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Integral Matrices with Given Row and Column Sums

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Let $P=(p_{ij})$ and $Q=(q_{ij})$ be $m \times n$ integral matrices, R and S be integral vectors. Let $\mathfrak{A}_P^Q(R, S)$ denote the class of all $m \times n$ integral matrices A with row sum vector R and column sum vector S satisfying $P \leqslant A \leqslant Q$. For a wide variety of classes $\mathfrak{A}_P^Q(R, S)$ satisfying our *main condition*, we obtain two necessary and sufficient conditions for the existence of a matrix in $\mathfrak{A}_P^Q(R, S)$. The first characterization unifies the results of Gale–Ryser, Fulkerson, and Anstee. Many other properties of (0, 1)-matrices with prescribed row and column sum vectors generalize to integral classes satisfying the main condition. We also study the decomposibility of integral classes satisfying the main condition. As a consequence of our decomposibility theorem, it follows a theorem of Beineke and Harary on the existence of a strongly connected digraph with given indegree and outdegree sequences. Finally, we introduce the incidence graph for a matrix in an integral class $\mathfrak{A}_P^Q(R, S)$ and study the invariance of an element in a matrix in terms of its incidence graph. Analogous to Minty's Lemma for arc colorings of a digraph, we give a very simple labeling algorithm to determine if an element in a matrix is invariant. By observing the relationship between invariant positions of a matrix and the strong connectedness of its incidence graph, we present a very short graph theoretic proof of a theorem of Brualdi and Ross on invariant sets of (0, 1)-matrices. Our proof also implies an analogous theorem for a class of tournament matrices with given row sum vector, as conjectured by the analogy between bipartite tournaments and ordinary tournaments.

1. INTRODUCTION

Let $R=(r_1, r_2, ..., r_m)$ and $S=(s_1, s_2, ..., s_n)$ be integral vectors with $r_1+r_2+\cdots+r_m=s_1+s_2+\cdots+s_n$. Let $P=(p_{ij})$ and $Q=(q_{ij})$ be two $m \times n$ integral matrices such that $p_{ij} \leqslant q_{ij}$ for all i and j, denoted $P \leqslant Q$. Let $\mathfrak{A}_P^Q(R, S)$ denote the class of all integral matrices A with row sum vector R and column sum vector S satisfying the lower bound and upper bound conditions $P \leqslant A \leqslant Q$. Although we may assume without loss of generality

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that $P=0$, we state our results for P to be a general integral matrix because this does not cause any complication instead, such formulations can make our statements look symmetric and easier to apply to special cases.

The most important special case of integral matrices is the class $\mathfrak{A}(R, S)$ of (0, 1)-matrices with row sum vector R and column sum vector S. Much work has been done on $\mathfrak{A}(R, S)$ since the independent work of Gale [16] and Ryser [27] in the 1950s. The reader is referred to the beautiful survey article of Brualdi [9] on (0, 1)-matrices. The fundamental result of Gale and Ryser is the discovery of the relationship between the *dominance order* or the *majorization order* on partitions of integers and the existence of a matrix in $\mathfrak{A}(R, S)$. This theorem turned out to be important to the theory of symmetric functions and the representation of the symmetric group. Furthermore, some properties of $\mathfrak{A}(R, S)$, even in graph theoretic nature, can be characterized by refinements of the dominance order. For example, Fulkerson [15] shortly observed that the result of Gale and Ryser can be generalized to the class of (0, 1)-matrices with given row and column sums and with zero trace. Note that a (0, 1)-matrix with zero trace is the adjacency matrix of a digraph. Anstee [1–3] studied more general classes of (0, 1)-matrices and integral matrices with boundary conditions. His network flows approach turned out to be quite successful, especially in algorithmic aspects.

In this paper, we are mainly concerned with integral classes $\mathfrak{A}_P^Q(R, S)$ satisfying our *main condition*. Many known classes of (0, 1)-matrices turn out to satisfy the main condition. We discover that many properties of (0, 1)-matrices with prescribed row and column sum vectors still hold for these integral classes. For example, we generalize the concepts of the conjugate matrix and the structure matrix to integral classes. In this way, we obtain two necessary and sufficient conditions for these classes to be nonempty. It seems quite surprising that we may even obtain the decomposition theorem for integral classes satisfying the main condition. Since many classes of (0, 1)-matrices studied before satisfy our main condition, our results unify many known results on (0, 1)-matrices and multiple bipartite graphs, for example, the Gale–Ryser theorem, Fulkerson's theorem on the existence of a simple directed graph with given indegree and outdegree sequences, and Beineke and Harary's theorem on the existence of a strongly connected directed graph with given indegree and outdegree sequences. Our matrix approach proof of Beineke and Harary's theorem is quite different from their original graph theoretic proof and seems much more natural. However, the problem of finding a necessary and sufficient condition for the existence of a k-strongly connected digraph with given indegree and outdegree sequences is still unsolved [23, 30]. Since the structure matrix of a class of (0, 1)-matrices is an alternate description of the dominance order on integer partitions, it would be interesting to incorporate the notion of a structure matrix and our integral generalization into the theory of representations of the symmetric group and the theory of symmetric functions [29]. We would also like to add that there is an interesting notion of majorization with respect to a poset [24].

Another objective of this paper is to introduce incidence graphs of a class $\mathfrak{A}_P^Q(R, S)$ (not necessarily satisfying the main condition). This is a different approach from that of network flows proposed by Anstee [2] and it turns out to be quite effective for studying invariant elements. Analogous to Minty's Lemma for arc colorings of a digraph [7], we give a very simple labeling algorithm to determine if an entry is an invariant element. By this incidence graph approach we give a very short proof of a theorem of Brualdi and Ross on invariant sets of a class of (0, 1)-matrices. Our proof implies an analogous theorem for a class of ordinary tournaments with given score list, or a class of tournament matrices with given row sum vector, as conjectured by the analogy between bipartite tournaments and ordinary tournaments [24, 28].

2. Conjugate Matrix and Structure Matrix

We first clarify the notation that is used throughout this paper. For a positive integer n, we use the common notation $[n]$ for the set $\{1, 2, ..., n\}$. For an $m \times n$ matrix A and $I \subseteq [m]$, $J \subseteq [n]$, we use $A[I, J]$ to denote the submatrix of A whose rows are indexed by I and whose columns are indexed by J. We use $\bar{I}$ to denote the set $[m] \backslash I$ and $\bar{J}$ to denote $[n] \backslash J$. We shall keep the notation $\mathfrak{A}(R, S)$ for the usual class of (0, 1)-matrices with row sum vector R and column sum vector S, but without any additional restriction on each entry. If P and Q are (0, 1)-matrices, $\mathfrak{A}_P(R, S)$ and $\mathfrak{A}^Q(R, S)$ denote the classes consisting all matrices in $\mathfrak{A}(R, S)$ covering P and covered by Q, respectively.

In the study of (0, 1)-matrices, the concept of the conjugate matrix (or the maximum matrix) plays an important role. We now give a generalization of this notion to integral classes.

Definition 2.1 (Conjugate Matrix, Conjugate Vector, and K-vector). For a class $\mathfrak{A}_P^Q(R, S)$, the conjugate matrix $A^* = (a_{ij}^*)$ is defined as the matrix with row sum vector R in which entries on the left are as big as possible subject to the condition $P \leqslant A^* \leqslant Q$. Formally, A^* is the matrix satisfying the following conditions:

1. $P \leqslant A^* \leqslant Q$ and A^* has row sum vector R.

2. For the ith row of A^* $(1 \leqslant i \leqslant m)$, there exists an integer k_i $(0 \leqslant k_i \leqslant n)$ such that $a_{ij}^* = q_{ij}$ for $j \leqslant k_i$, $a_{ij}^* < q_{ij}$ for $j = k_i + 1$, and $a_{ij}^* = p_{ij}$ for $j > k_i + 1$.

We call the vector $(k_1, k_2, ..., k_m)$ the K-vector of the class $\mathfrak{A}_P^Q(R, S)$ and denote the column sum vector of A^* by $S^* = (s_1^*, s_2^*, ..., s_n^*)$.

The K-vector of a class $\mathfrak{A}_P^Q(R, S)$ is actually a description of the right-most "nonzero" elements in all the rows of the conjugate matrix. As we shall see, the above definition of the K-vector seems to be a right concept for the study of integral classes. As a refined description of the right-most "nonzero" elements in the conjugate matrix, we have the following definition of the U-vector.

DEFINITION 2.2 (U-vector). The U-vector $(u_1, u_2, ..., u_m)$ associated with a class $\mathfrak{A}_P^Q(R, S)$ is defined as follows:

$$u_i = \begin{cases} k_i, & \text{if} \quad k_i = n \text{ or } a_{ij}^* = p_{ij} \text{ for } j = k_i + 1, \\ k_i + \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2.1}$$

The U-vector is a refinement of the K-vector in the sense that from the U-vector we can find out if the ith row in the conjugate matrix is full (u_i is an integer) or it is partially filled (u_i is not an integer), namely $p_{ij} < a_{ij}^* < q_{ij}$, where $j = k_i + 1$. A class is called K-monotone (U-monotone) if its K-vector (U-vector) is nonincreasing. By a permutation of rows, one can always transform a class into a K-monotone (U-monotone) class. Therefore, for any class $\mathfrak{A}_P^Q(R, S)$, we may assume without loss of generality that it is U-monotone and S is nonincreasing, and we shall simply call such a class a *monotone* class or a *normalized* class. In the present and the following sections, we restrict ourselves to monotone classes $\mathfrak{A}_P^Q(R, S)$ satisfying the following condition

$$s_i - c_i \geqslant s_j - d_j + m\Delta - 1, \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \tag{2.2}$$

where $(c_1, c_2, ..., c_n)$ and $(d_1, d_2, ..., d_n)$ are the column sum vectors of P and Q, respectively, Δ denotes the maximum entry of $Q - P$. We call the above condition (2.2) the *main condition*. As we shall see, many classes of (0, 1)-matrices which have been studied so far satisfy the main condition.

Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two sequences of real numbers. We say that X is majorized or dominated by Y, denoted by $X \prec Y$, if $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$ and

$$x_1 + x_2 + \cdots x_i \leqslant y_1 + y_2 + \cdots y_i, \qquad \text{for} \quad i = 1, 2, ..., n-1.$$

Note that X and Y are not required to be nonincreasing or nondecreasing.

THEOREM 2.3. *Suppose $\mathfrak{A}_P^Q(R, S)$ is a monotone class satisfying the main condition. Then $\mathfrak{A}_P^Q(R, S)$ is nonempty if and only if $S \prec S^*$.*

Proof. The necessity of the above theorem is clear. Now we prove the sufficiency, If $S^*=S$, then A^* is in $\mathfrak{A}_P^Q(R, S)$. So we may assume that $S^* \neq S$. Since S is nonincreasing and $S^* \succ S$, S and S^* must obey the following pattern

$$\begin{array}{ccccccccccc} s_1^* & \cdots & s_{l-1}^* & s_l^* & s_{l+1}^* & \cdots & s_{i-1}^* & s_i^* & \cdots & s_n^* \\ \geqslant & \cdots & \geqslant & > & \| & \cdots & \| & < & \cdots & \leqslant \\ s_1 \geqslant & \cdots \geqslant & s_{l-1} \geqslant & s_l \geqslant & s_{l+1} \geqslant & \cdots \geqslant & s_{i-1} \geqslant & s_i \geqslant & \cdots \geqslant & s_n \end{array}$$

where i is the minimum integer such that $s_i > s_i^*$ and l is the maximum integer such that $l < i$ and $s_l^* > s_l$. Set $d = \min(s_l^* - s_l,\ s_i - s_i^*)$ and

$$\begin{aligned} S^{(1)} &= (s_1', s_2', ..., s_n') \\ &= (s_1^*, ..., s_{l-1}^*, s_l^* - d, s_{l+1}^*, ..., s_{i-1}^*, s_i^* + d, ..., s_n^*). \end{aligned}$$

It is straightforward to check that $S \prec S^{(1)} \prec S^*$. Since we have $s_l' = s_l$ or $s_i' = s_i$, the number of components of $S^{(1)}$ which differ from the corresponding components in S is less than the number of components of S^* which differ from the corresponding components in S. Therefore, there exists a sequence of vectors $S^{(1)}, S^{(2)}, ..., S^{(t)}$ such that

$$S = S^{(t)} \prec S^{(t-1)} \prec \cdots \prec S^{(1)} \prec S^*.$$

Let $\delta_k = \min(a_{kl}^* - p_{kl},\ q_{ki} - a_{ki}^*)$. Clearly, $\delta_k \geqslant 0$. We are going to show that

$$\delta_k \geqslant a_{kl}^* - a_{ki}^* - p_{kl} + q_{ki} - \Delta. \tag{2.3}$$

Since Δ is the maximum entry of $Q-P$, it follows that

$$\Delta \geqslant q_{ki} - a_{ki}^* \qquad \text{and} \qquad \Delta \geqslant a_{kl}^* - p_{kl}.$$

Thus, we have

$$\begin{aligned} a_{kl}^* - a_{ki}^* - p_{kl} + q_{ki} - \Delta &= (a_{kl}^* - p_{kl}) - (\Delta - (q_{ki} - a_{ki}^*)) \\ &\leqslant a_{kl}^* - p_{kl}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} a_{kl}^* - a_{ki}^* - p_{kl} + q_{ki} - \Delta &= (q_{ki} - a_{ki}^*) - (\Delta - (a_{kl}^* - p_{kl})) \\ &\leqslant q_{ki} - a_{ki}^*. \end{aligned} \tag{2.5}$$

Hence (2.3) follows from (2.4) and (2.5). Since $\delta_k \geqslant 0$, it is easy to see that

$$p_{kl} \leqslant a_{kl}^* - \delta_k \leqslant q_{kl}, \tag{2.6}$$

$$p_{ki} \leqslant a_{ki}^* + \delta \leqslant q_{ki}. \tag{2.7}$$

Since $s_l^* > s_l$, from (2.3) and the main condition, we have

$$\begin{aligned}\sum_{k=1}^{m} \delta_k &\geqslant \sum_{k=1}^{m} a_{kl}^* - \sum_{k=1}^{m} a_{ki}^* - \sum_{k=1}^{m} p_{kl} + \sum_{k=1}^{m} q_{ki} - m\Delta \\ &= s_l^* - s_i^* - c_l + d_i - m\Delta \\ &\geqslant s_l + 1 - c_l + d_i - m\Delta - s_i^* \\ &\geqslant s_i - s_i^* \geqslant d.\end{aligned}$$

Therefore, there exist integers $\varepsilon_1, \varepsilon_2, ..., \varepsilon_m$ such that $0 \leqslant \varepsilon_k \leqslant \delta_k$ for $1 \leqslant k \leqslant m$ and $\sum_{k=1}^{m} \varepsilon_k = d$. Suppose $A^{(1)}$ is the matrix obtained from A^* by replacing a_{kl}^* by $a_{kl}^* - \varepsilon_k$ and replacing a_{ki}^* by $a_{ki}^* + \varepsilon_k$ for $1 \leqslant k \leqslant m$. From (2.6) and (2.7), it follows that $P \leqslant A^{(1)} \leqslant Q$. It is clear that the column sum vector of $A^{(1)}$ is $S^{(1)}$. Since the main condition remains the same for $S^{(1)}$, by repeating the above procedure we can get a matrix $A^{(2)}$ with column sum vector $S^{(2)}$ such that $P \leqslant A^{(2)} \leqslant Q$. In this way, we eventually obtain a matrix $A^{(t)}$ with column sum vector $S^{(t)} = S$. Clearly, $A^{(t)}$ is in $\mathfrak{A}_P^Q(R, S)$. This completes the proof. ▮

It is easy to verify that the above theorem unifies the Gale–Ryser theorem, Fulkerson's theorem on the existence of a simple digraph with given indegree and outdegree sequences, Anstee's characterization of the existence of a (0, 1)-matrix in a class $\mathfrak{A}_P(R, S)$, where P is a (0, 1)-matrix with at most one 1 in each column, and the necessary and sufficient condition for the existence of a nonnegative integral matrix with prescribed row and column sum vectors and with every entry not exceeding a constant p, namely a p-digraph with given indegree and outdegree sequences (see [12]).

In the study of (0, 1)-matrices, there has been an alternative way to represent the dominance order by using the notion of a structure matrix. It turned out that we may have similar results for integral classes satisfying the main condition. The definition of the structure matrix for an integral class goes as follows.

DEFINITION 2.4 (Structure Matrix). For a class $\mathfrak{A}_P^Q(R, S)$, its structure matrix $T = (t_{ij})$ is an $(m+1) \times (n+1)$ matrix defined by

$$t_{kl} = \sum_{\substack{i \leqslant k \\ j \leqslant l}} q_{ij} + \sum_{i > k} r_i - \sum_{j \leqslant l} s_j - \sum_{\substack{i > k \\ j > l}} p_{ij} \qquad (0 \leqslant k \leqslant m,\ 0 \leqslant l \leqslant n).$$

THEOREM 2.5. *Let $\mathfrak{A}_P^Q(R, S)$ be a monotone class satisfying the main condition and $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$. Then $\mathfrak{A}_P^Q(R, S)$ is nonempty if and only if its structure matrix is nonnegative.*

Proof. First we prove the necessity. For any matrix M, we denote by $\sigma(M)$ the sum of all entries of M. Let A be any matrix in $\mathfrak{A}_P^Q(R, S)$. Then for any $I=[k]$, $J=[l]$ we have

$$\begin{aligned} t_{kl} &= \sigma(Q[I, J]) + \sigma(A[\bar{I}, J \cup \bar{J}]) - \sigma(A[\bar{I} \cup I, J]) - \sigma(P[\bar{I}, \bar{J}]) \\ &= \sigma(Q[I, J]) - \sigma(A[I, J]) + \sigma(A[\bar{I}, \bar{J}]) - \sigma(P[\bar{I}, \bar{J}]) \\ &= \sigma((Q-A)[I, J]) + \sigma((A-P)[\bar{I}, \bar{J}]). \end{aligned}$$

Since $P \leqslant A \leqslant Q$, T is nonnegative.

Now we prove the sufficiency. Suppose $T \geqslant 0$ and $U=(u_1, u_2, ..., u_m)$ is the U-vector of the class $\mathfrak{A}_P^Q(R, S)$. For any integer f $(0 \leqslant f \leqslant n)$, set $F=[f]$ and

$$e = \max\{0, i \mid u_i \geqslant f, 1 \leqslant i \leqslant m\}. \tag{2.8}$$

Let $E=[e]$. Since $\mathfrak{A}_P^Q(R, S)$ is U-monotone, it follows that $u_i \geqslant f$, for $1 \leqslant i \leqslant e$ and $u_j < f$, for $e < j \leqslant m$. Thus, we have

$$A^*[E, F] = Q[E, F], \qquad A^*[\bar{E}, \bar{F}] = P[\bar{E}, \bar{F}].$$

It follows that

$$\sum_{i>l} r_i = \sigma(A^*[\bar{E}, [n]]) = \sigma(A^*[\bar{E}, F]) + \sigma(P[\bar{E}, \bar{F}]), \tag{2.9}$$

$$\sigma(Q(E, F]) + \sigma(A^*[\bar{E}, F]) = \sum_{j \leqslant f} s_j^*. \tag{2.10}$$

Since $t_{ef} \geqslant 0$, by (2.9) and (2.10) we get

$$t_{ef} = \sigma(Q[E, F]) + \sum_{i>e} r_i - \sum_{j \leqslant f} s_j - \sigma(P[\bar{E}, \bar{F}]) \geqslant 0.$$

That is,

$$\sum_{j \leqslant f} s_j^* - \sum_{j \leqslant f} s_j \geqslant 0.$$

By Theorem 2.3, we see that $\mathfrak{A}_P^Q(R, S) \neq \varnothing$. This completes the proof. ∎

COROLLARY 2.6 (Brualdi [9]). *Let R and S be nonincreasing integral vectors. Then $\mathfrak{A}(R, S)$ is nonempty if and only if, for $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant n$, we have*

$$kl + \sum_{i>k} r_i - \sum_{j \leqslant l} s_j \geqslant 0,$$

with equality for $k=0$ and $l=n$.

The above corollary can be restated for bipartite tournaments as follows.

COROLLARY 2.7 (Beineke and Moon [6]). *Let R and S be nonincreasing integral vectors. Then there is a bipartite tournament with score lists R and S if and only if, for $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant n$, we have*

$$\sum_{i=1}^{k} r_i + \sum_{j=1}^{l} s_j \geqslant kl,$$

with equality for $k = m$ and $l = n$.

We now discuss further properties of integral classes that satisfy the main condition. In [9], it is shown that any row or column of the structure matrix of a normalized class $\mathfrak{A}(R, S)$ of (0, 1)-matrices is a convex sequence. A sequence $a_1, a_2, ..., a_n$ is said to be *convex* provided that $a_{i-1} + a_{i+1} \geqslant 2a_i$ for any $1 < i < n$. It is easy to show that any convex sequence $a_1, a_2, ..., a_n$ is *unimodal*, that is, there exists k such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$ and $a_k \leqslant a_{k+1} \leqslant \cdots \leqslant a_n$. We observe that a row or column of the structure matrix of an integral class (even satisfying the main condition) may not be convex. For example, let $R = (3, 2, 2, 2)$, $S = (3, 2, 2, 2)$, and

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the class $\mathfrak{A}_P(R, S)$ contains the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The structure matrix of $\mathfrak{A}_P(R, S)$ equals

$$T = \begin{pmatrix} 8 & 5 & 3 & 2 & 0 \\ 5 & 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 4 & 5 \\ 0 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

Note that the second row $(3, 2, 2, 3, 3)$ of T is not a convex sequence (recall that the rows of T are indexed from 0 to 4). Nevertheless, we have a generalization to Theorem 4.2 in [9] concerning the unimodality of columns of a structure matrix.

THEOREM 2.8. *Let $\mathfrak{A}_P^Q(R, S)$ be a monotone class and let T be its structure matrix. Then the fth column $(0 \leqslant f \leqslant n)$ is a unimodal sequence with minimum component equal to*

$$t_{ef} = \sum_{j=1}^{f} s_j^* - \sum_{j=1}^{f} s_j, \tag{2.11}$$

where e is given by (2.8).

Proof. Let e be given by (2.8). First we prove that $t_{gf} \geqslant t_{kf}$ for $g < k \leqslant e$. Let $I_1 = [g]$, $I_2 = [k]$, $J = [f]$. Then

$$\bar{I}_1 \backslash \bar{I}_2 = I_2 \backslash I_1 = \{g+1, g+2, ..., k\}.$$

We have

$$\begin{aligned} t_{gf} - t_{kf} &= \sigma(Q[I_1, J]) + \sum_{i>g} r_i - \sum_{j \leqslant f} s_j - \sigma(P[\bar{I}_1, \bar{J}]) \\ &\quad - \sigma(Q[I_2, J]) - \sum_{i>k} r_i + \sum_{j \leqslant f} s_j + \sigma(P[\bar{I}_2, \bar{J}]) \\ &= -\sigma(Q[I_2 - I_1, J]) + \sum_{i \in I_2 - I_1} r_i - \sigma(P[I_2 - I_1, \bar{J}]). \end{aligned}$$

Since $u_i \geqslant f$ for $1 \leqslant i \leqslant e$, we have

$$\begin{aligned} \sum_{i \in I_2 - I_1} r_i &= \sigma(A^*[I_2 - I_1, J]) + \sigma(A^*[I_2 - I_1, \bar{J}]) \\ &= \sigma(Q[I_2 - I_1, J]) + \sigma(A^*[I_2 - I_1, \bar{J}]) \\ &\geqslant \sigma(Q[I_2 - I_1, J]) + \sigma(P[I_2 - I_1, \bar{J}]). \end{aligned}$$

Thus, $t_{gf} - t_{kf} \geqslant 0$. Similarly, we can prove $t_{gf} - t_{kf} \geqslant 0$ for $g > k \geqslant e$. Therefore, the fth column is unimodal with minimum element t_{ef}. Equation (2.11) then follows from (2.9) and (2.10). ∎

COROLLARY 2.9 (Brualdi [9]). *Let T be the structure matrix of a normal class $\mathfrak{A}(R, S)$ of $(0, 1)$-matrices. Then the minimum element of the fth column appears in the s_f^*th row.*

Similar to the proof of Theorem 2.8, we may show that each row of the structure matrix of any monotone integral class is unimodal.

3. Decomposability

It is well known that a class of (0, 1)-matrices can be decomposed into smaller classes whenever it contains an invariant 1 or an invariant 0. Moreover, such decomposability can be characterized by the conjugate vector and the column sum vector. However, there could be two possible generalizations of this notion to integral classes. In this section, we study the first generalization and give a characterization of such decomposability in terms of the conjugate vector for an integral class satisfying the main condition.

Definition 3.1 (Decomposable Classes). Suppose $\mathfrak{A}_P^Q(R, S)$ is nonempty. If there exist I, J such that $\varnothing \subseteq I \subseteq [m]$, $\varnothing \subseteq J \subseteq [n]$, and a matrix $A \in \mathfrak{A}_P^Q(R, S)$ satisfying $A[I, J] = Q[I, J]$ and $A[\bar{I}, \bar{J}] = P[\bar{I}, \bar{J}]$, then we say that $\mathfrak{A}_P^Q(R, S)$ is decomposable and we call the pair (I, J) a decomposition of the class $\mathfrak{A}_P^Q(R, S)$.

Clearly, the above definition is a generalization of the classical decomposability of (0, 1)-matrices in terms of invariant 1's and invariant 0's in the following sense. If $A \in \mathfrak{A}_P^Q(R, S)$ has a decomposition (I, J), then every matrix $B \in \mathfrak{A}_P^Q(R, S)$ must have the decomposition (I, J). Therefore, we can always break a decomposable class into smaller classes. For any integral class, it is easy to detect if it has decompositions of the form (I, J), when $I = \varnothing$ or $[m]$, or $J = \varnothing$ or $[n]$. We call such decompositions *trivial decompositions*. Clearly, an integral class $\mathfrak{A}_P^Q(R, S)$ has a trivial decomposition if and only if there exists a row sum r_i which equals the ith row sum of P or Q, or a column sum s_j which equals the jth column sum of P or Q. If a class of integral matrices has a trivial decomposition, we may always delete some rows or columns to make it a class without any trivial decomposition. An integral class is called a *reduced class* if it has no trivial decomposition. In other words, a class is reduced if every decomposition (I, J) satisfies $\varnothing \subset I \subset [m]$ and $\varnothing \subset J \subset [n]$. The following theorem gives a characterization of decomposable reduced integral classes satisfying the main condition.

Theorem 3.2. *Suppose $\mathfrak{A}_P^Q(R, S)$ is a reduced integral class satisfying the main condition. Then $\mathfrak{A}_P^Q(R, S)$ is decomposable if and only if there exists an integer f $(1 \leqslant f < n)$ such that*

$$\sum_{j=1}^{f} s_j^* = \sum_{j=1}^{f} s_j. \tag{3.1}$$

Proof. First we prove the sufficiency. Let $(u_1, u_2, ..., u_m)$ be the U-vector of $\mathfrak{A}_P^Q(R, S)$ and e be the integer given by (2.8). If (3.1) holds, then we have

$u_i \geqslant f$ for $1 \leqslant i \leqslant e$ and $u_i < f$ for $e < i \leqslant m$. Let $I = [e]$ and $J = [f]$. Then (I, J) is a decomposition.

We now prove the necessity. Suppose $\mathfrak{A}_P^Q(R, S)$ is a reduced class and it is decomposable and A is a matrix in $\mathfrak{A}_P^Q(R, S)$. Let (I, J) be a decomposition of $\mathfrak{A}_P^Q(R, S)$ such that $|J|$ is maximum. We aim to construct a decomposition of the form $([e], [f])$, although for the sake of (3.1) we only need to consider J. Let $\min(\bar{J}) = k$. Therefore, J can be partitioned into $J = J_1 \cup J_2$, where

$$J_1 = \{1, 2, ..., k-1\}$$

and every element in J_2 is greater than k. We prove that $A[\bar{I}, J_2] = P[\bar{I}, J_2]$. Since $|J|$ is maximum, we have $A[I, k] \neq Q[I, k]$, otherwise $(I, J \cup \{k\})$ would form a bigger decomposition. Hence

$$\sigma(A[I, k]) \leqslant \sigma(Q[I, k]) - 1. \tag{3.2}$$

Since $A[\bar{I}, k] = P[\bar{I}, k]$, we obtain

$$s_k = \sigma(A[I, k]) + \sigma(A[\bar{I}, k]) \leqslant \sigma(Q[I, k]) + \sigma(P[\bar{I}, k]) - 1. \tag{3.3}$$

Let $j \in J_2$. Since $j > k$ and $A[I, j] = Q[I, j]$, it follows that $\sigma(A[I, j]) = \sigma(Q[I, j])$ and

$$s_j = \sigma(A[I, j]) + \sigma(A[\bar{I}, j]) \geqslant \sigma(Q[I, j]) + \sigma(P[\bar{I}, j]).$$

Thus

$$\begin{aligned} s_j - d_j &= \sigma(A[I, j]) + \sigma(A[\bar{I}, j]) - \sigma(Q[I, j]) - \sigma(Q[\bar{I}, j]) \\ &\geqslant -\sigma(Q[\bar{I}, j]) + \sigma(P[\bar{I}, j]). \end{aligned} \tag{3.4}$$

From (3.3), (3.4), and the main condition, we obtain

$$\begin{aligned} &\sigma(Q[I, k]) - \sigma(P[I, k]) - 1 \\ &\quad = (\sigma(Q[I, k]) + \sigma(P[\bar{I}, k]) - 1) - (\sigma(P[I, k]) + \sigma(P[\bar{I}, k])) \\ &\quad \geqslant s_k - c_k \\ &\quad \geqslant s_j - d_j + m\Delta - 1 \\ &\quad \geqslant -\sigma(Q[\bar{I}, j]) + \sigma(P[\bar{I}, j]) + m\Delta - 1. \end{aligned}$$

Thus,

$$\sigma((Q-P)[I, k]) + \sigma((Q-P)[\bar{I}, j]) \geqslant m\Delta. \tag{3.5}$$

However, from the definition of Δ, we have

$$\sigma((Q-P)[I, k]) + \sigma((Q-P)[\bar{I}, j]) \leqslant m\Delta. \tag{3.6}$$

Inequalities (3.5) and (3.6) indicate that all the above inequalities from (3.2) to (3.6) hold with equalities. Hence, by (3.4), we have $A[\bar{I}, j] = P[\bar{I}, j]$; therefore,

$$A[\bar{I}, J_2] = P[\bar{I}, J_2]. \tag{3.7}$$

Since $A[\bar{I}, \bar{J}] = P[\bar{I}, \bar{J}]$, from (3.7) and the condition that $\mathfrak{A}_P^Q(R, S)$ is reduced, it follows that $J_1 = \{1, 2, ..., k-1\} \neq \emptyset$, namely $k > 1$. Up to now, we have shown that (I, J_1) also forms a decomposition of the class $\mathfrak{A}_P^Q(R, S)$.

Let $\min(\bar{I}) = g$. Then we may partition I into $I_1 \cup I_2$, where

$$I_1 = \{1, 2, ..., g-1\},$$

and I_2 consists of all elements of I which are greater than g. Since $\mathfrak{A}_P^Q(R, S)$ is U-monotone, for any $i \in I_2$, we have $g < i$ and therefore $u_g \geqslant u_i$. Since (I, J_1) is a decomposition, we have $A[g, \bar{J}_1] = P[g, \bar{J}_1]$ and $A[i, J_1] = Q[i, J_1]$. It follows that $u_g \leqslant |J_1|$ and $u_i \geqslant |J_1|$. Thus we have the following relations

$$u_i \geqslant |J_1| \geqslant u_g \qquad \text{and} \qquad u_g \geqslant u_i,$$

which imply $u_i = u_g$ for all $i \in I_2$. Furthermore, it follows that $A[I_2, \bar{J}_1] = P[I_2, \bar{J}_1]$. Since $A[\bar{I}, \bar{J}_1] = P[\bar{I}, \bar{J}_1]$ and $\mathfrak{A}_P^Q(R, S)$ is reduced, we may conclude that $I_1 \neq \emptyset$, namely $g > 1$. Hence we obtain a decomposition of the form $([e], [f])$. Choosing $f = |J_1|$, we get (3.1). This completes the proof. ∎

Anstee [2] considered the decomposability of a class of (0, 1)-matrices covered by a given matrix. We were kindly informed that his decomposition theorem (Theorem 3.4 in [2]) based on the notion of a t-vector does not hold in general. The special case of our Theorem 3.2 restricted to (0, 1)-matrices can be regarded as a correction of Anstee's result. The following corollary is an application of Theorem 3.2 to strongly connected digraphs with given indegree and outdegree sequences. We adopt the usual convention that I denotes the $n \times n$ identity matrix and J denotes the $n \times n$ matrix with every entry being 1.

COROLLARY 3.3 (Beineke and Harary [5]). *Let $R = (r_1, r_2, ..., r_n)$ and $S = (s_1, s_2, ..., s_n)$ be two nonnegative integral sequences satisfying $0 < r_i$, $s_i < n-1$ for all i, and $S^* = (s_1^*, s_2^*, ..., s_n^*)$ be the conjugate vector of the class $\mathfrak{A}^{J-I}(R, S)$. Then there exists a strongly connected digraph with outdegree sequence R and indegree sequence S, if and only if $r_1 + r_2 + \cdots + r_n = s_1 + s_2 \cdots + s_n$ and for $1 \leqslant i < n$,*

$$s_1 + s_2 + \cdots s_i < s_1^* + s_2^* + \cdots + s_i^*.$$

Proof. Let G be a digraph with outdegree sequence R and indegree sequence S and A be the adjacency matrix of G. Clearly, A is a matrix in the class $\mathfrak{A}^{J-I}(R, S)$, and any matrix in this class determines a digraph with outdegree sequence R and indegree sequence S. Since any digraph without directed cycles must have a vertex whose outdegree is zero, G is not strongly connected if and only if there exists a strongly connected component of G such that there is no arc from a vertex in this component to an outside vertex (equivalently, the condensation of a digraph has a vertex with outdegree zero, see [19]). It follows that G is strongly connected if and only if A is not decomposable in the class $\mathfrak{A}^{J-I}(R, S)$. The proof is therefore complete by Theorem 2.3. ∎

By the properties of the structure matrix of a monotone class satisfying the main condition, we may have a structure matrix version of the decomposability theorem. One may also obtain a structure matrix version of the above theorem of Beineke and Harary.

THEOREM 3.4. *Let $\mathfrak{A}_P^Q(R, S)$ be a reduced monotone class satisfying the main condition and $T = (t_{ij})$ be its structure matrix. Then $\mathfrak{A}_P^Q(R, S)$ is decomposable if and only if there exists an element $t_{ef} = 0$ for some $1 \leqslant f < n$.*

4. INCIDENCE GRAPHS

Motivated by the study of invariant positions in a class of (0, 1)-matrices, we consider the second generalization of the concept of decomposability to integral classes. We introduce the notion of the incidence graph of a matrix in an integral class, which turns out to be appropriate for the study of invariant positions of an integral class. In particular, for incidence graphs for a class of (0, 1)-matrices, the concept of invariant positions is equivalent to that of strongly connected incidence graphs or strongly connected bipartite tournaments. From this point of view, we give a very short graph theoretic proof of a theorem of Brualdi and Ross concerning invariant sets of a class of (0,1)-matrices. Our proof essentially shows that an analogous theorem is also true for tournament matrices with given row sum vector.

DEFINITION 4.1 (Boundary, Restricted, Movable Elements). Let $A = (a_{ij})$ be a matrix in the class $\mathfrak{A}_P^Q(R, S)$. We say that a_{ij} is a lower bound if $a_{ij} = p_{ij}$ and a_{ij} is an upper bound if $a_{ij} = q_{ij}$. An element a_{ij} is called a restricted element if $p_{ij} = a_{ij} = q_{ij}$. For an element a_{ij}, if there exists another matrix $B = (b_{ij})$ such that $b_{ij} > a_{ij}$, then we say that a_{ij} is upward movable. We may similarly define a_{ij} to be downward movable.

DEFINITION 4.2 (Invariant Elements). Let $A = (a_{ij})$ be a matrix in the class $\mathfrak{A}_P^Q(R, S)$. An element a_{ij} is called an invariant element if it is not restricted, and it is neither upward movable nor downward movable. We also call (i, j) an invariant position of the class $\mathfrak{A}_P^Q(R, S)$ if a_{ij} is an invariant element.

DEFINITION 4.3 (Incidence Graphs). Let $A = (a_{ij})$ be a matrix in the class $\mathfrak{A}_P^Q(R, S)$. We associate with A a directed bipartite graph $G(A) = (X, Y, E)$, where

1. $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$.
2. There are $a_{ij} - p_{ij}$ arcs from x_i to y_j and $q_{ij} - a_{ij}$ arcs from y_j to x_i.

We shall call $G(A)$ the incidence graph of A with respect to the class $\mathfrak{A}_P^Q(R, S)$.

It is straightforward to verify the following basic properties of an incidence graph $G(A)$ of a matrix in $\mathfrak{A}_P^Q(R, S)$. Let $(a_1, a_2, ..., a_m)$ and $(b_1, b_2, ..., b_m)$ be the row sum vectors of P and Q. Then the outdegree of x_i is $d^+(x_i) = r_i - a_i$ and the indegree of x_i is $d^-(x_i) = b_i - r_i$. Similarly, we have $d^+(y_j) = d_j - s_j$ and $d^-(y_j) = s_j - c_j$; recall that c_j and d_j are the jth column sums of P and Q. Thus, the outdegrees and indegrees of the incidence graph $G(A)$ are independent of the choice of the matrix A. Moreover, given a class $\mathfrak{A}_P^Q(R, S)$, a matrix A can be recovered from its incidence graph $G(A)$. For a matrix $A \in \mathfrak{A}_P^Q(R, S)$, a_{ij} is an upper bound if there is no arc from y_j to x_i in $G(A)$, and a_{ij} is a lower bound if there is no arc from x_i to y_j, and a_{ij} is restricted if there is no arc between x_i and y_j.

From now on, by a path or cycle, we mean a directed path or cycle which does not contain repeated arcs, but it may contain the same vertex more than once, and by an elementary path or cycle we mean a directed path or cycle which does not contain any vertex more than once.

THEOREM 4.4. *Let A and B be two matrices in $\mathfrak{A}_P^Q(R, S)$. Then $G(A)$ can be expressed as*

$$G(A) = G_0 \cup C_1 \cup C_2 \cup \cdots \cup C_t \tag{4.1}$$

where C_i's are edge-disjoint elementary directed cycles of length greater than 2 and $G(B)$ can be obtained from $G(A)$ by reversing all the arcs in $C_1, C_2, ..., C_t$.

Proof. Let $G(A) = (X, Y, E)$, and let G_0 be the bipartite digraph with vertex set $X \cup Y$ and with the common arcs of $G(A)$ and $G(B)$. Let $G^*(A)$ and $G^*(B)$ be the digraphs obtained from $G(A)$ and $G(B)$ by removing the arcs in G_0. It is easy to see that there are $q_{ij} - p_{ij}$ arcs (regardless of direction) between x_i and y_i in both $G(A)$ and $G(B)$. After we remove all the common arcs between x_i and y_j in $G(A)$ and $G(B)$, the remaining arcs must be in the same direction between any two vertices in $G(A)$ and $G(B)$. Thus, $G^*(A)$ and $G^*(B)$ do not contain any 2-cycle. Moreover, it is clear that the remaining arcs between x_i and y_j in $G(A)$ are of the opposite direction to those remaining arcs in $G(B)$. For any vertex $v \in X \cup Y$, the outdegree and indegree of v in $G(A)$ are the same as those in $G(B)$, therefore, the outdegree and indegree of v in $G^*(A)$ are the same as those in $G^*(B)$. Since $G^*(B)$ is obtained from $G^*(A)$ by reversing all the arcs, it follows that the outdegree of v in $G^*(A)$ equals its indegree in $G^*(A)$, namely $G^*(A)$ is an Eulerian digraph or a balanced graph (see [21]). Since every Eulerian digraph can be decomposed into edge-disjoint elementary cycles, this completes the proof. ∎

Like the chordless cycle interchange defined in [2], we define a chordless cycle in an incidence graph as an elementary cycle in which there is no arc between any two nonadjacent vertices. By Theorem 4.4, we may easily prove the following digraph version of the interchange theorem of Anstee.

COROLLARY 4.5 (Anstee [2]). *Let A and B be two matrices in $\mathfrak{A}_p^Q(R, S)$. Then the incidence graph $G(B)$ can be obtained from the incidence graph $G(A)$ by a sequence of operations of reversing chordless cycles.*

Proof. By Theorem 4.4, we may express $G(A)$ in the form of (4.1). Clearly, we only need to show that we can transform $G(A)$ to $G_0 \cup C_1 \cup C_2 \cup \cdots \cup C_{t-1} \cup C_t'$, where C_t' is the reverse of C_t, by a sequence of operations of reversing chordless cycles. We shall use induction on the length of C_t. While C_t is a 4-cycle, it is chordless. Suppose the assertion is true for any cycle of length less than $2k$. When C_t contains $2k$ vertices and it is not chordless, then there exists an arc $e = (v_i, v_j)$ between two nonadjacent vertices of C_t. Since e and the directed path from v_j to v_i on C_t form a cycle of smaller length, by induction, we may reverse this cycle and leave all other arcs unchanged by reversing some chordless cycles. As a result, the arc e becomes (v_j, v_i). Since (v_j, v_i) and the path from v_i to v_j on C_t form a cycle of smaller length, we may reverse this cycle by reversing some chordless cycles. Obviously, the arc (v_j, v_i) will change back to (v_i, v_j). The proof is complete by induction. ∎

The following corollary of Theorem 4.4 gives a characterization of movable elements in terms of incidence graphs.

COROLLARY 4.6. *Let A be a matrix in $\mathfrak{A}_p^Q(R, S)$. Then the following statements are equivalent:*

1. *a_{ij} is neither an upper bound nor upward movable.*

2. *In $G(A)$, no arc from y_j to x_i is contained in an elementary cycle of length greater than 2.*

3. *Let G_{ij} be the graph obtained from $G(A)$ by removing all the arcs from x_i to y_j. Then no arc from y_j to x_i belongs to a strongly connected component of G_{ij}.*

When P and Q are $(0, 1)$-matrices, an incidence graph does not contain any 2-cycle. Therefore, we may have the following simple characterization of invariant positions in such a class $\mathfrak{A}_P^Q(R, S)$ of $(0, 1)$-matrices.

COROLLARY 4.7. *Let P and Q be $(0, 1)$-matrices satisfying $P \leqslant Q$. Let A be a matrix in $\mathfrak{A}_P^Q(R, S)$. Then $\mathfrak{A}_P^Q(R, S)$ has no invariant position if and only if $G(A)$ is a vertex-disjoint union of strongly connected digraphs.*

For a class $\mathfrak{A}(R, S)$ of $(0, 1)$-matrices, the above corollary reduces to the fact that the class $\mathfrak{A}(R, S)$ has no invariant position if and only if there exists a strongly connected bipartite tournament with score lists R and $\bar{S} = (m - s_1, m - s_2, ..., m - s_n)$. This fact can be derived from the two equivalent characterizations in terms of R and S. The following theorem gives a matrix characterization of movableness which displays a connection between movableness and decomposability studied in the previous section.

THEOREM 4.8. *Let A be a matrix in $\mathfrak{A}_P^Q(R, S)$. Then we have*

1. *Suppose a_{ij} is not a lower bound. Then a_{ij} is not downward movable if and only if there exist $I \subseteq [m]$, $J \subseteq [n]$ such that $i \in I$, $j \in J$, and every element in $A[I, J]$ possibly except a_{ij} is an upper bound and every element in $A[\bar{I}, \bar{J}]$ is a lower bound.*

2. *Suppose a_{ij} is not an upper bound. Then a_{ij} is not upward movable if and only if there exist $I \subseteq [m]$, $J \subseteq [n]$ such that $i \in I$, $j \in J$ and every element in $A[I, J]$ possibly except a_{ij} is a lower bound and every element in $A[\bar{I}, \bar{J}]$ is an upper bound.*

In the following proof of Theorem 4.8, we shall use a corollary of Minty's Arc Coloring Lemma. Let $G = (V, E)$ be a digraph allowing multiple arcs, and S be a subset V. The set of all arcs joining a vertex in S and a vertex in $V \backslash S$ is called a cocircuit of G, denoted $\omega(S)$. Let e be any arc of G. From Minty's Lemma [7, p. 14], it follows that either e is contained in an elementary cycle, or e is contained in a cocircuit in which all the arcs have the same direction.

Proof of Theorem 4.8. Since 1 and 2 are dual to each other, we only give the proof of 1. The sufficiency of 1 is obvious. Let us prove the necessity. Suppose a_{ij} is neither a lower bound nor downward movable. Let $G(A)$ be the incidence graph of A and G'_{ij} be the graph obtained from $G(A)$ by removing all arcs from y_j to x_i (if there are any). Since a_{ij} is not a lower bound, there exists at least one arc from x_i to y_j. By Corollary 4.6, it follows that G'_{ij} is not strongly connected and no elementary cycle in G'_{ij} contains an arc from x_i to y_i. Therefore, by Minty's Lemma, there exists a cocircuit in G'_{ij} containing all arcs from x_i to y_j in which all arcs have the same direction. Let $\omega(X_1 \cup Y_1)$ be such a cocircuit, where $X_1 \subseteq X$ and $Y_1 \subseteq Y$. Let I be the index set of X_1 and J be the index set of $Y \setminus Y_1$. Since all the arcs in $\omega(X_1 \cup Y_1)$ have the same direction, it follows that every element in $A[I, J]$ except possibly for a_{ij} is an upper bound, and every element in $A[\bar{I}, \bar{J}]$ is a lower bound. ∎

If P and Q are (0, 1)-matrices, then there is at most one arc between any two vertices x_i and y_j in an incidence graph; namely, any incidence graph is a simple digraph. Thus, if $\mathfrak{A}_P^Q(R, S)$ has an invariant position, then it must be decomposable. Moveover, it is clear that the above corollary reduces to Ryser's theorem [9, Theorem 6.2] on the decomposition of classes of (0, 1)-matrices with invariant positions. We now give a very simple labeling algorithm to determine if an element in a matrix A is upward movable or downward movable in an integral class $\mathfrak{A}_P^Q(R, S)$. This algorithm is analogous to Minty's Lemma, so we omit its proof. We only give the algorithm for testing downward movableness, since the algorithm for testing upward movableness is just the dual.

ALGORITHM. Let $A = (a_{ij})$ be a matrix in a class $\mathfrak{A}_P^Q(R, S)$. Let a_{ij} be an element which is not a lower bound.

1. Label the ith row of A with the symbol $*$.

2. If the kth row of A has already been labeled and there exists a column, say the lth column, such that a_{kl} is not an upper bound and $(k, l) \neq (i, j)$, then label the lth column of A with the symbol $*$.

3. If the lth column of A has been labeled and there exists an element a_{kl} which is not a lower bound, then label the kth row of A with the symbol $*$.

Repeat the above procedure until we cannot label any more rows or columns. If the jth column eventually gets labeled, then a_{ij} is downward movable; otherwise, a_{ij} is not downward movable. Let I be the index set of labeled rows and J be the index set of unlabeled columns. Then I and J are just the desired index sets in the first part of Theorem 4.8.

We now turn to invariant sets of a class of (0, 1)-matrices which have been studied by Brualdi and Ross [11], and Anstee [3]. By the characterization of invariant positions in terms of strongly connected incidence graphs, we give a very short graph theoretic proof of a theorem of Brualdi and Ross [11] concerning invariant sets.

DEFINITION 4.9 (Invariant Sets). Let $\mathfrak{A}(R, S)$ be a nonempty class of (0, 1)-matrices. An invariant set of $\mathfrak{A}(R, S)$ is a pair (I, J) such that $I \subseteq [m]$, $J \subseteq [n]$ and for any $A \in \mathfrak{A}(R, S)$ the row and column sums of $A[I, J]$ do not depend on the choice of A. An invariant set (I, J) is said nontrivial if $(I, J) \neq (\emptyset, \emptyset)$, $([m], [n])$.

THEOREM 4.10 (Brualdi and Ross [11]). *Let $\mathfrak{A}(R, S)$ be a nonempty class of* (0, 1)*-matrices. Then $\mathfrak{A}(R, S)$ has a nontrivial invariant set if and only if it has an invariant position.*

Proof. The sufficiency of the theorem is obvious. We now prove the necessity. Suppose $\mathfrak{A}(R, S)$ does not have any invariant position, namely, the incidence graph of any matrix in $\mathfrak{A}(R, S)$ is strongly connected. We aim to show that there is no nontrivial invariant set. Let A be a matrix in $\mathfrak{A}(R, S)$ and $G = (X, Y, E)$ be the incidence graph of A. Let $W = X_I \cup Y_J$, where X_I is the subset of X indexed by I and Y_J is the subset of Y indexed by J, I and J satisfy $(I, J) \neq (\emptyset, \emptyset)$, $([m], [n])$. Let H be the induced subgraph of G on the vertex set W and $\bar{H}$ be the directed graph obtained from G by removing all arcs in H (note that $\bar{H}$ is not the induced graph on $(X \cup Y) \setminus W$). If H is not strongly connected, then there is an arc in H contained in an elementary cycle C such that C also contains some arcs in $\bar{H}$. Therefore, there exist two consecutive arcs (W, u) and (u, v) on C such that (W, u) is in H and (u, v) is in $\bar{H}$. Reversing this cycle C, suppose the resulting digraph is the incidence graph of a matrix B. Clearly, we have either $(u, v) = (x_i, y_j)$ or $(u, v) = (y_j, x_i)$. Thus, $A[I, J]$ and $B[I, J]$ either have different row sum vectors or have different column sum vectors. That is, (I, J) is not an invariant set. If $\bar{H}$ is not strongly connected, a similar agument shows that (I, J) cannot be an invariant set. We now consider the case when both H and $\bar{H}$ are strongly connected. Suppose C is any cycle in H, and u, v are two vertices on C. Since $\bar{H}$ is strongly connected and u, v are also vertices in $\bar{H}$, there exists an elementary path P from u to v in $\bar{H}$. If we walk from u along the path P, we shall eventually reach a vertex on C because v is on C. Suppose v' is the first vertex we meet on C. Combining the path from u to v' on P and the path from v' to u on C, we obtain an elementary cycle containing some arcs in H and some arcs in $\bar{H}$. This implies that (I, J) cannot be an invariant set. ∎

Finally, we remark that the above graph theoretic proof of Theorem 4.10 implies an analogous theorem on invariant sets of a class of tournament matrices with given row sums, or a class of tournaments with prescribed score list. The analogy between bipartite tournaments and ordinary tournaments has been known since the analogy between the Gale–Ryser theorem [16, 27] and Landau's theorem [22] on tournament score lists.

Let $R=(r_1, r_2, ..., r_n)$ be a vector of integers. A tournament matrix $A=(a_{ij})$ is a skew symmetric (0, 1)-matrix, i.e., for any i and j, $a_{ij}=-a_{ji}$. A tournament matrix can be regarded as the adjacency matrix of a tournament. Let $\mathfrak{T}(R)$ be the set of all tournament matrices with row sum vector R. An *invariant position* (i, j) of $\mathfrak{T}(R)$ is such a position that for any $A \in \mathfrak{T}(R)$, a_{ij} remains invariant. A subset $I \subseteq [n]$ is said to be an *invariant set* if for any $A \in \mathfrak{T}(R)$, the row sum vector of $A[I, I]$ is independent of the choice of A, and I is said to be nontrivial if $I \neq \varnothing$, $[n]$. Then we have the following analog of Theorem 4.10.

THEOREM 4.11. *Let $\mathfrak{T}(R)$ be a nonempty class of tournament matrices with row sum vector R. Then it has a nontrivial invariant set if and only if it has an invariant position.*

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