

$$R(w(|x|)\, Y_{kl}(x/|x|))(\omega, s) = h(s)\, Y_{kl}(\omega),$$

where

$$h(s) = \frac{\Gamma(n-2)\,\Gamma(l+1)\, 2\pi^{(n-1)/2}}{\Gamma(n-2+l)\,\Gamma((n-1)/2)} \int_s^{\infty} C_l^{\lambda}(s/r)\, w(r)\, r^{n-2}(1-(s/r)^2)^{(n-3)/2}\, dr \tag{3.10}$$

for $s > 0$ and $\lambda = (n-2)/2$. For $n = 2$ the integral for $h(s)$ becomes [3]

$$h(s) = 2\int_s^{\infty} T_l(s/r)\, w(r)(1-(s/r)^2)^{-1/2}\, dr. \tag{3.11}$$

These can be easily proven using the Funk Hecke theorem [22, pp. 514–515]. Using (3.6) and (3.10) or (3.11) as well as formula 7.311.2 on page 826 of [10], one can see for $j = 0, 1, 2, \ldots$ that

$$R(|x|^{-n-l-2j}\, Y_{kl}(x/|x|))(\omega, s) = B_{lj}\, s^{-1-l-2j}\, Y_{kl}(\omega), \tag{3.12}$$

where

$$B_{lj} = \frac{\pi^{n/2}\Gamma(2j+l+1)}{\Gamma(j+l+n/2)\, 2^{2j+l}\Gamma(j+1)} \tag{3.13}$$

for $s > 0$. Using the orthogonality relations of the C_l^{λ} and T_l and (3.9)–(3.11) we can see

$$R(|x|^{-n-l+2j}\, Y_{kl}(x/|x|))(\omega, s) = 0 \quad \text{for } j = 1, 2, \ldots, \lfloor l/2 \rfloor. \tag{3.14}$$

R. M. Perry has calculated the coefficients of Q_m and found [20, Eq. (3.26)]

$$Q_m(\alpha, \beta, t) = \sum_{j=0}^{m} q_{mj}(\alpha, \beta)\, t^j, \tag{3.15}$$

where

$$q_{mj}(\alpha, \beta) = \frac{(-1)^{m-j}\,\Gamma(\alpha+\beta+m+j+1)}{\Gamma(m-j+1)\,\Gamma(j+1)\,\Gamma(\alpha+j+1)} \times \left(\frac{(\alpha+\beta+2m+1)\,\Gamma(m+1)\,\Gamma(\alpha+m+1)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\beta+m+1)}\right)^{1/2}. \tag{3.16}$$

We now calculate Rf^p_{klm}. For $m < [l/2]$ the function f^p_{klm} is a linear combination of terms of the form $|x|^{-n-l+2j} Y_{kl}(x/|x|)$ for $j = 1, 2,\dots [l/2]$. By the calculation (3.14), $Rf^p_{klm} = 0$. This proves (3.7).

For the calculation of (3.8) we assume l is even. From the definition of f^p_{klm}, (3.3), the expansion of Q_m, (3.15), and the calculation (3.12), we see that

$$R(f^p_{klm})(\omega, s) = Y_{kl}(\omega)\, s^{-l-1} \sum_{j=0}^{m'} B_{lj} q_{m,j+l/2}(-1/2, p)\, s^{-2j}, \tag{3.17}$$

where $m' = m - [l/2] = m - l/2$. However, a simple check using the definition of $R^p_{lm'}$, (3.9), B_{lj}, (3.13), and $q_{m,j+l/2}$, (3.16), shows that $B_{lj} q_{m,j+l/2}(-1/2, p) = R^p_{lm'} q_{m',j}(l + (n-2)/2,\ p - (n-1)/2)$. Using this result in (3.17) we prove (3.8) for l even. The proof for l odd is similar and is left to the reader.

Some of the functions and constants in this section are related to those of Perry in [20]. Our polynomials $Q_m(\alpha, \beta, t)$ are Perry's $Q^*_m(\alpha, \beta, t)$; our functions f^p_{klm} correspond to Perry's f^p_{mn}, where our l is equivalent to Perry's m, and our m is Perry's n. Finally, our $R^p_{lm'}$ is Perry's R^p_{mn}.

Remark 3.6. We now explain how the singular value decomposition combined with the support theorem for the Radon transform gives an exact inversion method for functions of compact support.

Let $f(x) \in L^2_p(E)$ be equal to zero for $|x| \geqslant M > 1$ and assume $Rf(\omega, s) = \sum_{klm} e_{klm}\, g_{klm}(\omega, s)$ in $L^2_p(E')$. Then, using the singular value decomposition, we can recover part of the orthogonal expansion of f, namely,

$$f_R = \sum_{k,l,m} (e_{klm}/R^p_{lm})\, f^p_{kl(m+[l/2])}, \tag{3.18}$$

which is the projection of f onto the orthogonal complement of the null space of the exterior transform. Now $f = f_R + f_N$, where f_N is the projection of f onto the null space of the exterior transform. Because $f(x) = 0$ for $|x| \geqslant M$,

$$f_N(x) = -f_R(x) \qquad \text{for} \quad |x| \geqslant M. \tag{3.19}$$

By Lemma 3.7 below, Eq. (3.19) uniquely determines $f_N(x)$ for all $x \in E$, and therefore we can recover $f(x)$. Simply put, the method to recover f is: (1) use (3.18) to recover $f_R(x)$, and (2) use Eq. (3.19) to find $f_N(x)$.

There are ways this method can be implemented practically. In Section 5 of [20] Perry outlines a method to recover f_R numerically. To recover f_N we can use the coefficient of Y_{kl} for f_R to recover this coefficient for f_N, $(f_N)_{kl}$ (see (2.4) and (2.5)). By the nature of the null space of R, Corollary 3.4, $(f_N)_{kl}(r)$ is a polynomial of degree less than $n + l$ in $1/r$ and there are various ways to find this polynomial from the given function $(f_R)_{kl}(r)$ for

$r \geqslant M$ [14]. For small values of l least squares is effective and not too noise sensitive. For larger l a regularization procedure is necessary. Details will appear elsewhere.

We finally need to check.

LEMMA 3.7. *Equation* (3.19) *uniquely determines* $f_N(x)$ *for* $x \in E$.

Proof. Assume f_N is in the null space of R. It is sufficient to check that if $f_N(x)$ is zero for $|x| \geqslant M$ and Rf_N is zero on E', then f_N is zero on E. But this is immediate from the support theorem for the Radon transform [12, 15, 18].

4. THE SPHERICAL RADON TRANSFORM

We now produce a singular value decomposition for the Radon transform that integrates over spheres through the origin. This transform is intimately related to the classical Radon transform on R^n (4.7) and is of interest in partial differential equations [6]. It is defined as

$$SMf(y) = 1/\omega_n \int_{\omega \in S^{n-1}} f(y/2 + |y|\, \omega/2)\, d\omega, \tag{4.1}$$

where ω_n is the volume of S^{n-1} in its standard measure; $SMf(y)$ is the mean or average of f over the sphere centered at $y/2$ and containing the origin and y.

In order to state the singular value decomposition let B be the closed unit ball in R^n, and define

$$F^p_{klm}(x) = |x|^{2-n}\, Q_m(-1/2, p, |x|^2)\, Y_{kl}(x/|x|) \tag{4.2}$$

for l even, and

$$F^p_{klm}(x) = |x|^{3-n}\, Q_m(1/2, p, |x|^2)\, Y_{kl}(x/|x|) \tag{4.3}$$

for l odd, and let

$$G^p_{klm}(x) = |x|^{2+l-n}\, Q_m(l + (n-2)/2, p-(n-1)/2, |x|^2)\, Y_{kl}(x/|x|), \tag{4.4}$$

where $l, m = 0, 1, 2,..., k = 1, 2,..., N(l)$.

We have the following singular value decomposition.

THEOREM 4.1. *Let* $p > (n-3)/2$ *then* $\{F^p_{klm}\}$ *is an orthonormal basis of* $L^2(B, 2|x|^{n-3}(1-|x|^2)^p\, dx)$ *and* $\{G^p_{klm}\}$ *is an orthonormal basis of* $L^2(B, 2|x|^{2n-4}(1-|x|^2)^{p-(n-1)/2}\, dx)$. *The Radon transform SM on* R^n *satisfies*

$$SMF^p_{klm} = 0 \qquad \textit{for} \quad m < [l/2], \tag{4.5}$$

and, for $m \geqslant [l/2]$,

$$SMF^p_{klm} = (2^{n-1}R^p_{lm'}/\omega_n)\, G^p_{klm'} \tag{4.6}$$

where $m' = m - [l/2]$ *and* $R^p_{lm'}$ *is given by* (3.9).

Just as for the classical exterior Radon transform, this singular value decomposition shows SM is a compact operator with null space the closure of the span of functions $|x|^{2-n+j}\, Y_{kl}(x/|x|)$, where $0 \leqslant j < l$ and $j - l$ is even. A range characterization similar to Corollary 3.5 and an inversion method similar to that in Remark 3.6 are also immediate from Theorem 4.1. The inversion method is valid for functions supported away from the origin.

Allan Cormack has singular value decompositions and other nice results for Radon transforms that integrate over certain curves in the plane, including the transforms R and SM on R^2 [5]. The null space calculation for SM is related to results of the author in [21].

Proof of Theorem 4.1. The proof rests on the relation between SM and R, (4.7) below, as well as Theorem 3.2. The sets $\{F^p_{klm}\}$ and $\{G^p_{klm}\}$ are easily proven to be orthonormal bases. Define the maps J and K as follows:

$$\begin{aligned} J &: L^2_p(E) \to L^2(B, 2\,|x|^{n-3}\,(1-|x|^2)^p\,dx), \\ Jf(x) &= |x|^{2-2n} f(x/|x|^2), \\ K &: L^2_p(E') \to L^2(B, 2\,|x|^{2n-4}\,(1-|x|^2)^{p-(n-1)/2}\,dx), \\ Kg(x) &= (2^{n-1}/\omega_n)\,|x|^{1-n}\, g(x/|x|, 1/|x|). \end{aligned}$$

It is straightforward to show $Jf^p_{klm} = F^p_{klm}$ and $Kg^p_{klm} = (2^{n-1}/\omega_n)\, G^p_{klm}$. Therefore J is an isometry and K is a factor of $(2^{n-1}/\omega_n)$ times an isometry.

The key to the proof of Theorem 4.1 is the following commutative diagram:

$$\begin{array}{ccc} L^2_p(E) & \xrightarrow{\quad R \quad} & L^2_p(E') \\ \big\downarrow J & & \big\downarrow K \\ L^2(B, 2\,|x|^{n-3}\,(1-|x|^2)^p\,dx) & \xrightarrow{SM} & L^2(B, 2\,|x|^{2n-4}\,(1-|x|^2)^{p-(n-1)/2}\,dx). \end{array} \tag{4.7}$$

Under inversion in the unit sphere, $x \to x/|x|^2$, the sphere $S(y)$ with center $y/2$ and containing the origin and y gets mapped to the hyperplane $H(y/|y|, 1/|y|)$. To show that the diagram above commutes, one only needs to write the integral of a function f over the sphere $S(y)$ as an integral of $f(x/|x|^2)$ over the hyperplane $H(y/|y|, 1/|y|)$. This is outlined in [21, Lemma 4.4]. By tracing the functions f^p_{klm} through the diagram (4.7) using the singular value decomposition for R one finishes the proof of the theorem.

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