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Recursive method to solve the problem of "Gambling with God"

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ABSTRACT

Suppose Alice gambles with God who is the dealer. There are n total rounds in the game and God can choose any m rounds to win and the other $n - m$ rounds to lose. At first Alice has holdings a. In each round, Alice can increase her holdings by q times the amount she wagers if she wins. So what strategy should Alice take to ensure the maximum total holdings in the end? And how much is the total final holdings? It is called the "Gambling with God" problem. In this paper, a recursive method is proposed to solve the problem, which shows the extensive application of recursive methods.

1. Introduction

Suppose Alice gambles with God who is the dealer. There are n total rounds in the game and God can choose any m rounds to win and the other $n - m$ rounds to lose. God's goal is to minimize Alice's winnings. Alice has holdings a at the beginning. In each round, Alice can increase her holdings by q times the amount she wagers if she wins. So what strategy should Alice take to ensure the maximum total holdings when the game is over? And how much is the total final holdings? This problem is called the "Gambling with God" problem. It was first proposed in an old American newspaper in a simpler case of $q = 1$, $a = 100$, $n = 10$ and $m = 1$. The newspaper's answer was 9309. Unfortunately, it was not easy to find the newspaper again. In this paper we generalize the original problem to the above version with general q, n and m.

2. Main result

In order to understand the problem quickly, we consider the simplest case of $n = 1$ and $m = 0$. Alice gambles with God for only one round and will definitely win. Then Alice should wager all her holdings a. Because the odds are q to 1, Alice will gain $(q + 1)a$ as the final holdings. In general, if $m = 0$, for any $n \geq 1$, Alice will gamble with God for n rounds and never lose. Then Alice will wager all she has in each round. So her final holdings would be $(q + 1)^n a$.

Consider the case of $m = 1$, which means that God can choose any one round to win. We have the following theorem.

Theorem 1. *When $m = 1$, if Alice gambles with God for n rounds, the best way for Alice to win as much as she can is to wager $\frac{(n-1)q}{nq+1}a$ in the first round, and the final holdings would be $\frac{(q+1)^n}{nq+1}a$.*

Proof. We prove it by induction.

First we consider the simplest case of $n = 1$. Alice only gambles one round with God and knows she would lose it definitely. Then she will wager $0 = \frac{(1-1)q}{q+1}a$. Her final holdings would be $a = \frac{(q+1)^1}{q+1}a$. The result is true.

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Now assume that the assertion is true when $n = k$. Then consider the case of $n = k + 1$. Alice will gamble $k + 1$ rounds with God and God will win only one round. Suppose Alice wagers x in the first round. There will be two possibilities:

1. Alice wins the first round. Then Alice has the holdings $a + qx$. There are k rounds later and God will only win one round. So Alice will gain the final holdings $\frac{(q+1)^k}{kq+1}(a + qx)$ according to the induction hypothesis.
2. Alice loses the first round. Then Alice has the holdings $a - x$. There are k rounds later and God will lose each round. So Alice will wager all she has for each round and gain the final holdings $(q + 1)^k(a - x)$.

In order to gain the biggest holdings, the following equality should hold,

$$\frac{(q+1)^k}{kq+1}(a+qx) = (q+1)^k(a-x).$$

Solving this equation, we get $x = \frac{kq}{(k+1)q+1}a$. The final holdings will be

$$(q+1)^k(a-x) = \frac{(q+1)^{k+1}}{(k+1)q+1}a.$$

The result is true, which, from the principle of induction, proves the theorem. □

For the case of $m = 2$, the result will be more complex.

Theorem 2. *When $m = 2$, if Alice gambles with God for $n \geq 2$ rounds, the best way for Alice to win as much as she can is to wager $\frac{\binom{n-1}{2}q}{\binom{n}{2}q^2+nq+1}a$ in the first round. The final holdings would be $\frac{(q+1)^n}{\binom{n}{2}q^2+nq+1}a$.*

Proof. We will argue by induction too.

This result is trivial for $n = 2$. Assume the assertion is true when $n = k$. Then consider the case of $n = k + 1$. Alice will gamble $k + 1$ rounds with God and God will win only two rounds. Suppose Alice wagers x in the first round. There are two possibilities:

1. Alice wins the first round. Then Alice has the holdings $a + qx$. There are k rounds later and God will win two of them. So Alice will gain the final holdings $\frac{(q+1)^k}{\binom{k}{2}q^2+kq+1}(a + qx)$ according to the induction hypothesis.
2. Alice loses the first round. Then Alice has the holdings $a - x$. There are k rounds later and God will win only one round. According to Theorem 1, Alice will gain the final holdings $\frac{(q+1)^k}{kq+1}(a - x)$.

In order to gain the biggest final holdings, the following equality should hold,

$$\frac{(q+1)^k}{\binom{k}{2}q^2+kq+1}(a+qx) = \frac{(q+1)^k}{kq+1}(a-x).$$

Solving this equation, we get

$$x = \frac{\binom{k}{2}q^2}{\binom{k+1}{2}q^2+(k+1)q+1}a.$$

The final holdings will be

$$\frac{(q+1)^k}{kq+1}(a-x) = \frac{(q+1)^{k+1}}{\binom{k+1}{2}q^2+(k+1)q+1}a.$$

The result is true, which completes the proof. □

For the general m, the problem becomes complicated. To simplify the discussion, we first denote some symbols. Suppose Alice will gamble with God for n rounds and God can choose any m rounds to win and the other $n - m$ rounds to lose. Let $f(n, m)$, $n \geq m \geq 0$ represent the ratio between the holdings Alice will wager in the first round and her original holdings before the gambling. Let $S(n, m)$, $n \geq m \geq 0$ represent the ratio between the final holdings of Alice and her original holdings. It is easy to see that

$$f(n,0) = 1, \qquad S(n,0) = (q+1)^n.$$

From Theorems 1 and 2, we can know

$$f(n,1)=\frac{(n-1)q}{nq+1},\qquad S(n,1)=\frac{(q+1)^n}{nq+1}.$$

$$f(n,2)=\frac{\binom{n-1}{2}q}{\binom{n}{2}q^2+nq+1},\qquad S(n,2)=\frac{(q+1)^n}{\binom{n}{2}q^2+nq+1}.$$

For general m, we can analyze similarly to the proof of the Theorems 1 and 2. Suppose that Alice wagers $f(n,m)a$ in the first round. There will be two possibilities:

1. Alice wins the first round. Then Alice has the holdings $a+qf(n,m)a=(1+qf(n,m))a$. There are $n-1$ rounds later and God will win m rounds of them. So Alice will gain the final holdings $S(n-1,m)(1+qf(n,m))a$.
2. Alice loses the first round. Then Alice has the holdings $a-f(n,m)a=(1-f(n,m))a$. There are $n-1$ rounds later and God will win $m-1$ rounds. So Alice will gain the final holdings $S(n-1,m-1)(1-f(n,m))a$.

In order to gain the biggest final holdings (i.e., $S(n,m)a$), the following equalities should hold

$$S(n-1,m)(1+qf(n,m))a=S(n-1,m-1)(1-f(n,m))a=S(n,m)a.$$

That is

$$S(n,m)=S(n-1,m)(1+qf(n,m))=S(n-1,m-1)(1-f(n,m)). \tag{1}$$

From the latter equality, we can get

$$f(n,m)=\frac{S(n-1,m-1)-S(n-1,m)}{S(n-1,m-1)+qS(n-1,m)}. \tag{2}$$

Substituting it into (1), we can get

$$S(n,m)=\frac{(q+1)S(n-1,m-1)S(n-1,m)}{S(n-1,m-1)+qS(n-1,m)}. \tag{3}$$

So now we obtain the recursive formula of $S(n,m)$.

In fact, we can generalize $S(n,m)$ with $0\le n<m$. In those cases, Alice will gamble with God for n rounds but lose $m>n$ rounds, i.e., Alice will never win. It is obvious that Alice will wager 0 each time. So her final holdings will be equal to her original holdings, which means $S(n,m)=1$, $0\le n<m$.

If we denote $T(n,m)=\frac{(q+1)^n}{S(n,m)}$, (3) can be modified as

$$T(n,m)=T(n-1,m)+qT(n-1,m-1),\quad n>m\ge 1. \tag{4}$$

It is easy to see that $T(n,0)\equiv 1$ because $S(n,0)=(q+1)^n$. And $T(n,m)=(q+1)^m$ when $n\le m$ because $S(n,m)=1$, $n\le m$. Then we can easily verify that (4) holds for any $n\ge 0$ and $m\ge 0$ instead of the restriction of $n>m\ge 1$.

We need to note that the recursive formula (4) is used often, especially in the papers discussing on the Pascal matrix [2,3,7,5,4,1].

From the above recursive formula, we could obtain the expression of $T(n,m)$.

Theorem 3.

$$T(n,m)=\sum_{k=0}^{m}\binom{n}{k}q^k. \tag{5}$$

Proof. It is not difficult to prove it by induction on m. But here we use the generating function method. For the readers being interested in this method, see [6, chap. 5].

Consider the generating function $g(x,y)$ of $T(n,m)$, i.e.,

$$g(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T(n,m)x^ny^m.$$

Then we can get

$$\begin{aligned}g(x,y)&=\sum_{m=0}^{\infty}T(0,m)y^m+\sum_{n=1}^{\infty}T(n,0)x^n+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[T(n-1,m)+qT(n-1,m-1)]x^ny^m\\&=\sum_{m=0}^{\infty}y^m+\sum_{n=1}^{\infty}T(n-1,0)x^n+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T(n-1,m)x^ny^m+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}qT(n-1,m-1)x^ny^m\end{aligned}$$

$$= \frac{1}{1-y} + x\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(n,m)x^n y^m + qxy\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(n,m)x^n y^m$$
$$= \frac{1}{1-y} + xg(x,y) + qxyg(x,y).$$

So

$$\begin{aligned} g(x,y) &= \frac{1}{(1-x-qxy)(1-y)} = \sum_{n=0}^{\infty}(x+qxy)^n \sum_{r=0}^{\infty} y^r \\ &= \sum_{n=0}^{\infty} x^n(1+qy)^n \sum_{r=0}^{\infty} y^r = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{n}{k} q^k y^k \sum_{r=0}^{\infty} y^r \\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\sum_{r=0}^{\infty}\binom{n}{k} q^k y^{k+r} = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{n}{k} q^k y^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\binom{n}{k} q^k\right] x^n y^m. \end{aligned}$$

By the theory of generating function, the coefficient of $x^n y^m$ in $g(x,y)$ is

$$T(n,m) = \sum_{k=0}^{m}\binom{n}{k} q^k.$$

So we complete the proof. □

From (5) we know that $S(n,m) = \frac{(q+1)^n}{\sum_{k=0}^{m}\binom{n}{k}q^k}$. Then from (1) or (2) we can get

$$f(n,m) = \frac{\binom{n-1}{m} q^m}{\sum_{k=0}^{m}\binom{n}{k} q^k}.$$

Thus we have the final result.

Theorem 4. *Suppose Alice gambles with God for n rounds with the odds being q to* 1*, and God can choose any m rounds to win and the other n − m rounds to lose. If Alice has the original holdings a, the best way for Alice to win is to wager* $f(r,s)w = \frac{\binom{r-1}{s}q^s}{\sum_{k=0}^{s}\binom{r}{k}q^k} w$ *in each round, where r represents the number of rounds left, s represents the number of rounds that God will win in the r rounds, and w represents the total holdings Alice has when she begins this round. The final holdings Alice could gain would be* $S(n,m)a = \frac{(q+1)^n}{\sum_{k=0}^{m}\binom{n}{k}q^k} a$.

We need to note here that if Alice does not gamble with this strategy, God could make her final holdings less than $S(n,m)a$. The method is that God chooses to win when Alice wagers more than $f(r,s)w$ in some round, or God chooses to lose when Alice wagers less than $f(r,s)w$ in some round.

If we let $q = 1$, this means that, Alice will gain two times her wager in each round if she win. Then we have the special result of $f(n,m)$ and $S(n,m)$.

Theorem 5. *When the odds are* 1 *to* 1*,*

$$f(n,m) = \frac{\binom{n-1}{m}}{\sum_{k=0}^{m}\binom{n}{k}},$$

$$S(n,m) = \frac{2^n}{\sum_{k=0}^{m}\binom{n}{k}}. \tag{6}$$

Substituting $n = 10$ and $m = 1$ into (6), we get $S(10,1) \approx 93.09$. So if the original holdings is 100, the final holdings will be $S(10,1) \times 100 \approx 9309$, which is exact the answer of the newspaper.

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