Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of
MATHEMATICAL

www.elsevier.com/locate/jmaa

Weak linear bilevel programming problems: existence of solutions via a penalty method

Abdelmalek Aboussoror [a,*], Abdelatif Mansouri [b]

[a] Centre des Etudes Universitaires de SAFI, Université Cadi Ayyad, Sidi Bouzid, B.P. 4162, SAFI, Morocco
[b] Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, MAD, B.P. 2390, Marrakech, Morocco

Received 19 May 2004

Available online 27 January 2005

Submitted by J.P. Dauer

Abstract

We are concerned with a class of weak linear bilevel programs with nonunique lower level solutions. For such problems, we give via an exact penalty method an existence theorem of solutions. Then, we propose an algorithm.

Keywords: Bilevel programming; Linear programming; Duality; Penalty methods

1. Introduction

Let us consider the following weak linear bilevel programming problem:

$$(S):\quad \operatorname*{Min}_{\substack{x\in X\\ x\geqslant 0}} \sup_{y\in\mathcal{M}(x)} F(x,y)=c^T x+d_1^T y,$$

* Corresponding author.
E-mail address: aboussororabdel@hotmail.com (A. Aboussoror).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.09.033

where $\mathcal{M}(x)$ is the set of solutions to the lower level problem

$$\mathcal{P}(x): \quad \operatorname*{Min}_{\substack{y\in\mathbb{R}^m_+ \\ Ax+By\leqslant b}} \quad f(x,y)=d_2^T y$$

with $c\in\mathbb{R}^n$, $d_1,d_2\in\mathbb{R}^m$, $b\in\mathbb{R}^p$, $A\in\mathbb{R}^{p\times n}$, $B\in\mathbb{R}^{p\times m}$, X is a closed subset of $\mathbb{R}^n$, and T stands for transpose. Set

$$X^+=\{x\in X \mid x\geqslant 0\},$$

and for $x\in X^+$,

$$Y(x)=\left\{y\in\mathbb{R}^m_+ \mid By\leqslant b-Ax\right\}.$$

The problem (S) called also a weak linear Stackelberg problem, corresponds to a static uncooperative two player game, where a leader plays against a follower. The leader knowing the objective function f and the constraints of the follower, selects first a strategy x in X^+, in order to minimize his objective function F. Then, for this announced strategy, the follower reacts optimally by selecting a strategy y in $Y(x)$. The formulation of the problem that we consider is called a pessimistic formulation. It corresponds to the case where the solution set $\mathcal{M}(x)$ is not always a singleton, and the leader provides himself against the possible worst choice of the follower in $\mathcal{M}(x)$. So, he minimizes the function $\operatorname{Sup}_{y\in\mathcal{M}(x)} F(x,y)$.

Note that several papers have been devoted to weak bilevel problems dealing with different subjects (existence of solutions, approximation, regularization...); we cite, for example, [1–5]. The reader is also referred to the annotated bibliography on bilevel optimization given in [6,7].

As is well known, weak bilevel programming problems are difficult to solve on both the theoretical and the numerical aspects. In this paper, for the problem (S), we will give an existence theorem of solutions via an exact penalty method. This penalty method that we present is inspired from [8,9], where the authors consider a strong linear bilevel programming problem. Finally, we give an algorithm. In [9], White and Anandalingam developed a penalty function approach that gives global solutions, while in [8], they only obtain local solutions. However, in [9], some trouble have been identified by Campelo et al. [10]. Then, they have given a new resolution of the considered problem under a weaker assumption than the two assumptions used in [9], which one of them is nonvalid.

The paper consists of four sections. In Section 2, we present our penalty method. In Section 3, we give preliminary results and establish our main result (Theorem 3.3) on the existence of solutions to (S). Finally in Section 4, we propose an algorithm.

2. The exact penalty method

The exact penalty method that we will give is based on the use of the duality gap in the lower level. First, remark that in the definition of the objective function f, we have ignored a term of the form $e^T x$, since for a given x, $e^T x$ is a constant in the follower's problem $\mathcal{P}(x)$. Throughout the paper, we assume that the following assumptions are satisfied.

(H_1) For any $x \in X^+$, $Y(x) \neq \emptyset$, and there exists a compact subset Z of $\mathbb{R}^m$, such that $Y(x) \subset Z$.
(H_2) The set X^+ is a polytope.

For $x \in X^+$, set

$$v(x) = \operatorname*{Sup}_{y \in \mathcal{M}(x)} d_1^T y.$$

Then, (S) can be written as

$$\operatorname*{Min}_{x \in X^+} \big[c^T x + v(x)\big].$$

Let $\mathcal{D}(x)$ denote the follower's dual problem of $\mathcal{P}(x)$, i.e.,

$$\mathcal{D}(x): \qquad \operatorname*{Max}_{\substack{z \in \mathbb{R}^p_+ \\ B^T z \geqslant -d_2}} \ (Ax - b)^T z$$

and let

$$\pi(x, y, z) = d_2^T y - (Ax - b)^T z,$$

denote the duality gap.

Remark 2.1. We have that y solves $\mathcal{P}(x)$, and z solves $\mathcal{D}(x)$ if and only if (y, z) is a solution of the following system:

$$\begin{cases} d_2^T y - (Ax - b)^T z = 0, \\ By \leqslant b - Ax, \\ B^T z \geqslant -d_2, \\ y \in \mathbb{R}^m_+, \quad z \in \mathbb{R}^p_+. \end{cases}$$

Thus, $v(x)$ is also the optimal value of the following linear maximization problem

$$\tilde{\mathcal{P}}(x): \quad \begin{cases} \operatorname{Max} d_1^T y, \\ (y, z) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+, \\ \text{subject to} \quad \begin{cases} d_2^T y - (Ax - b)^T z = 0, \\ By \leqslant b - Ax, \\ B^T z \geqslant -d_2. \end{cases} \end{cases}$$

For $k \in \mathbb{R}_+$, we consider the following penalized problem of $\tilde{\mathcal{P}}(x)$, where the nonnegative duality gap is introduced in the objective function of $\tilde{\mathcal{P}}(x)$, by the penalty parameter k,

$$\tilde{\mathcal{P}}_k(x): \quad \begin{cases} \operatorname{Max}\{d_1^T y - k[d_2^T y - (Ax - b)^T z]\}, \\ (y, z) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+, \\ \text{subject to} \quad \begin{cases} By \leqslant b - Ax, \\ B^T z \geqslant -d_2, \end{cases} \end{cases}$$

and denote by $v_k(x)$ its optimal value. Then, the dual problem of $\tilde{\mathcal{P}}_k(x)$ is

$$\tilde{\mathcal{D}}_k(x):\quad \begin{cases} \text{Min}[d_2^T t + (b - Ax)^T u], \\ (t,u) \in \mathbb{R}_+^m \times \mathbb{R}_+^p, \\ \text{subject to} \quad \begin{cases} -B^T u \leqslant kd_2 - d_1, \\ Bt \leqslant k(b - Ax). \end{cases} \end{cases}$$

Under assumption (H_1), we will see later (Lemma 3.1), that for any $x \in X^+$, the problem $\tilde{\mathcal{P}}_k(x)$ has a solution. So, from the theory of linear programming, $v_k(x)$ is the common optimal value of $\tilde{\mathcal{P}}_k(x)$ and $\tilde{\mathcal{D}}_k(x)$. Then, in the first level we get the following intermediate penalized problem:

$$(\tilde{S}_k):\quad \begin{cases} \text{Min}[c^T x + d_2^T t + (b - Ax)^T u], \\ (x,t,u) \in X^+ \times \mathbb{R}_+^m \times \mathbb{R}_+^p, \\ \text{subject to} \quad \begin{cases} -B^T u \leqslant kd_2 - d_1, \\ Bt \leqslant k(b - Ax). \end{cases} \end{cases}$$

Finally, we obtain the following penalized problem of (S):

$$(S_k):\quad \underset{x \in X^+}{\text{Min}} \big[c^T x + v_k(x)\big].$$

3. Preliminaries and main results

In this section, we first give preliminary results and establish our main result on the existence of solutions (Theorem 3.3). Finally, we give an algorithm.

Set

$$\mathcal{Q} = \big\{z \in \mathbb{R}_+^p \mid B^T z \geqslant -d_2\big\},$$

and let $(\tilde{S})$ be the strong bilevel programming problem corresponding to (S), i.e., the problem

$$(\tilde{S}):\quad \underset{x \in X^+}{\text{Min}} \inf_{y \in \mathcal{M}(x)} \big[c^T x + d_1^T y\big].$$

In the sequel, we will work with its equivalent form, i.e., the problem

$$(\bar{S}):\quad \underset{\substack{x \in X^+ \\ y \in \mathcal{M}(x)}}{\text{Min}} \big[c^T x + d_1^T y\big].$$

Let the following assumption which was introduced in [10]:

(H^*) $\mathcal{Q} \neq \emptyset$, and the following relaxed problem of $(\bar{S})$:

$$\underset{\substack{(x,y) \in X^+ \times \mathbb{R}_+^m \\ Ax + By \leqslant b}}{\text{Min}} \big[c^T x + d_1^T y\big]$$

has a solution.

Then, we have the following proposition.

Proposition 3.1. *Let assumption* (H^*) *hold. Then, the problem* $(\bar{S})$ *has a solution.*

See [10] for a proof.

Corollary 3.1. *Let assumptions* (H_1) *and* (H_2) *hold. Then, the problem* $(\bar{S})$ *has a solution.*

Proof. We can easily see that under assumptions (H_1) and (H_2), the assumption (H^*) is satisfied, and the result follows. □

Let us introduce the following notations. For $k \in \mathbb{R}_+$, set

$$\mathcal{U}_k = \left\{u \in \mathbb{R}^p_+ \mid -B^T u \leqslant kd_2 - d_1\right\},$$
$$\mathcal{Z}_k = \left\{(x,t) \in X^+ \times \mathbb{R}^m_+ \mid Bt \leqslant k(b - Ax)\right\},$$

and for $(k,x) \in \mathbb{R}_+ \times X^+$, set

$$\mathcal{Z}_k(x) = \left\{t \in \mathbb{R}^m_+ \mid Bt \leqslant k(b - Ax)\right\}.$$

In the sequel, for a subset $\mathcal{A}$ of $\mathbb{R}^q$, we shall denote by $V(\mathcal{A})$ the set of vertices of $\mathcal{A}$. Set

$$\hat{F}(x,t,u) = c^T x + d_2^T t + (b - Ax)^T u.$$

For $k \in \mathbb{R}_+$, let $\theta_k(.)$ be the marginal function defined on $\mathcal{U}_k$, by

$$\theta_k(u) = \operatorname*{Inf}_{(x,t)\in\mathcal{Z}_k} \hat{F}(x,t,u).$$

Then, we have the following result.

Theorem 3.1. *Let* $k \in \mathbb{R}_+$. *Suppose that assumptions* (H_1) *and* (H_2) *are satisfied. Then, the problem*

$$\operatorname*{Min}_{u\in\mathcal{U}_k} \theta_k(u)$$

has at least one solution in $V(\mathcal{U}_k)$.

Proof. First note that $\theta_k(.)$ is a concave function (see, for example, [11]). Otherwise, using the fact that $v_k(x)$ is the value of $\tilde{\mathcal{P}}_k(x)$ and $\tilde{\mathcal{D}}_k(x)$, we obtain

$$\begin{aligned}
\inf_{u\in\mathcal{U}_k} \theta_k(u) &= \inf_{x\in X^+} \inf_{\substack{t\in\mathcal{Z}_k(x)\\ u\in\mathcal{U}_k}} \left[c^T x + d_2^T t + (b - Ax)^T u\right] \\
&= \inf_{x\in X^+}\Bigg[c^T x + \sup_{\substack{(y,z)\in\mathbb{R}^m_+\times\mathbb{R}^p_+\\ By\leqslant b-Ax\\ B^T z\geqslant -d_2}} \left(d_1^T y - k\pi(x,y,z)\right)\Bigg] \\
&\geqslant \inf_{x\in X^+}\left[c^T x + d_1^T y - k\pi(x,y,z)\right]
\end{aligned}$$

for all $(y,z) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+$, such that $By \leqslant b - Ax$, $B^T z \geqslant -d_2$. In particular, let y^* be a solution of $\mathcal{P}(x)$, and z^* be a solution of $\mathcal{D}(x)$. Then, since $\pi(x, y^*, z^*) = 0$, we get

$$\inf_{u\in\mathcal{U}_k} \theta_k(u) \geqslant \inf_{x\in X^+}\left[c^T x + d_1^T y^*\right] \geqslant \inf_{\substack{x\in X^+\\ y\in\mathcal{M}(x)}} \left[c^T x + d_1^T y\right].$$

Using Corollary 3.1, we deduce that the function $\theta_k(.)$ is bounded from bellow by the finite optimal value of the problem $(\bar{S})$. Since the function $\theta_k(.)$ is concave and the set $\mathcal{U}_k$ is a polyhedron, then result follows by using [11, Corollary 32.3.4]. $\square$

According to the notations introduced above, the problem $(\tilde{S}_k)$ can be written as

$$(\tilde{S}_k):\qquad \operatorname*{Min}_{\substack{(x,t)\in\mathcal{Z}_k\\ u\in\mathcal{U}_k}} \left[c^T x + d_2^T t + (b - Ax)^T u\right].$$

Then, we have the following theorem.

Theorem 3.2. *Let $k \in \mathbb{R}_+$, and let assumptions* (H_1) *and* (H_2) *hold. Then,*

(1) *the problem $(\tilde{S}_k)$ has at least one solution in $V(\mathcal{Z}_k) \times V(\mathcal{U}_k)$,*
(2) *the problem (S_k) has at least one solution in $V(X^+)$.*

Proof. (1) Let $u_k^* \in V(\mathcal{U}_k)$ be a solution to the problem (Theorem 3.1)

$$\operatorname*{Min}_{u\in\mathcal{U}_k} \theta_k(u).$$

We have

$$\begin{aligned}\inf_{(x,t)\in\mathcal{Z}_k}\left[c^T x + d_2^T t + (b - Ax)^T u_k^*\right] &= \inf_{\substack{(x,t)\in\mathcal{Z}_k\\ u\in\mathcal{U}_k}}\left[c^T x + d_2^T t + (b - Ax)^T u\right]\\ &\geqslant \inf_{\substack{x\in X^+\\ y\in\mathcal{M}(x)}}\left[c^T x + d_1^T y\right],\end{aligned}$$

where the last inequality follows from the proof of Theorem 3.1. Then, for the same reasons as in Theorem 3.1, and by applying [11, Corollary 32.3.4], we deduce that the problem

$$\operatorname*{Min}_{(x,t)\in\mathcal{Z}_k} \hat{F}(x, t, u_k^*)$$

has a solution $(x_k^*, t_k^*) \in V(\mathcal{Z}_k)$, and hence $(x_k^*, t_k^*, u_k^*) \in V(\mathcal{Z}_k) \times V(\mathcal{U}_k)$ is a solution of $(\tilde{S}_k)$.

(2) It is obvious that x_k^* which is in $V(X^+)$, solves (S_k). $\square$

Set

$$X^* = \left\{(x, y) \in X^+ \times \mathbb{R}_+^m \mid Ax + By \leqslant b\right\},$$

and define the function

$$g(x, y, z) = d_1^T y - k\left[d_2^T y - (Ax - b)^T z\right] = d_1^T y - k\pi(x, y, z).$$

Then, we have the following lemmas.

Lemma 3.1. *Let $k \in \mathbb{R}_+$, and $x \in X^+$. Assume that assumption* (H_1) *is satisfied. Then, the problem $\tilde{\mathcal{P}}_k(x)$ has a solution in $V(Y(x)) \times V(\mathcal{Q})$.*

Proof. We have

$$\sup_{\substack{(y,z)\in\mathbb{R}_+^m\times\mathbb{R}_+^p\\ By\leqslant b-Ax\\ B^Tz\geqslant -d_2}}\left[d_1^T y-k\pi(x,y,z)\right]\leqslant \sup_{\substack{y\in\mathbb{R}_+^m\\ By\leqslant b-Ax}} d_1^T y=\max_{\substack{y\in\mathbb{R}_+^m\\ By\leqslant b-Ax}} d_1^T y,$$

where the equality follows from the fact that the set $Y(x)$ is a polytope (see assumption (H_1)). That is the function $g(x,.,.)$ is bounded from above. Then, using [11, Corollary 32.3.4], we deduce that the problem $\tilde{\mathcal{P}}_k(x)$ admits a solution (y_k,z_k) in $V(Y(x))\times V(\mathcal{Q})$. □

Lemma 3.2. *Let assumptions* (H_1) *and* (H_2) *hold. Let* (x_k), $x_k\in V(X^+)$, *be a sequence of solutions of the problems* (S_k), $k\in\mathbb{R}_+$. *Then, there exists* $k_1\in\mathbb{R}_+$, *such that for all* $k\geqslant k_1$,

(1) $\pi(x_k,y_k,z_k)=0$, *for all* $(y_k,z_k)\in V(Y(x_k))\times V(\mathcal{Q})$, *solution of* $\tilde{\mathcal{P}}_k(x_k)$,
(2) $v_k(x_k)=v(x_k)$.

Proof. (1) Let $(y_k,z_k)\in V(Y(x_k))\times V(\mathcal{Q})$ be a solution of the problem $\tilde{\mathcal{P}}_k(x_k)$ (Lemma 3.1). Then,

$$v_k(x_k)=d_1^T y_k-k\pi(x_k,y_k,z_k)\geqslant d_1^T y-k\pi(x_k,y,z),\quad \forall(y,z)\in Y(x_k)\times\mathcal{Q}.$$

In particular, let y and z be solutions of $\mathcal{P}(x_k)$ and $\mathcal{D}(x_k)$, respectively. Then, $\pi(x_k,y,z)=0$, and from the above inequality we deduce that

$$0\leqslant\pi(x_k,y_k,z_k)\leqslant\frac{d_1^T(y_k-y)}{k}\leqslant\frac{\|d_1\|_2(\|y_k\|_2+\|y\|_2)}{k},$$

where $\|.\|_2$ denotes the euclidean norm. Since $(y,y_k)\in Y(x_k)\times Y(x_k)\subset Z\times Z$, which is a compact set, there exists $M>0$, such that $\|d_1\|_2(\|y_k\|_2+\|y\|_2)\leqslant M$, and hence

$$0\leqslant\pi(x_k,y_k,z_k)\leqslant\frac{M}{k}.$$

So

$$\lim_{k\to+\infty}\pi(x_k,y_k,z_k)=0.$$

Using the fact that $(x_k,y_k,z_k)\in V(X^*)\times V(\mathcal{Q})$ (because $V(X^*)=V(X^+)\times V(Y(x_k))$), and that $V(X^*)\times V(\mathcal{Q})$ is a finite set, it follows that there exists $k_1\in\mathbb{R}_+$, such that

$$\pi(x_k,y_k,z_k)=0,\quad \forall k\geqslant k_1.$$

(2) We have

$$v_k(x_k)=d_1^T y_k-k\pi(x_k,y_k,z_k)=d_1^T y_k,\quad \text{for all } k\geqslant k_1.$$

Let us show that (y_k,z_k) solves $\tilde{\mathcal{P}}(x_k)$. First, we remark that (y_k,z_k) is a feasible point of $\tilde{\mathcal{P}}(x_k)$. Now, let (y,z) be a feasible point of $\tilde{\mathcal{P}}(x_k)$ and let us show that

$$d_1^T y_k\geqslant d_1^T y.$$

Since (y,z) is also a feasible point of $\tilde{\mathcal{P}}_k(x_k)$, it follows that

$$d_1^T y_k - k\pi(x_k, y_k, z_k) = d_1^T y_k \geqslant d_1^T y - k\pi(x_k, y, z) = d_1^T y, \quad k \geqslant k_1,$$

where the last equality follows from the fact that $\pi(x_k, y, z) = 0$, since (y,z) is a feasible point of $\tilde{\mathcal{P}}(x_k)$. That is (y_k, z_k) solves $\tilde{\mathcal{P}}(x_k)$. Then,

$$v(x_k) = d_1^T y_k = v_k(x_k), \quad \text{for all } k \geqslant k_1. \qquad \square$$

Lemma 3.3. *Let assumptions* (H_1) *and* (H_2) *hold. Then, there exists* $k_2 \in \mathbb{R}_+$*, such that*

$$v_k(x) \leqslant v(x), \quad \forall x \in X^+, \ \forall k \geqslant k_2.$$

Proof. Let $x \in X^+$. For $k \in \mathbb{R}_+$, let $(y_k, z_k) \in V(Y(x)) \times V(\mathcal{Q})$ be a solution of $\tilde{\mathcal{P}}_k(x)$. Then, with a similar arguments as in Lemma 3.2, we can show that there exists $k_2 \in \mathbb{R}_+$, such that

$$\pi(x, y_k, z_k) = 0, \quad \forall k \geqslant k_2.$$

So (y_k, z_k) is a feasible point of $\tilde{\mathcal{P}}(x)$, and

$$v_k(x) = d_1^T y_k - k\pi(x, y_k, z_k) = d_1^T y_k, \quad \forall k \geqslant k_2.$$

Then,

$$v(x) \geqslant d_1^T y_k = v_k(x), \quad \forall k \geqslant k_2. \qquad \square$$

Now, we are able to establish the following theorem which shows that the penalty is exact.

Theorem 3.3. *Let assumptions* (H_1) *and* (H_2) *hold. Let* (x_k)*,* $x_k \in V(X^+)$*, be a sequence of solutions of the problems* (S_k)*,* $k \in \mathbb{R}_+$*. Then, there exists* $k^* \in \mathbb{R}_+$*, such that for all* $k \geqslant k^*$*,* x_k *solves* (S)*.*

Proof. Since x_k ($x_k \in V(X^+)$) is a solution of (S_k) (see Theorem 3.2), we have

$$c^T x_k + v_k(x_k) \leqslant c^T x + v_k(x), \quad \forall x \in X^+.$$

Let $k^* = \max(k_1, k_2)$. Then, by Lemma 3.3, and (2) of Lemma 3.2, for all $k \geqslant k^*$, we obtain

$$c^T x_k + v(x_k) = c^T x_k + v_k(x_k) \leqslant c^T x + v_k(x) \leqslant c^T x + v(x), \quad \forall x \in X^+.$$

That is, for all $k \geqslant k^*$, x_k is a solution of the original problem (S). $\square$

The following theorem and remark will be used for a test of optimality in the algorithm.

Theorem 3.4. *Assume that assumptions* (H_1) *and* (H_2) *are satisfied. Let* $k \in \mathbb{R}_+$*, and* $(u, u') \in \mathcal{U}_k \times \mathcal{U}_k$*. Let* $(x_k(u'), t_k(u'))$ *be a solution to the problem*

$$\min_{(x,t) \in \mathcal{Z}_k} \hat{F}(x, t, u').$$

Then,

$$\theta_k(u) \leqslant \theta_k(u') + \big(b - Ax_k(u')\big)^T (u - u').$$

Proof. We have

$$\theta_k(u') = c^T x_k(u') + d_2^T t_k(u') + \big(b - Ax_k(u')\big)^T u' \tag{3.1}$$

and

$$\theta_k(u) \leqslant c^T x + d_2^T t + (b - Ax)^T u, \quad \forall (x,t) \in \mathcal{Z}_k.$$

Then,

$$\theta_k(u) \leqslant c^T x_k(u') + d_2^T t_k(u') + \big(b - Ax_k(u')\big)^T u. \tag{3.2}$$

From (3.1), we have

$$c^T x_k(u') + d_2^T t_k(u') = \theta_k(u') - \big(b - Ax_k(u')\big)^T u'.$$

Finally, the inequality (3.2) implies that

$$\theta_k(u) \leqslant \theta_k(u') + \big(b - Ax_k(u')\big)^T (u - u'). \qquad \square$$

From Theorem 3.4, we deduce the following remark.

Remark 3.1. Set

$$\alpha_k(u') = \min_{u \in \mathcal{U}_k} \big(b - Ax_k(u')\big)^T (u - u').$$

If $\alpha_k(u') < 0$, then

$$u' \notin \operatorname{argmin}\big\{\theta_k(u)\colon u \in \mathcal{U}_k\big\}.$$

The following algorithm is inspired from the algorithm given in [8].

4. The algorithm

Initialization $i = 0$,

choose $k > 0$ (k large), $u_k^0 \in \mathcal{U}_k$, and $\lambda > 0$.

Iteration $i = 1, 2, \ldots$

(1) Compute $(x_k^i, t_k^i) \in \operatorname{argmin}\{c^T x + d_2^T t + (b - Ax)u_k^i\colon (x,t) \in \mathcal{Z}_k\}$.
(2) Compute $\alpha_k^i = \min\{(b - Ax_k^i)^T (u - u_k^i)\colon u \in \mathcal{U}_k\}$, and a solution $u_{k,i}^*$.

Optimality test

(3) If $\alpha_k^i \geqslant 0$, then, $u_k^i \in \operatorname{argmin}\{\theta_k(u)\colon u \in \mathcal{U}_k\}$, and go to (a).
  (a) Compute a solution $(y(x_k^i), z(x_k^i))$ of the linear program $\tilde{\mathcal{P}}_k(x_k^i)$:
  (a_1) If $\pi(x_k^i, y(x_k^i), z(x_k^i)) = 0$, then x_k^i solves the problem (S),
  (a_2) If $\pi(x_k^i, y(x_k^i), z(x_k^i)) > 0$, put $k \leftarrow k + \lambda$, and go to (1).

(4) If $\alpha_k^i < 0$, set $u_k^{i+1} = u_{k,i}^*$. Then, put $i \leftarrow i + 1$, and go to (1).

In (3), (a_1) of the algorithm, the penalty parameter k is increased by discrete small steps λ.

Remark 4.1. We note that all results remain valid if replace the term $c^T x$ by a concave function $\tilde{g}(x)$.

Acknowledgment

The authors are thankful to one of the referees for his remarks and suggestions that have improved an earlier version of the paper.

References

[1] A. Aboussoror, Weak bilevel programming problems: existence of solutions, Adv. Math. Res. 1 (2002) 83–92.
[2] M.B. Lignola, J. Morgan, Topological existence and stability for Stackelberg problems, J. Optim. Theory Appl. 84 (1995) 145–169.
[3] P. Loridan, J. Morgan, On strict ϵ-solutions for a two-level optimization problem, in: Proceedings of the International Conference on Operations Research, vol. 90, Springer-Verlag, Berlin, 1992, pp. 165–172.
[4] P. Loridan, J. Morgan, Weak via strong Stackelberg problem: new results, J. Global Optim. 8 (1996) 263–287.
[5] R. Lucchetti, F. Mignanego, G. Pieri, Existence theorem of equilibrium points in Stackelberg games with constraints, Optimization 18 (1987) 857–866.
[6] S. Dempe, Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints, Optimization 52 (2003) 333–359.
[7] L.N. Vicente, P.H. Calamai, Bilevel and multi-level programming: a bibliography review, J. Global Optim. 5 (1994) 291–306.
[8] G. Anandalingam, D.J. White, A solution method for the linear static Stackelberg problem using penalty functions, IEEE Trans. Automat. Control 35 (1990) 1170–1173.
[9] D.J. White, G. Anandalingam, A penalty function for solving bi-level linear programs, J. Global Optim. 3 (1993) 397–419.
[10] M. Campelo, S. Dantas, S. Scheimberg, A note on a penalty function approach for solving bi-level linear programs, J. Global Optim. 16 (2000) 245–255.
[11] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton, NJ, 1970.