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On Warnaar's elliptic matrix inversion and Karlsson–Minton-type elliptic hypergeometric series

Hjalmar Rosengren[a,*], Michael Schlosser[b,1]

[a] *Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg SE-412 96, Sweden*
[b] *Institut für Mathematik der Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria*

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Abstract

Using Krattenthaler's operator method, we give a new proof of Warnaar's recent elliptic extension of Krattenthaler's matrix inversion. Further, using a theta function identity closely related to Warnaar's inversion, we derive summation and transformation formulas for elliptic hypergeometric series of Karlsson–Minton type. A special case yields a particular summation that was used by Warnaar to derive quadratic, cubic and quartic transformations for elliptic hypergeometric series. Starting from another theta function identity, we derive yet different summation and transformation formulas for elliptic hypergeometric series of Karlsson–Minton type. These latter identities seem quite unusual and appear to be new already in the trigonometric (i.e., $p=0$) case.

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1. Introduction

Matrix inversions provide a fundamental tool for studying hypergeometric and basic hypergeometric (or q-) series. For instance, they underlie the celebrated Bailey transform [1]. For multiple hyper-

* Corresponding author. Tel.: 46 31 7725358; 46 31 161973.
E-mail addresses: hjalmar@math.chalmers.se (H. Rosengren), schlosse@ap.univie.ac.at (M. Schlosser)
URLs: http://www.math.chalmers.se/~hjalmar, http://www.mat.univie.ac.at/~schlosse.
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geometric series, multidimensional matrix inversions have similarly proved to be a powerful tool, see [2,3,14–18,23–25].

Recently, a new class of generalized hypergeometric series was introduced, the elliptic hypergeometric series of Frenkel and Turaev [6]. In [31], Warnaar found an elliptic matrix inversion and used it to obtain several new quadratic, cubic and quartic summation and transformation formulas for elliptic hypergeometric series.

Warnaar's matrix inversion can be stated as follows [31, Lemma 3.2]. If

$$f_{nk} = \frac{\prod_{j=k}^{n-1} \theta(a_j c_k)\theta(a_j/c_k)}{\prod_{j=k+1}^{n} \theta(c_j c_k)\theta(c_j/c_k)} \tag{1.1a}$$

and

$$g_{kl} = \frac{c_l \theta(a_l c_l)\theta(a_l/c_l)}{c_k \theta(a_k c_k)\theta(a_k/c_k)} \frac{\prod_{j=l+1}^{k} \theta(a_j c_k)\theta(a_j/c_k)}{\prod_{j=l}^{k-1} \theta(c_j c_k)\theta(c_j/c_k)}, \tag{1.1b}$$

then the infinite lower-triangular matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are *inverses* of each other, i.e., the orthogonality relations

$$\sum_{k=l}^{n} f_{nk} g_{kl} = \delta_{nl}, \quad \text{for all } n, l \in \mathbb{Z} \tag{1.2}$$

and (equivalently)

$$\sum_{k=l}^{n} g_{nk} f_{kl} = \delta_{nl}, \quad \text{for all } n, l \in \mathbb{Z} \tag{1.3}$$

hold. In (1.1a) and (1.1b), $\theta(x)$ is the *theta function*, defined by

$$\theta(x) = \theta(x; p) := \prod_{j=0}^{\infty} (1 - xp^j)(1 - p^{j+1}/x),$$

for $|p| < 1$.

Note that $\theta(x)$ reduces to $1 - x$ for $p = 0$. In this case Warnaar's matrix inversion reduces to a result of Krattenthaler [13, Corollary], which in turn generalizes a large number of previously known explicit matrix inversions.

The present paper can be viewed as a spin-off of an attempt to obtain multivariable extensions of Warnaar's matrix inversion and use these to study elliptic hypergeometric series related to classical root systems. This led us to discover several aspects of Warnaar's result which are interesting already in the one-variable case. Multivariable extensions of these ideas are postponed to future publications.

Warnaar's proof of his inversion is based on Eq. (1.3), which is obtained as a special case of a more general identity, the latter being easily proved by induction. This approach seems difficult (though interesting) to generalize to the multivariable case. On the other hand, as was pointed out in [21], the identity (1.2) for Warnaar's inversion is equivalent to a partial fraction-type expansion for theta functions due to Gustafson, (2.2) below. This leads to a short proof of Warnaar's (and thus also Krattenthaler's) matrix inversion, which is described in Section 2.

In another direction, Krattenthaler's proof of the case $p=0$ used a certain "operator method", cf. Lemma 3.1 below. In Section 3 we extend Krattenthaler's proof to the elliptic case. This requires some non-obvious steps, essentially because addition formulas for theta functions are more complicated than those for trigonometric functions implicitly used by Krattenthaler.

We hope that both the elementary proof of Warnaar's inversion given in Section 2 and the operator proof given in Section 3 will be useful for finding multivariable extensions.

Apart from the matrix inversion (1.1), another important tool in Warnaar's paper is the identity [31, Theorem 4.1] (see (1.11) below for the notation), which we write as

$$\sum_{k=0}^{N} \frac{\theta(aq^{2ks})}{\theta(a)} \frac{(a, q^{-Ns}, b, a/b; q^s)_k}{(q^s, aq^{(N+1)s}, aq^s/b, bq^s; q^s)_k} \frac{(cq^N, aq/c; q)_{sk}}{(aq^{1-N}/c, c; q)_{sk}} q^{sk} = \frac{(aq^s, q^s; q^s)_N}{(bq^s, aq^s/b; q^s)_N} \frac{(c/b, bc/a; q)_N}{(c, c/a; q)_N}. \tag{1.4}$$

Here, s is a positive and N a nonnegative integer. In [31], this was obtained by combining (1.2) for the inverse pair (1.1) with a certain bibasic summation. Identity (1.4) was then applied, with $s=2$, 3 and 4, to obtain quadratic, cubic and quartic elliptic hypergeometric identities, respectively.

A characteristic property of (1.4) is that certain quotients of numerator and denominator parameters (such as b over bq^s and cq^N over c) are integral powers of q. Classical and basic hypergeometric series with the analogous property have been called *Karlsson–Minton-type* and (q-)IPD-type (for Integral Parameter Differences) series. A seminal result for such series is Minton's summation formula [19]

$${}_{r+2}F_{r+1}\left(\begin{matrix}-N, b, c_1+m_1, \dots, c_r+m_r \\ b+1, c_1, \dots, c_r\end{matrix}; 1\right) = \frac{N!}{(b+1)_N} \prod_{i=1}^{r} \frac{(c_i-b)_{m_i}}{(c_i)_{m_i}}, \tag{1.5}$$

where it is assumed that m_i are nonnegative integers with $|m| := \sum_i m_i \leqslant N$. This has been extended to nonterminating, bilateral and well-poised series [4,5,7,8,11,27] and further to multiple series [20,22,26]. However, for elliptic hypergeometric series, (1.4) has until now been an isolated result.

At first sight, (1.4) looks somewhat different from known Karlsson–Minton-type identities. However, writing

$$(x; q)_{sk} = (x, xq, \dots, xq^{s-1}; q^s)_k, \tag{1.6}$$

it is not hard to check that it can be obtained as a special case of the more conventional summation formula

$$\sum_{k=0}^{N} \frac{\theta(aq^{2k})}{\theta(a)} \frac{(a, q^{-N}, b, a/b; q)_k}{(q, aq^{N+1}, aq/b, bq; q)_k} q^k \prod_{j=1}^{r} \frac{(c_j q^{m_j}, aq/c_j; q)_k}{(aq^{1-m_j}/c_j, c_j; q)_k} = \frac{(aq, q; q)_N}{(bq, aq/b; q)_N} \prod_{j=1}^{r} \frac{(c_j/b, c_j b/a; q)_{m_j}}{(c_j, c_j/a; q)_{m_j}}, \quad |m| = N \tag{1.7}$$

(with q replaced by q^s). This result will be proved in Section 4. When $p=0$, (1.7) reduces to a special case of an identity of Gasper [8, Eq. (5.13)], which in turn contains (1.5) as a degenerate case.

Gasper's proof of (1.7) in the case $p=0$ does not immediately extend to the elliptic case. A different proof was given by Chu [5], who independently obtained and generalized Gasper's identity by recognizing it as a special case of a partial fraction expansion. In Section 4 we use Chu's method to generalize (1.7) in a different direction, namely, to a multiterm Karlsson–Minton-type transformation, Theorem 4.1. It is obtained as a special case of Gustafson's identity (2.2), or equivalently of (1.2) for Warnaar's inversion. Theorem 4.1 may be viewed as an elliptic analogue of Sears' transformation for well-poised series, cf. Remark 4.4.

In Section 5, we repeat the analysis of Section 4, starting from a different elliptic partial fraction identity, (5.1). This leads to some exotic summation and transformation formulas for Karlsson–Minton-type elliptic hypergeometric series, which appear to be new also when $p = 0$.

Finally, in the Appendix we give an alternative proof of (1.7), using induction on N. We hope that the two proofs we give for this identity will both be useful for finding multivariable extensions of (1.7), and of related quadratic, cubic and quartic identities from [31].

Notation: We have already introduced the theta function $\theta(x)=\theta(x;\ p)$. The nome p is fixed throughout and will be suppressed from the notation. We sometimes write

$$\theta(x_1, \dots, x_n) := \theta(x_1)\cdots\theta(x_n) \tag{1.8}$$

for brevity. We will frequently use the following two properties of theta functions:

$$\theta(x) = -x\,\theta(1/x) \tag{1.9}$$

and the *addition formula*

$$\theta(xy, x/y, uv, u/v) - \theta(xv, x/v, uy, u/y) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u) \tag{1.10}$$

(cf. [32, Example 5, p. 451]).

We denote *elliptic shifted factorials* by

$$(a;q)_k := \theta(a)\theta(aq)\cdots\theta(aq^{k-1}), \tag{1.11a}$$

and write

$$(a_1, \dots, a_n; q)_k := (a_1;q)_k \cdots (a_n;q)_k. \tag{1.11b}$$

These symbols satisfy similar identities as in the case $p = 0$ [9, Appendix I]. In particular, we mention that

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \left(\frac{b}{a}\right)^k \frac{(a;q)_n (q^{1-n}/b;q)_k}{(b;q)_n (q^{1-n}/a;q)_k} \tag{1.12}$$

and

$$\frac{(a;q)_n}{(b;q)_n} = \left(\frac{a}{b}\right)^n \frac{(q^{1-n}/a;q)_n}{(q^{1-n}/b;q)_n}. \tag{1.13}$$

2. Warnaar's matrix inversion and elliptic partial fractions

In this section, we give an easy proof of Warnaar's matrix inversion. Since the case $n = l$ is trivial, it is enough to prove that the left-hand side of (1.2) vanishes for $n > l$. Writing this side out explicitly gives

$$\sum_{k=l}^{n} \frac{\prod_{j=k}^{n-1}\theta(a_jc_k)\theta(a_j/c_k)}{\prod_{j=k+1}^{n}\theta(c_jc_k)\theta(c_j/c_k)} \frac{c_l\theta(a_lc_l)\theta(a_l/c_l)}{c_k\theta(a_kc_k)\theta(a_k/c_k)} \frac{\prod_{j=l+1}^{k}\theta(a_jc_k)\theta(a_j/c_k)}{\prod_{j=l}^{k-1}\theta(c_jc_k)\theta(c_j/c_k)}$$
$$= c_l\theta(a_lc_l)\theta(a_l/c_l)\sum_{k=l}^{n}\frac{1}{c_k}\frac{\prod_{j=l+1}^{n}\theta(a_jc_k)\theta(a_j/c_k)}{\prod_{j=l,\ j\neq k}^{n}\theta(c_jc_k)\theta(c_j/c_k)}.$$

Thus, it is enough to prove that

$$\sum_{k=l}^{n}\frac{1}{c_k}\frac{\prod_{j=l+1}^{n-1}\theta(a_jc_k)\theta(a_j/c_k)}{\prod_{j=l,\ j\neq k}^{n}\theta(c_jc_k)\theta(c_j/c_k)}=0, \quad n>l, \tag{2.1}$$

where (as a matter of relabeling) we may assume $l=1$.

We are now reduced to a theta function identity of Gustafson [10, Lemma 4.14], which we write as

$$\sum_{k=1}^{n}\frac{a_k\prod_{j=1}^{n-2}\theta(a_kb_j)\theta(a_k/b_j)}{\prod_{j=1,\ j\neq k}^{n}\theta(a_ka_j)\theta(a_k/a_j)}=0, \quad n\geqslant 2. \tag{2.2}$$

The case $p=0$ is equivalent to an elementary partial fraction expansion, so we refer to (2.2) as an elliptic partial fraction identity. To identify (2.1) with (2.2) it is enough to replace c_j with a_j, a_j with b_{j-1} and then use (1.9) repeatedly.

Gustafson's proof of (2.2) uses Liouville's theorem and is thus analytic in nature. We refer to [21] for an elementary proof (using only (1.9) and (1.10)), as well as some further comments on this identity.

3. An operator proof of Warnaar's matrix inversion

In [12] Krattenthaler gave a method for solving Lagrange inversion problems, which are closely connected with the problem of inverting lower-triangular matrices. In particular, Krattenthaler applied this method in [13] to derive a very general matrix inversion, namely, the $p=0$ case of (1.1). In the following, we provide a proof of Warnaar's elliptic matrix inversion using Krattenthaler's operator method. Like in Warnaar's proof, the essential ingredient is the addition formula (1.10).

By a *formal Laurent series* we mean a series of the form $\sum_{n\geqslant k}a_nz^n$, for some $k\in\mathbb{Z}$. Given the formal Laurent series $a(z)$ and $b(z)$, we introduce the bilinear form $\langle\ ,\ \rangle$ by

$$\langle a(z),b(z)\rangle=[z^0](a(z)\cdot b(z)),$$

where $[z^0]c(z)$ denotes the coefficient of z^0 in $c(z)$. Given any linear operator L acting on formal Laurent series, L^* denotes the adjoint of L with respect to $\langle\ ,\ \rangle$; i.e. $\langle La(z),b(z)\rangle=\langle a(z),L^*b(z)\rangle$ for all formal Laurent series $a(z)$ and $b(z)$. We need the following special case of [12, Theorem 1].

Lemma 3.1. *Let $F=(f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix with $f_{kk}\neq 0$ for all $k\in\mathbb{Z}$. For $k\in\mathbb{Z}$, define the formal Laurent series $f_k(z)$ and $g_k(z)$ by $f_k(z)=\sum_{n\geqslant k}f_{nk}z^n$ and $g_k(z)=\sum_{l\leqslant k}g_{kl}z^{-l}$,* *where $(g_{kl})_{k,l\in\mathbb{Z}}$ is the uniquely determined inverse matrix of F. Suppose that for $k \in \mathbb{Z}$ a system of equations of the form*

$$Uf_k(z) = w_k V f_k(z) \tag{3.1}$$

holds, where U, V are linear operators acting on formal Laurent series, V being bijective, and $(w_k)_{k\in\mathbb{Z}}$ is an arbitrary sequence of different nonzero constants. Then, if $h_k(z)$ is a solution of the dual system

$$U^* h_k(z) = w_k V^* h_k(z) \tag{3.2}$$

with $h_k(z) \not\equiv 0$ for all $k \in \mathbb{Z}$, the series $g_k(z)$ is given by

$$g_k(z) = \frac{1}{\langle f_k(z), V^* h_k(z)\rangle} V^* h_k(z). \tag{3.3}$$

In order to prove Warnaar's elliptic extension of Krattenthaler's matrix inversion (1.1), we set $f_k(z) = \sum_{n\geqslant k} f_{nk} z^k$ with f_{nk} given as in (1.1a). Obviously, for $n \geqslant k$,

$$\theta(c_n c_k, c_n/c_k) f_{nk} = \theta(a_{n-1}c_k, a_{n-1}/c_k) f_{n-1,k}. \tag{3.4}$$

We now introduce a "multiplier" after which we apply the addition formula for theta functions and separate the variables depending on n and on k appearing in (3.4). Namely, we multiply both sides of (3.4) by $\theta(uv, u/v)$ where u, v are two new auxiliary independent variables, which gives

$$\theta(c_n c_k, c_n/c_k, uv, u/v) f_{nk} = \theta(a_{n-1}c_k, a_{n-1}/c_k, uv, u/v) f_{n-1,k}. \tag{3.5}$$

Next, we apply the addition formula (1.10) to each side of (3.5) and obtain

$$\begin{aligned}&\left[\theta(c_n v, c_n/v, uc_k, u/c_k) + \frac{u}{c_k}\,\theta(vc_k, c_k/v, c_n u, c_n/u)\right] f_{nk}\\ &\quad = \left[\theta(a_{n-1}v, a_{n-1}/v, uc_k, u/c_k) + \frac{u}{c_k}\,\theta(vc_k, c_k/v, a_{n-1}u, a_{n-1}/u)\right] f_{n-1,k}.\end{aligned}$$

If we define the linear operators $\mathscr{A}$ and $\mathscr{C}$ by $\mathscr{A}z^k = a_k z^k$ and $\mathscr{C}z^k = c_k z^k$, for all $k \in \mathbb{Z}$, this may be rewritten in the form

$$\begin{aligned}&\left[\theta(\mathscr{C}v, \mathscr{C}/v, uc_k, u/c_k) + \frac{u}{c_k}\,\theta(vc_k, c_k/v, \mathscr{C}u, \mathscr{C}/u)\right] f_k(z)\\ &\quad = z\left[\theta(\mathscr{A}v, \mathscr{A}/v, uc_k, u/c_k) + \frac{u}{c_k}\,\theta(vc_k, c_k/v, \mathscr{A}u, \mathscr{A}/u)\right] f_k(z),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&[\theta(\mathscr{C}v, \mathscr{C}/v) - z\,\theta(\mathscr{A}v, \mathscr{A}/v)] f_k(z)\\ &\quad = \frac{u\,\theta(vc_k, c_k/v)}{c_k\,\theta(uc_k, u/c_k)}[z\,\theta(\mathscr{A}u, \mathscr{A}/u) - \theta(\mathscr{C}u, \mathscr{C}/u)] f_k(z),\end{aligned} \tag{3.6}$$

valid for all $k \in \mathbb{Z}$.

Equation (3.6) is a system of equations of type (3.1) with

$$U=\theta(\mathscr{C}v,\mathscr{C}/v)-z\,\theta(\mathscr{A}v,\mathscr{A}/v),$$

$$V=z\,\theta(\mathscr{A}u,\mathscr{A}/u)-\theta(\mathscr{C}u,\mathscr{C}/u)$$

and

$$w_k=\frac{u\,\theta(vc_k,c_k/v)}{c_k\,\theta(uc_k,u/c_k)}.$$

The dual equations (3.2) for the auxiliary formal Laurent series $h_k(z)=\sum_{l\leqslant k}h_{kl}z^{-l}$ in this case read

$$\begin{aligned}&[\theta(\mathscr{C}^*v,\mathscr{C}^*/v)-\theta(\mathscr{A}^*v,\mathscr{A}^*/v)\,z]h_k(z)\\&\quad=\frac{u\,\theta(vc_k,c_k/v)}{c_k\,\theta(uc_k,u/c_k)}[\theta(\mathscr{A}^*u,\mathscr{A}^*/u)\,z-\theta(\mathscr{C}^*u,\mathscr{C}^*/u)]h_k(z).\end{aligned}\tag{3.7}$$

Since $\mathscr{A}^*z^{-k}=a_kz^{-k}$ and $\mathscr{C}^*z^{-k}=c_kz^{-k}$, by comparing coefficients of z^{-l} in (3.7) we obtain

$$\begin{aligned}&\left[\theta(c_lv,c_l/v,uc_k,u/c_k)+\frac{u}{c_k}\,\theta(vc_k,c_k/v,c_lu,c_l/u)\right]h_{kl}\\&\quad=\left[\theta(a_lv,a_l/v,uc_k,u/c_k)+\frac{u}{c_k}\,\theta(vc_k,c_k/v,a_lu,a_l/u)\right]h_{k,l+1},\end{aligned}$$

which, after application of the addition formula (1.10) and dividing both sides by $\theta(uv,u/v)$, reduces to

$$\theta(c_lc_k,c_l/c_k)h_{kl}=\theta(a_lc_k,a_l/c_k)h_{k,l+1}.$$

If we set $h_{kk}=1$, we get

$$h_{kl}=\frac{\prod_{j=l}^{k-1}\theta(a_jc_k,a_j/c_k)}{\prod_{j=l}^{k-1}\theta(c_jc_k,c_j/c_k)}.$$

Taking into account (3.3), we compute

$$\begin{aligned}V^*h_k(z)&=[\theta(\mathscr{A}^*u,\mathscr{A}^*/u)\,z-\theta(\mathscr{C}^*u,\mathscr{C}^*/u)]h_k(z)\\&=\sum_{l\leqslant k}\left[\frac{\theta(c_lc_k,c_l/c_k)}{\theta(a_lc_k,a_l/c_k)}\theta(a_lu,a_l/u)-\theta(c_lu,c_l/u)\right]\frac{\prod_{j=l}^{k-1}\theta(a_jc_k,a_j/c_k)}{\prod_{j=l}^{k-1}\theta(c_jc_k,c_j/c_k)}\,z^{-l}\\&=\sum_{l\leqslant k}\theta(c_kv,c_k/v)\frac{a_l\,\theta(a_lc_l,c_l/a_l)}{c_k\,\theta(a_lc_k,a_l/c_k)}\,\frac{\prod_{j=l}^{k-1}\theta(a_jc_k,a_j/c_k)}{\prod_{j=l}^{k-1}\theta(c_jc_k,c_j/c_k)}\,z^{-l},\end{aligned}\tag{3.8}$$

where we again have used the addition formula (1.10). Now, since $f_{kk}=1$, the pairing $\langle f_k(z),V^*h_k(z)\rangle$ is simply the coefficient of z^{-k} in (3.8). Thus, (3.3) reads

$$g_k(z)=-\frac{1}{\theta(c_kv,c_k/v)}V^*h_k(z),\tag{3.9}$$

where $g_k(z)=\sum_{l\leqslant k}g_{kl}z^{-l}$. Hence, extracting coefficients of z^{-l} in (3.9) we obtain exactly (1.1b).

4. Elliptic Karlsson–Minton-type identities

As was mentioned in the introduction, we can obtain a generalization of the Karlsson–Minton-type identities (1.4) and (1.7) as a special case of the partial fraction identity (2.2). To this end, we make the substitutions

$$(a_1,\dots,a_n)\mapsto(a_1,a_1q,\dots,a_1q^{l_1},\dots,a_s,a_sq,\dots,a_sq^{l_s}), \tag{4.1a}$$

$$(b_1,\dots,b_{n-2})$$
$$\mapsto(b_1,b_1q^{1/y_1},\dots,b_1q^{(m_1-1)/y_1},\dots,b_r,b_rq^{1/y_r},\dots,b_rq^{(m_r-1)/y_r}) \tag{4.1b}$$

in (2.2), with m_i and l_i nonnegative and y_i positive integers satisfying

$$|l|+s=|m|+2. \tag{4.2}$$

The resulting special case of (2.2) may be written

$$\sum_{i=1}^{s}\sum_{k=0}^{l_i}\frac{a_iq^k\prod_{j=1}^{r}\prod_{t=0}^{m_j-1}\theta(a_iq^kb_jq^{t/y_j},a_iq^k/b_jq^{t/y_j})}{\prod_{t=0,\,t\neq k}^{l_i}\theta(a_i^2q^{k+t},q^{k-t})\prod_{j=1,\,j\neq i}^{s}\prod_{t=0}^{l_j}\theta(a_iq^ka_jq^t,a_iq^k/a_jq^t)}=0.$$

It is now straightforward to rewrite the products in t in terms of elliptic shifted factorials, giving

$$\prod_{t=0}^{m_j-1}\theta(a_iq^kb_jq^{t/y_j})=(a_ib_jq^k;q^{1/y_j})_{m_j}=(a_ib_j;q^{1/y_j})_{m_j}\frac{(a_ib_jq^{m_j/y_j};q^{1/y_j})_{y_jk}}{(a_ib_j;q^{1/y_j})_{y_jk}},$$

and similarly

$$\prod_{t=0}^{m_j-1}\theta(a_iq^k/b_jq^{t/y_j})=(a_iq^{(1-m_j)/y_j}/b_j;q^{1/y_j})_{m_j}\frac{(a_iq^{1/y_j}/b_j;q^{1/y_j})_{y_jk}}{(a_iq^{(1-m_j)/y_j}/b_j;q^{1/y_j})_{y_jk}},$$

$$\frac{1}{\prod_{t=0,\,t\neq k}^{l_i}\theta(a_i^2q^{k+t})}=\frac{1}{(a_i^2q;q)_{l_i}}\frac{\theta(a_i^2q^{2k})}{\theta(a_i^2)}\frac{(a_i^2;q)_k}{(a_i^2q^{l_i+1};q)_k},$$

$$\frac{1}{\prod_{t=0,\,t\neq k}^{l_i}\theta(q^{k-t})}=\frac{1}{(q^{-l_i};q)_{l_i}}\frac{(q^{-l_i};q)_k}{(q;q)_k},$$

$$\frac{1}{\prod_{t=0}^{l_j}\theta(a_iq^ka_jq^t)}=\frac{1}{(a_ia_j;q)_{l_j+1}}\frac{(a_ia_j;q)_k}{(a_ia_jq^{l_j+1};q)_k},$$

$$\frac{1}{\prod_{t=0}^{l_j}\theta(a_iq^k/a_jq^t)}=\frac{1}{(a_iq^{-l_j}/a_j;q)_{l_j+1}}\frac{(a_iq^{-l_j}/a_j;q)_k}{(a_iq/a_j;q)_k}.$$

We thus arrive at the following result.

Theorem 4.1. *Let $l_1, \dots, l_s$ and $m_1, \dots, m_r$ be nonnegative integers such that $|l| + s = |m| + 2$, and let $y_1, \dots, y_r$ be positive integers. Then the following identity holds*:

$$\sum_{i=1}^{s} \frac{a_i \prod_{j=1}^{r} (a_i b_j, a_i q^{(1-m_j)/y_j}/b_j; q^{1/y_j})_{m_j}}{(a_i^2 q, q^{-l_i}; q)_{l_i} \prod_{j=1,\, j\neq i}^{s} (a_i a_j, a_i q^{-l_j}/a_j; q)_{l_j+1}}$$

$$\times \sum_{k=0}^{l_i} \frac{\theta(a_i^2 q^{2k})}{\theta(a_i^2)} q^k \prod_{j=1}^{s} \frac{(a_i a_j, a_i q^{-l_j}/a_j; q)_k}{(a_i q/a_j, a_i a_j q^{l_j+1}; q)_k}$$

$$\times \prod_{j=1}^{r} \frac{(a_i b_j q^{m_j/y_j}, a_i q^{1/y_j}/b_j; q^{1/y_j})_{y_j k}}{(a_i q^{(1-m_j)/y_j}/b_j, a_i b_j; q^{1/y_j})_{y_j k}} = 0.$$

Remark 4.2. It is clear from the proof that Theorem 4.1 is actually equivalent to its special case when $y_j \equiv 1$. This may be checked directly using (1.6). However, in view of the work of Warnaar [31], the form given above seems more useful for potential application to quadratic and higher identities.

Remark 4.3. In principle, one can obtain an even more general identity by replacing (4.1) with a substitution involving independent bases, that is,

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_1 q_1^{l_1}, \dots, a_s, \dots, a_s q_s^{l_s}),$$

$$(b_1, \dots, b_{n-2}) \mapsto (b_1, \dots, b_1 p_1^{m_1-1}, \dots, b_r, \dots, b_r p_r^{m_r-1}).$$

However, the inner sums in the resulting identity will not be elliptic hypergeometric.

Remark 4.4. In the basic case, $p=0$, Theorem 4.1 may be obtained as a special case of Sears' transformation for well-poised series [28]. More precisely, if we start from the special case given in [9, Exercise 4.7], replace r by $r+s$ and choose the parameters $(b_1, \dots, b_{r+s})$ there as

$$(q^{-l_1}/a_1, \dots, q^{-l_s}/a_s, q^{m_1+1}/a_{s+1}, \dots, q^{m_r+1}/a_{r+s}),$$

we obtain an identity equivalent to the case $p=0$ of Theorem 4.1. This is exactly the case of Sears' transformation when all series involved are terminating, very-well-poised and balanced. Since these restrictions are natural in the elliptic case [29], we may view Theorem 4.1 as an elliptic analogue of Sears' transformation.

For applications, the case $s=2$ of Theorem 4.1 seems especially useful, and we give it explicitly in the following corollary. We have made the substitutions $(a_1, a_2, l_1, l_2, b_j) \mapsto (\sqrt{a}, b/\sqrt{a}, N, L, c_j/\sqrt{a})$ and used (1.13) to simplify some of the factors.

Corollary 4.5. *Let L, N and $m_1, \dots, m_r$ be nonnegative integers with $|m| = L + N$, and let $y_1, \dots, y_r$ be positive integers. Then,*

$$\begin{aligned}
&\sum_{k=0}^{N} \frac{\theta(aq^{2k})}{\theta(a)} \frac{(a, q^{-N}, b, aq^{-L}/b; q)_k}{(q, aq^{N+1}, aq/b, bq^{L+1}; q)_k} q^k \prod_{j=1}^{r} \frac{(c_j q^{m_j/y_j}, aq^{1/y_j}/c_j; q^{1/y_j})_{y_j k}}{(aq^{(1-m_j)/y_j}/c_j, c_j; q^{1/y_j})_{y_j k}} \\
&\quad = \frac{(aq, q; q)_N}{(bq, aq/b; q)_N} \frac{(bq, bq/a; q)_L}{(b^2q/a, q; q)_L} \prod_{j=1}^{r} \frac{(c_j/b, c_j b/a; q^{1/y_j})_{m_j}}{(c_j, c_j/a; q^{1/y_j})_{m_j}} \\
&\quad\quad \times \sum_{k=0}^{L} \frac{\theta(b^2 q^{2k}/a)}{\theta(b^2/a)} \frac{(b^2/a, q^{-L}, b, bq^{-N}/a; q)_k}{(q, q^{L+1}b^2/a, bq/a, bq^{N+1}; q)_k} q^k \\
&\quad\quad \times \prod_{j=1}^{r} \frac{(bc_j q^{m_j/y_j}/a, bq^{1/y_j}/c_j; q^{1/y_j})_{y_j k}}{(bq^{(1-m_j)/y_j}/c_j, bc_j/a; q^{1/y_j})_{y_j k}}.
\end{aligned}$$

If we let $L = 0$ in Corollary 4.5 we obtain the following summation formula.

Corollary 4.6. *Let $y_1, \dots, y_r$ be positive integers and $m_1, \dots, m_r$ be nonnegative integers with $m_1 + \dots + m_r = N$. Then the following identity holds*:

$$\begin{aligned}
&\sum_{k=0}^{N} \frac{\theta(aq^{2k})}{\theta(a)} \frac{(a, q^{-N}, b, a/b; q)_k}{(q, aq^{N+1}, aq/b, bq; q)_k} q^k \prod_{j=1}^{r} \frac{(c_j q^{m_j/y_j}, aq^{1/y_j}/c_j; q^{1/y_j})_{y_j k}}{(aq^{(1-m_j)/y_j}/c_j, c_j; q^{1/y_j})_{y_j k}} \\
&\quad = \frac{(aq, q; q)_N}{(bq, aq/b; q)_N} \prod_{j=1}^{r} \frac{(c_j/b, c_j b/a; q^{1/y_j})_{m_j}}{(c_j, c_j/a; q^{1/y_j})_{m_j}}.
\end{aligned}$$

Note that the case $r = 1$ of Corollary 4.6 is equivalent to (1.4), and that the case $y_j \equiv 1$ is (1.7).

5. Some exotic Karlsson–Minton-type identities

Besides (2.2), we are aware of another elliptic partial fraction expansion, namely,

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{n} \theta(a_k/b_j)}{\prod_{j=1,\, j\neq k}^{n} \theta(a_k/a_j)} = 0, \quad a_1 \cdots a_n = b_1 \cdots b_n, \tag{5.1}$$

which goes back at least to the 1898 treatise of Tannery and Molk [30, p. 34]. Again, we refer to [21] for an elementary proof and some further comments.

It does not seem possible to obtain a matrix inversion from (5.1) in a similar way as Warnaar's inversion was obtained from (2.2) in Section 2. However, it is straightforward to imitate the analysis of Section 4 and obtain Karlsson–Minton-type summation and transformation formulas from (5.1). The resulting identities seem quite exotic and appear to be new even in the case $p = 0$.

Thus, we make the substitutions (4.1) into (5.1). In place of (4.2), we now have the two conditions $|l| + s = |m|$ and

$$q^{\binom{l_1+1}{2}+\cdots+\binom{l_s+1}{2}} a_1^{l_1+1} \cdots a_s^{l_s+1} = q^{\frac{1}{y_1}\binom{m_1}{2}+\cdots+\frac{1}{y_r}\binom{m_r}{2}} b_1^{m_1} \cdots b_r^{m_r}. \tag{5.2}$$

Clearly, the resulting transformation can be obtained from Theorem 4.1 by deleting the factor $a_i q^k$, together with all factors involving products (rather than quotients) of the parameters a_i, b_i. This gives the following result.

Theorem 5.1. *Let $l_1, \dots, l_s$ and $m_1, \dots, m_r$ be nonnegative integers and $y_1, \dots, y_r$ be positive integers. Assume that $|l| + s = |m|$, and that* (5.2) *holds. Then,*

$$\sum_{i=1}^{s} \frac{\prod_{j=1}^{r} (a_i q^{(1-m_j)/y_j}/b_j; q^{1/y_j})_{m_j}}{(q^{-l_i}; q)_{l_i} \prod_{j=1,\, j\neq i}^{s} (a_i q^{-l_j}/a_j; q)_{l_j+1}} \times \sum_{k=0}^{l_i} \prod_{j=1}^{s} \frac{(a_i q^{-l_j}/a_j; q)_k}{(a_i q/a_j; q)_k} \prod_{j=1}^{r} \frac{(a_i q^{1/y_j}/b_j; q^{1/y_j})_{y_j k}}{(a_i q^{(1-m_j)/y_j}/b_j; q^{1/y_j})_{y_j k}} = 0.$$

Next we write down the case $s = 2$ of Theorem 5.1 explicitly. For this we make the substitutions

$$(a_1, a_2, l_1, l_2, b_j) \mapsto (bq^L, 1, N, L, bq^{(Ly_j - m_j + 1)/y_j}/c_j).$$

(Since we may multiply all a_j and b_j in (5.1) with a common factor, the assumption $a_2 = 1$ is no restriction.) This yields that if $|m| = N + L + 2$ and

$$q^{\binom{L+1}{2}} b^{L+1} = q^{\binom{N+1}{2}+\frac{1}{y_1}\binom{m_1}{2}+\cdots+\frac{1}{y_r}\binom{m_r}{2}} c_1^{m_1} \cdots c_r^{m_r}, \tag{5.3}$$

then

$$\frac{\prod_{j=1}^{r} (c_j; q^{1/y_j})_{m_j}}{(q^{-N}; q)_N (b; q)_{L+1}} \sum_{k=0}^{N} \frac{(q^{-N}, b; q)_k}{(q, bq^{L+1}; q)_k} \prod_{j=1}^{r} \frac{(c_j q^{m_j/y_j}; q^{1/y_j})_{y_j k}}{(c_j; q^{1/y_j})_{y_j k}} + \frac{\prod_{j=1}^{r} (q^{-L} c_j/b; q^{1/y_j})_{m_j}}{(q^{-L}; q)_L (q^{-L-N}/b; q)_{N+1}} \sum_{k=0}^{L} \frac{(q^{-L}, q^{-L-N}/b; q)_k}{(q, q^{1-L}/b; q)_k} \times \prod_{j=1}^{r} \frac{(q^{(m_j - Ly_j)/y_j} c_j/b; q^{1/y_j})_{y_j k}}{(q^{-L} c_j/b; q^{1/y_j})_{y_j k}} = 0.$$

To make this look nicer we replace k by $L - k$ in the second sum. After repeated application of (1.12) and some further simplification, we arrive at the following transformation formula.

Corollary 5.2. *Let L, N and $m_1, \dots, m_r$ be nonnegative integers with $|m| = N + L + 2$, and let $y_1, \dots, y_r$ be positive integers. Then, assuming also* (5.3), *one has the identity*

$$\sum_{k=0}^{N} \frac{(q^{-N}, b; q)_k}{(q, bq^{L+1}; q)_k} \prod_{j=1}^{r} \frac{(c_j q^{m_j/y_j}; q^{1/y_j})_{y_j k}}{(c_j; q^{1/y_j})_{y_j k}}$$

$$= b^{N+1} \frac{(q;q)_N (bq;q)_L}{(bq;q)_N (q;q)_L} \prod_{j=1}^{r} \frac{(c_j/b; q^{1/y_j})_{m_j}}{(c_j; q^{1/y_j})_{m_j}}$$

$$\times \sum_{k=0}^{L} \frac{(q^{-L}, b; q)_k}{(q, bq^{N+1}; q)_k} \prod_{j=1}^{r} \frac{(bq^{1/y_j}/c_j; q^{1/y_j})_{y_j k}}{(bq^{(1-m_j)/y_j}/c_j; q^{1/y_j})_{y_j k}}.$$

When $L = 0$, Corollary 5.2 reduces to the following summation formula.

Corollary 5.3. *Let $m_1, \dots, m_r$ and N be nonnegative integers with $|m| = N + 2$, and let $y_1, \dots, y_r$ be positive integers. Then, assuming also*

$$b = q^{\binom{N+1}{2} + \frac{1}{y_1}\binom{m_1}{2} + \cdots + \frac{1}{y_r}\binom{m_r}{2}} c_1^{m_1} \cdots c_r^{m_r},$$

one has the identity

$$\sum_{k=0}^{N} \frac{(q^{-N}, b; q)_k}{(q, bq; q)_k} \prod_{j=1}^{r} \frac{(c_j q^{m_j/y_j}; q^{1/y_j})_{y_j k}}{(c_j; q^{1/y_j})_{y_j k}} = b^{N+1} \frac{(q;q)_N}{(bq;q)_N} \prod_{j=1}^{r} \frac{(c_j/b; q^{1/y_j})_{m_j}}{(c_j; q^{1/y_j})_{m_j}}.$$

The evaluation in Corollary 5.3 looks so unusual that it is worth pointing out that we believe it is free from misprints. In particular, the factor q^k is not missing from the left-hand side. Like for other results in the paper, special cases have been confirmed by numerical calculations.

Using the results of [29] it is easy to check that the sum in Corollary 5.3 is *modular* (that is, invariant under a natural action of $\mathrm{SL}(2, \mathbb{Z})$ on (p, q)-space) and, in particular, *balanced* in the sense of Spiridonov. However, the special case $p = 0$ is not balanced in the usual sense of basic hypergeometric series [9]. This is another indication of the importance of modular invariance and Spiridonov's balancing condition for elliptic hypergeometric series.

Appendix A. An alternative proof of (1.7)

Gasper's proof of the case $p = 0$ of (1.7) uses induction on r. As was remarked in the introduction, this proof does not immediately extend to the general case. However, we were able to find a proof by induction on N, which is different in details from Gasper's proof, but closer to standard methods for basic hypergeometric series [9] than the proof given in Section 4. We include this proof here since it may have independent interest and be useful for generalizations, for instance, to multiple series. For brevity, we will write $(a)_k = (a; q)_k$ for the elliptic shifted factorials in (1.11a).

To start our inductive proof of (1.7), we assume that it holds for fixed N and consider the sum

$$S=\sum_{k=0}^{N+1}\frac{\theta(aq^{2k})}{\theta(a)}\frac{(a,q^{-N-1},b,a/b)_k}{(q,aq^{N+2},aq/b,bq)_k}\,q^k\prod_{j=1}^{r}\frac{(c_jq^{m_j},aq/c_j)_k}{(aq^{1-m_j}/c_j,c_j)_k},$$

where $m_1+\cdots+m_r=N+1$. By symmetry, we may assume $m_r\geqslant 1$.

We multiply the sum S termwise by

$$\begin{aligned}1=&\frac{1}{\theta(aq^{N+1},q^{-N-1},c_rq^{m_r+k-1},aq^{1-m_r+k}/c_r)}\\&\times[\theta(aq^{k+N+1},q^{k-N-1},c_rq^{m_r-1},aq^{1-m_r}/c_r)\\&-q^{-N-1}\theta(aq^k,q^k,c_rq^{m_r+N},aq^{2-m_r+N}/c_r)],\end{aligned}$$

which is equivalent to (1.10) with the replacements

$$(u,v,x,y)\mapsto(\sqrt{a},q^{m_r-1}c_r/\sqrt{a},q^{N+1}\sqrt{a},q^k\sqrt{a}).$$

Since the factors $\theta(q^{k-N-1})$ and $\theta(q^k)$ vanish at the end-points $k=N+1$ and $k=0$, respectively, this gives an identity of the form

$$S=\sum_{k=0}^{N}(\cdots)+\sum_{k=1}^{N+1}(\cdots).$$

Replacing k by $k+1$ in the last sum and simplifying gives

$$\begin{aligned}S=&\sum_{k=0}^{N}\frac{\theta(aq^{2k})}{\theta(a)}\frac{(a,q^{-N},b,a/b,c_rq^{m_r-1},aq/c_r)_k}{(q,aq^{N+1},aq/b,bq,aq^{2-m_r}/c_r,c_r)_k}\,q^k\prod_{j=1}^{r-1}\frac{(c_jq^{m_j},aq/c_j)_k}{(aq^{1-m_j}/c_j,c_j)_k}\\&-q^{-N}\frac{\theta(aq,aq^2,b,a/b,c_rq^{m_r+N},aq/c_r,aq^{2-m_r+N}/c_r)}{\theta(aq/b,bq,aq^{N+1},aq^{N+2},aq^{1-m_r}/c_r,aq^{2-m_r}/c_r,c_r)}\prod_{j=1}^{r-1}\frac{\theta(c_jq^{m_j},aq/c_j)}{\theta(aq^{1-m_j}/c_j,c_j)}\\&\times\sum_{k=0}^{N}\frac{\theta(aq^{2k+2})}{\theta(aq^2)}\frac{(aq^2,q^{-N},bq,aq/b,c_rq^{m_r},aq^2/c_r)_k}{(q,aq^{N+3},aq^2/b,bq^2,aq^{3-m_r}/c_r,c_rq)_k}\,q^k\\&\times\prod_{j=1}^{r-1}\frac{(c_jq^{1+m_j},aq^2/c_j)_k}{(aq^{2-m_j}/c_j,c_jq)_k}.\end{aligned}$$

Both sums are now evaluated by the induction hypothesis, giving

$$\begin{aligned}S=&\frac{(aq,q)_N}{(bq,aq/b)_N}\frac{(c_r/b,c_rb/a)_{m_r-1}}{(c_r,c_r/a)_{m_r-1}}\prod_{j=1}^{r-1}\frac{(c_j/b,c_jb/a)_{m_j}}{(c_j,c_j/a)_{m_j}}\\&\times\left\{1-\frac{\theta(b,a/b,c_rq^{N+m_r},aq^{N+2-m_r}/c_r)}{\theta(bq^{N+1},aq^{N+1}/b,c_rq^{m_r-1},aq^{1-m_r}/c_r)}\right\}.\end{aligned}$$

Using again (1.10), this time with

$$(u, v, x, y) \mapsto (\sqrt{a}, q^{m_r-1}c_r/\sqrt{a}, q^{N+1}\sqrt{a}, b/\sqrt{a}),$$

we find that the factor within brackets equals

$$\frac{\theta(q^{N+1}, aq^{N+1}, q^{m_r-1}c_r/b, q^{m_r-1}c_r b/a)}{\theta(bq^{N+1}, aq^{N+1}/b, q^{m_r-1}c_r, q^{m_r-1}c_r/a)},$$

and thus

$$S = \frac{(aq, q)_{N+1}}{(bq, aq/b)_{N+1}} \prod_{j=1}^{r} \frac{(c_j/b, c_j b/a)_{m_j}}{(c_j, c_j/a)_{m_j}}.$$

This completes our alternative proof of (1.7).

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