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Algebraic entropy and the action of mapping class groups on character varieties

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Abstract

We extend the definition of algebraic entropy to endomorphisms of affine varieties. We then calculate the algebraic entropy of the action of elements of mapping class groups on various character varieties, and show that it is equal to a quantity we call the spectral radius, a generalization of the dilatation of a pseudo-Anosov mapping class. Our calculations are compatible with all known calculations of the topological entropy of this action.

Keywords: Mapping class group; Character variety; Algebraic entropy

1. Introduction

Let $S = S_{g,b}$ be an oriented surface of genus g with $b \geqslant 1$ boundary components. The *Mapping class group of* S, which we denote by $\mathrm{Mod}(S)$, consists of isotopy classes of orientation-preserving diffeomorphisms of S which fix the boundary components pointwise.

Choose a basepoint $p_0 \in S$, and let $\pi = \pi_1(S, p_0)$. Given an algebraic group G, one can construct the variety: $Hom(\pi, G)$, which is called the *G-representation variety of* π. The group G acts algebraically on this variety by conjugation. The categorical quotient of the representation variety is called the *G-character variety of* π, which we will often denote by $\mathfrak{X} := Hom(\pi, G)//G$.

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The mapping class group $\mathrm{Mod}(S)$ acts algebraically on $\mathfrak{X}$. Given $f \in \mathrm{Mod}(S)$, our goal in this paper is to calculate an algebraic invariant that gives a measure of the complexity of the action of f on $\mathfrak{X}$.

In the study of dynamical systems, there are several different measures of complexity called entropy – topological entropy and measure theoretic entropy being two common examples. In general, one expects actions with high entropy to be more complicated than actions with low entropy. For our purposes we wish to use a measure that captures the algebraic nature of the action. In [1], Bellon and Viallet define a notion called algebraic entropy for algebraic endomorphisms of affine space, which measures the growth rate of the degrees of iterates of the map.

The variety $\mathfrak{X}$ is affine, but there is no preferred way to embed it into affine space. One of the goals of this paper is to give an intrinsic natural extension of Bellon and Viallet's concept of algebraic entropy to algebraic self maps of affine varieties. This is the invariant we study. As a caution to the reader, we mention that there is a different dynamical invariant, due to Gromov, which is called algebraic entropy. We define all the terms we use, so no confusion should arise.

Let $\mathrm{e}_{\mathrm{alg}}(f)$ be the algebraic entropy of f, which is defined in Section 3, and let $\rho(f)$ is the *spectral radius of f*, a generalization of the log of the dilatation of a pseudo-Anosov element, which is defined in Section 2. We prove the following theorem.

Theorem 1. *Let $K = \mathbb{R}$ or $\mathbb{C}$ and G be one of the following groups:*

$$SL_N(K),\ GL_N(K),\ O_N(\mathbb{R})(N \geqslant 3),\ SO_N(\mathbb{R})(n \geqslant 3),\ U_N,\ SU_2,\ Sp_{2N}(\mathbb{R}).$$

Let S be a surface with free fundamental group, and let $f \in \mathrm{Mod}(S)$. The mapping class f acts on the G character variety of S, and one has that

$$\mathrm{e}_{\mathrm{alg}}(f) = \rho(f).$$

The topological entropy of mapping class group actions on character varieties has been calculated by Fried for the case $S = S_{1,1}$ and $G = SU(2)$ [5] and by Cantat and Loray for reduced character varieties (these are character varieties where the traces of boundary components are fixed) in the case $S = S_{0,4}$, $G = SL_2(\mathbb{C})$ [3]. The algebraic entropy was calculated by Brown for the case $S = S_{1,1}$ and $G = SU(2)$ and a specific embedding of $\mathfrak{X}$ [2]. In all of the above cases, the entropy calculated was equal to $\rho(f)$.

The paper is organized as follows. In Section 2 we define the concept of spectral radius and show how to calculate it for many elements of the mapping class group. In Section 3 we define the concept of algebraic entropy. In Section 4 we discuss the basics of character varieties and define the action of mapping class groups on them. Section 5 is devoted to the proof of Theorem 1, divided into the proof of two inequalities.

2. The spectral radius of a mapping class

2.1. Mapping class groups

Let $S = S_{g,b}$ be a surface of genus g with b boundary components (in this paper we will always assume that $b \geqslant 1$). Let $\mathrm{Diff}^+(S)$ be the group of orientation-preserving diffeomorphisms of S that are the identity on the boundary components. The *mapping class group of S* is the group $\mathrm{Mod}(S) = \pi_0(\mathrm{Diff}^+(S))$.

In what follows we will make no notational distinctions between simple closed curves and their homotopy classes. Also, we will assume that a base point is chosen on the boundary of S. This allows us to identify any $f \in \mathrm{Mod}(S)$ with an element of $\mathrm{Aut}(\pi)$. All of the information about mapping class groups used in this paper can be found in [4].

Definition. Let $f \in \mathrm{Mod}(S)$. Let $\mathcal{S}$ be a generating set for π. For $w \in \pi$, let $|w|_{\mathcal{S},red}$ be its cyclically reduced word length with respect to $\mathcal{S}$. Define the *spectral radius of f with respect to* $\mathcal{S}$ to be the quantity:

$$\rho^{\mathcal{S}}(f) := \sup_{\alpha\in\pi} \limsup_{n\to\infty} \frac{1}{n} \log\bigl(\bigl|f^{\circ n}\alpha\bigr|_{\mathcal{S},red}\bigr).$$

First notice that the above definition does not depend on the choice of the base point. Indeed, after changing the base point, the action of f on π is changed by composition with an inner automorphism. This clearly does not change cyclically reduced word lengths. The next proposition shows that the dependence on the set $\mathcal{S}$ can be dropped.

Proposition 2.1. *Given any element $f \in \mathrm{Mod}(S)$, and any two generating sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of π, the following equality holds:*

$$\rho^{\mathcal{S}_1}(f) = \rho^{\mathcal{S}_2}(f).$$

Proof. Recall that a map $\Phi : X \to Y$ between metric spaces is called a *quasi-isometry* if there exist positive constants K, C, D such that for every $x, y \in X$:

$$\frac{1}{K} d_X(x, y) - C \leqslant d_Y\bigl(f(x), f(y)\bigr) \leqslant K d_X(x, y) + C$$

and such that for every $w \in Y$, $d_Y(w, \Phi(X)) \leqslant D$.

The generating sets $\mathcal{S}_1$ and $\mathcal{S}_2$ define two word metrics on π. It is well known that the two metric spaces defined in this way are quasi-isometric. Any element $w \in \pi$ acts on π by left translation. It is well known that the translation length of this action in the metric given by $|.|_{\mathcal{S}_i}$ $(i = 1, 2)$ is $|w|_{\mathcal{S}_i,red}$.

Suppose K, C, D are the quasi-isometry constants for the quasi-isomorphism between $(\pi, |.|_{\mathcal{S}_1})$ and $(\pi, |.|_{\mathcal{S}_2})$. Using the characterization of $|w|_{\mathcal{S}_i,red}$ as a translation length, it is clear that

$$\frac{1}{K} |w|_{\mathcal{S}_1,red} + C \leqslant |w|_{\mathcal{S}_1,red} \leqslant K |w|_{\mathcal{S}_1,red} + C.$$

Given any constants A, B it's true that

$$\sup_{\alpha\in\pi} \limsup_{n\to\infty} \frac{1}{n} \log\bigl(A\bigl|f^{\circ n}\alpha\bigr|_{\mathcal{S}_i,red} + B\bigr) = \sup_{\alpha\in\pi} \limsup_{n\to\infty} \frac{1}{n} \log\bigl(\bigl|f^{\circ n}\alpha\bigr|_{\mathcal{S}_i,red}\bigr).$$

And thus $\rho^{\mathcal{S}_1}(f) = \rho^{\mathcal{S}_2}(f)$, as required. □

Since the definition of spectral radius does not depend on the generating set, we will suppress the $\mathcal{S}$ in the notation, and use $\rho(f)$ for the *spectral radius of* f.

2.2. Calculating the spectral radius

Our next goal is to calculate spectral radius for many elements of the mapping class group.

A *multicurve* in S is a finite collection of homotopy classes of mutually disjoint simple closed curves in S, none of which is homotopic to a boundary component. The mapping class group acts on the set of multicurves. This action can be used to classify elements of the mapping class group as follows.

Let $f \in \mathrm{Mod}(S)$. Exactly one of the following is true.

1. The order of f is finite.
2. The order of f is infinite, and there exists a multicurve M such that $f(M) = M$. In this case f is called *reducible*.
3. For every multicurve, M, in S, $f(M) \neq M$. In this case f is called *pseudo-Anosov*.

We will consider a particularly well behaved subclass of $\mathrm{Mod}(S)$, called pure elements. We say that an element $f \in \mathrm{Mod}(S)$ is *pure* if there exists a diffeomorphism ϕ of S in the homotopy class of f, and a (possibly empty) one dimensional submanifold c of S with the following properties:

1. None of the components of c are null-homotopic or homotopic to boundary components of S.
2. $\phi|_c = id$.
3. ϕ does not rearrange the components of $S \backslash c$.
4. On each component of S_c, the surface obtained by cutting S along c, ϕ induces a diffeomorphism which is homotopic either to the identity or to a pseudo-Anosov.

Note that any pseudo-Anosov mapping class is pure. In [6], it is proved that $\mathrm{Mod}(S)$ contains a finite index subgroup consisting entirely of pure elements. An example of such a group is the kernel of the action of $\mathrm{Mod}(S)$ on $H_1(S, \mathbb{Z}/3\mathbb{Z})$.

A theorem of Thurston describes a canonical geometric element contained in a pseudo-Anosov mapping class f. Using this element, one can attach an algebraic integer $\lambda = \lambda(f) > 1$ to f called the *dilatation of* f. To any pure element, one can attach a collection of dilatations, one for each component of S_c on which f acts as a pseudo-Anosov. We call the maximum one of these dilatations the *dilatation of* f, and denote it $\lambda(f)$. We now proceed to calculate the spectral radius of any pure element of the mapping class group.

Lemma 2.2. *Let $f \in \mathrm{Mod}(S)$ be a pure element, and let α be the isotopy class of a simple closed curve on S. Let g be a Riemannian metric on S. If we denote by $l_g(.)$ the g-length of a curve of an isotopy class of curves, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log\bigl(l_g\bigl(f^{\circ n}(\alpha)\bigr)\bigr) \leqslant \log\bigl(\lambda(f)\bigr).$$

Furthermore, there exists a simple closed curve α for which the above inequality is an equality.

Proof. When f is a pseudo-Anosov element with dilatation λ, then $\rho(f) = \log(\lambda)$, and the inequality in the claim of the lemma is an equality for every curve. The proof of this fact can be found for instance in [4, Theorem 13.20].

Suppose that f is reducible. Let c and ϕ be the one dimensional submanifold, and the diffeomorphism associated with f. Suppose that α is transverse to c. The finite set $\alpha \cap c$ is fixed by ϕ. Thus, $\alpha \backslash \phi$ consists of a collection of arcs: $\alpha_1, \ldots, \alpha_p$ whose endpoints are fixed by ϕ. We view each of these arcs as being a subset of one of the components of the surface obtained from S by cutting along c. The result quoted in [4] can be restated to apply to arcs whose endpoints are fixed by f. The same proof carries through with only notational changes. One has that

$$l_g\bigl(\phi^{\circ n}(\alpha)\bigr) \leqslant \sum_{i=1}^{p} l_g\bigl(\phi^{\circ n}(\alpha_i)\bigr).$$

The first part of the result is now clear. To see the second part, choose a curve contained in a subsurface with boundary on which the action of ϕ has dilatation $\lambda(f)$. □

Proposition 2.3. *Let $f \in \mathrm{Mod}(S)$ be a pure element. Then*

$$\rho(f) = \log\bigl(\lambda(f)\bigr).$$

Proof. Choose a generating set $\mathcal{S}$ of π. This defines a word metric on π. A choice of a hyperbolic metric g on S, and a basepoint $x_0 \in \mathbb{H}^2$, the upper half plane, defines an embedding of the Cayley graph into the hyperbolic plane. This embedding induces a new metric on the graph. It is well known that these two metrics are quasi-isometric. Every element $\alpha \in \pi$ acts as an isometry on the Cayley graph of π. Furthermore, α acts as an isometry on $\mathbb{H}^2$ which preserves the embedded Cayley graph. The actions of α on the embedded Cayley graph and the abstract Cayley graph are conjugate.

For every n, and every curve γ, the curve $f^{\circ n}(\gamma)$ corresponds to an isometry on $\mathbb{H}^2$. The translation length of this isometry is given by $l_g(f^{\circ n}(\gamma))$. The curve $f^{\circ n}(\gamma)$ also acts on the Cayley graph of π by left multiplication. The translation length of that action is $|f^{\circ n}(\gamma)|_{\mathcal{S},red}$. The first assertion of the lemma now clearly follows from quasi-isometry, and the fact that

$$\lim_{n\to\infty} \frac{1}{n} \log\bigl(l_g\bigl(f^{n\circ}(\gamma)\bigr)\bigr) \leqslant \rho(f).$$

The second assertion follows immediately from the second part of Lemma 2.2. □

3. Algebraic entropy

Let f be an endomorphism of $\mathbb{A}^N$. Define the *dynamical degree of* f as

$$\Delta(f) = \lim_{n\to\infty} \deg\bigl(f^{\circ n}\bigr)^{\frac{1}{n}}.$$

In [1] Bellon and Viallet define the *algebraic entropy* of f as

$$\mathrm{e_{alg}}(f) = \log\bigl(\Delta(f)\bigr).$$

Algebraic entropy is meant to be an algebraic approximation of topological entropy. To see this, consider the following heuristic argument: the topological entropy of f can often be estimated by calculating the exponential growth rate of the number of isolated fixed points of $f^{\circ n}$.

For a polynomial automorphism f of $\mathbb{C}^m$, the number of isolated fixed points of f is at most $\deg(f)$. Thus, calculating the exponential growth rate of the degrees of f^n can be seen as estimating the topological entropy of f.

In this paper we are concerned with endomorphisms of character varieties, which are affine varieties. Given a variety V equipped with an endomorphism f, we wish to give a definition that provides an algebraic approximation of the topological entropy of f. The naive approach is to embed V to affine space, extend f to an endomorphism of affine space and calculate its algebraic entropy. The problem with this approach is that neither the embeddings nor the extensions are canonical, and one can get many different results in this way. Our goal in this section is to give an intrinsic invariant which generalizes algebraic entropy.

Definition. Let $V \subset \mathbb{A}^N$ be a subvariety of affine N-space. Let $f : \mathbb{A}^N \to \mathbb{A}^M$ be a morphism. Let $\mathfrak{R}_{f,V}$ be the set of morphisms, $g : \mathbb{A}^N \to \mathbb{A}^M$ such that $g|_V = f|_V$. Define the *degree of f relative to V* as the quantity:

$$\deg(f; V) := \min_{g \in \mathfrak{R}_{f,V}} \deg(g).$$

Definition. Let $V \subset \mathbb{A}^N$ be a subvariety of affine N-space. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a morphism. Define the *algebraic entropy of f relative to V* as the quantity:

$$\mathrm{e}_{\mathrm{alg}}(f; V) = \limsup_{n\to\infty} \frac{1}{n} \log \deg\bigl(f^{\circ n}; V\bigr).$$

Suppose now that V is an affine variety, and $f : V \to V$ is a morphism. V can be embedded in many ways into affine space, and f can be extended in many ways to a morphism of affine space. For each embedding of V and each extension of f we can calculate the algebraic entropy relative to V. The next proposition shows that the above choices don't affect the algebraic entropy.

Proposition 3.1. *Let V be an affine variety, and let $f : V \to V$ be a morphism. Let $\iota_1 : V \to \mathbb{A}^{N_1}$ and $\iota_2 : V \to \mathbb{A}^{N_2}$ be two affine embeddings of V. Let $g_i : \mathbb{A}^{N_i} \to \mathbb{A}^{N_i}$ $(i = 1, 2)$ be morphisms such that: $g_i(\iota_i(V)) = \iota_i(V)$, and $\iota_i^*(g_i) = f$. Then*

$$\mathrm{e}_{\mathrm{alg}}\bigl(g_1; \iota_1(V)\bigr) = \mathrm{e}_{\mathrm{alg}}\bigl(g_2; \iota_2(V)\bigr).$$

Proof. First note that if $\iota_1 = \iota_2$ then the claim is trivial by the definition of relative algebraic entropy.

For $i = 1, 2$ the maps ι_i can be written in coordinates as

$$\iota_i = (x_{j,i})_{j=1}^{N_i}.$$

Since each ι_i is an embedding, then each of the sets $X_i = \{x_{1,i}, \dots, x_{N_i,i}\}$ generates the ring of functions of V. Thus, for each $j = 1, \dots, N_2$, we can non-canonically write $x_{j,2}$ as a polynomial in the elements of X_1. Using this, we get a morphism $p_1 : \mathbb{A}^{N_1} \to \mathbb{A}^{N_2}$, such that $p_1 \circ \iota_1 = \iota_2$. Say that the degree of p_1 is D_1. By similar reasoning, there is a morphism $p_2 : \mathbb{A}^{N_2} \to \mathbb{A}^{N_1}$ of degree D_2 such that $p_1 \circ \iota_2 = \iota_1$. Given an endomorphism $\tau : \mathbb{A}^{N_2} \to \mathbb{A}^{N_2}$ of degree t, we can construct the endomorphism $p_2 \circ \tau \circ p_1 : \mathbb{A}^{N_1} \to \mathbb{A}^{N_1}$. The degree of this morphism is at most tD_1D_2.

For any integer n, we take $\tau = g_2^{\circ n}$. The resulting endomorphism is clearly an extension of $f^{\circ n}$ from $\iota_1(V)$ to $\mathbb{A}^{N_1}$. By the definition of degree relative to a subvariety, we have that

$$\deg\bigl(g_1^{\circ n}; \iota_1(V)\bigr) \leqslant D_1 D_2\bigl(\deg\bigl(g_2^{\circ n}; \iota_2(V)\bigr)\bigr).$$

Thus

$$\mathrm{e}_{\mathrm{alg}}\bigl(g_1; \iota_1(V)\bigr) \leqslant \mathrm{e}_{\mathrm{alg}}\bigl(g_2; \iota_2(V)\bigr).$$

Reversing the roles played by the two spaces, we get the result. □

Using Proposition 3.1, we can now define an intrinsic notion of algebraic entropy.

Definition. Let V be an affine variety, and let $f : V \to V$ be a morphism. Let $\iota : V \to \mathbb{A}^N$ be an affine embedding and let $g : \mathbb{A}^N \to \mathbb{A}^N$ be a morphism such that $g(\iota(V)) = \iota(V)$ and $\iota^*(g) = f$. Define the *algebraic entropy of f* to be the quantity:

$$\mathrm{e}_{\mathrm{alg}}(f) = \mathrm{e}_{\mathrm{alg}}\bigl(g; \iota(V)\bigr).$$

Notice that if $V = \mathbb{A}^N$, then the above definition agrees with the regular definition of algebraic entropy.

4. The mapping class group action on character varieties

4.1. Representation varieties and character varieties

Suppose that $\pi \cong F_n$. Let G be a linear reductive algebraic group defined over the field K. Fix, once and for all, a faithful linear representation of G. Let

$$\mathfrak{R} = \mathfrak{R}(S, G) = Hom(\pi, G) \cong G^n.$$

The set $\mathfrak{R}$ has a natural structure as a variety. We call $\mathfrak{R}$ the *G representation variety of S*.

G acts algebraically on $\mathfrak{R}$ by componentwise conjugation. Consider the ring of invariants under this action, $F[\mathfrak{R}]^G$. Define:

$$\mathfrak{X} = \mathfrak{X}(S, G) = \mathfrak{R}//G := \mathrm{spec}\bigl(F[\mathfrak{R}]^G\bigr).$$

We call this variety the *G character variety of S*. We think of it as the set of characters of representations of π into G.

The group $\mathrm{Aut}(\pi)$ acts on $\mathfrak{R}$ in the following way: given a representation $\phi \in \mathfrak{R}$, an element $\alpha \in \pi$ and an automorphism $f \in \mathrm{Aut}(\pi)$, define:

$$f(\phi)(\alpha) := \phi\bigl(f(\alpha)\bigr).$$

Let $\mathrm{Out}(\pi)$ be the group of outer automorphisms of π. The action of $\mathrm{Aut}(\pi)$ on $\mathfrak{R}$ descends to an action of $\mathrm{Out}(\pi)$ on $\mathfrak{X}$. Since $\mathrm{Mod}(S)$ can be viewed as a subgroup of $\mathrm{Out}(\pi)$, we get an action of $\mathrm{Mod}(S)$ on π.

4.2. *Generating the ring of invariants*

A theorem which gives an explicit generating set for the ring $K[\mathfrak{R}]^G$ is often called a *first fundamental theorem for G-invariants of n matrices*, where n is the rank of π. A first fundamental theorem for $SL_2(\mathbb{C})$, $SL_2(\mathbb{R})$, and SU_2 is known since the work of Fricke. In [7], Procesi proves a first fundamental theorem of GL_N, SL_N, O_N, $U(N)$, and $Sp_{2N}(\mathbb{R})$ for m matrices. In [8], Rogara proves a first fundamental theorem of $SO_N(\mathbb{R})$ invariants for n matrices.

The most common functions that are given as generators are called *trace functions*. Given an element $\alpha \in \pi$, we can define a function tr_α on $\mathfrak{R}$ by

$$\mathrm{tr}_\alpha(\phi) = \mathrm{trace}\big(\phi(\alpha)\big).$$

Choosing a generating set $\mathcal{A} = \{X_1, \dots, X_n\}$ for π identifies $\mathfrak{R}$ as a subset of $M^n_{N\times N} \cong \mathbb{A}^{N^2 n}$. Under this identification, to any word w in the elements of $\mathcal{A}$ one can associate the function tr_w, which is a homogeneous polynomial. Note that given an element $\alpha \in \pi$, it may be possible to write α in several different ways as a word in the elements of $\mathcal{A}$, and thus tr_α can be extended in more than one way to a function on $M^n_{N\times N}$.

Since the functions tr_α are conjugation invariant, we can view tr_α as an element of $\mathbb{C}[\mathfrak{X}]$, i.e. as a regular function on $\mathfrak{X}$. For GL_N, the set of trace functions generate the ring of invariants. In fact, only finitely many trace functions are required to generate the ring. For the other cases, slightly more complicated functions are needed. For example, for the case $G = O_N$, one needs to take traces of words in the elements of $\mathcal{A}$ and their transposes. For Sp_{2N}, one needs to add symplectic transposes. These functions are all homogeneous functions on the coordinates of the matrices representing elements of π. Formally, we use the following fact:

For any generating set $\mathcal{A}$, there exists an integer L, finitely many functions: $h_i : F_{\mathcal{A}} \to K[X_1, \dots, X_{N^2 n}]$ $(i = 1, \dots, p)$ whose images are all homogeneous of degree at most L, and a finite subset $\{w_1, \dots, w_p\} \subset \pi$ such that:

1. For any i, and $\alpha \in \pi$, the function $h_i(\alpha)$ is homogeneous of degree at most L in the coordinates of the matrix representing α, and is invariant under conjugation.
2. The collection $h_i(\alpha_i)$ generates the ring of invariants.

On a first reading, we suggest that the reader think of all of the functions h_i as being the trace function, and the $h_i(w_i)$ as being the traces of finitely many words. In this paper we use these functions to find affine embeddings of $\mathfrak{X}$.

Example. Let $S = S_{1,1}$ be a one holed torus, and let $G = SL_2(\mathbb{R})$. Choose 2 simple closed curves on S whose intersection number is 1. Call these curves X and Y, and let $Z = XY$. It is well known that the map $\mathrm{Tr} : \mathfrak{X} \to \mathbb{R}^3$ given by

$$\mathrm{Tr}(\chi) = \big(\mathrm{tr}_X(\chi), \mathrm{tr}_Y(\chi), \mathrm{tr}_Z(\chi)\big)$$

is an isomorphism. The action of the Dehn twist about X on π is given by

$$T_X(X) = X, \qquad T_X(Y) = YX.$$

In the above trace coordinates, the action is given by

$$(x, y, z) \to (x, z, xz - y).$$

Now consider the action of $T_X^{\circ 2}$. In coordinates, we can write it out as

$$(x, y, z) \to \bigl(x, xz - y, x^2z - xy - z\bigr).$$

The action of $T_X^{\circ 3}$ is given by

$$(x, y, z) \to \bigl(x, x^2z - yx - z, x^3z - x^2y + y\bigr).$$

In general, it is simple to see that $\deg(T_X^{\circ n} = n)$, and thus $\mathrm{e}_{\mathrm{alg}}(T_X) = 0$. This agrees with the fact that $\rho(T_X) = 0$.

5. Proof of Theorem 1

5.1. The upper bound

Proposition 5.1 *(The upper bound). In the notation of Theorem* 1*:*

$$\mathrm{e}_{\mathrm{alg}}(f) \leqslant \rho(f).$$

Proof. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ be a generating set for π. Let $\mathfrak{R}$ be the G-representation variety of π. Without loss of generality we assume that G is a subgroup of some GL_N. The set $\mathcal{A}$ determines an embedding $\iota : \mathfrak{R} \to M^n_{N\times N}$ given by

$$\iota(\rho) = \bigl(\rho(\alpha_1), \ldots, \rho(\alpha_n)\bigr).$$

There is an obvious isomorphism $M^n_{N\times N} \cong \mathbb{A}^{N^2 n}$.

As discussed in the previous section, there is an integer L, finitely many functions: $h_i : F_{\mathcal{A}} \to K[X_1, \ldots, X_{N^2 r}]$ $(i = 1, \ldots, p)$ whose images are all homogeneous of degree at most L, and a finite subset $\{w_1, \ldots, w_p\} \subset \pi$ such that any element of $F[\mathfrak{R}]^G$ can be written as a polynomial in $h_1(w_1), \ldots, h_m(w_m)$.

Let w be a word of length l in the elements of $\mathcal{A}$. For $1 \leqslant i \leqslant p$ we assign to w the function $h_i(w)$ on $\mathbb{A}^{N^2 n}$. If we think of w as an element of π and not just a word, we see that this function is an extension of $h_i(w)$ from $\mathfrak{R}$ to all of $\mathbb{A}^{N^2 n}$. Writing out matrix multiplication in coordinates, we see that $h_i(w)$ is a homogeneous function of degree at most lL.

Since all of the functions $h_i(w)$ are invariant under conjugation, we can deduce that given $w \in \pi$, $|w|_{\mathcal{A},red} = l$, then $h_i(w)$ can be written as a homogeneous function of degree at most lL on $\mathbb{A}^{N^2 n}$.

Now, given $w \in \pi$, with $|w|_{\mathcal{A},red} = l$, we have the function $h_i(w)$ can be written as a polynomial in $h_1(w_1), \ldots, h_p(w_p)$, each of which is a homogeneous function of degree at least 1.

Since degree is additive under multiplication of homogeneous polynomials, we have that $h_i(w)$ can be written as a polynomial of degree at most lL in $h_1(w_1), \dots, h_p(w_p)$.

Define an affine embedding $\kappa : \mathfrak{X} \to \mathbb{A}^m$ by

$$\kappa(\chi) = \big(h_1(w_1)(\chi), \dots, h_p(w_p)(\chi)\big).$$

Given an integer m, we can write the action of $f^{\circ m}$ in coordinates as

$$\kappa \circ f^{\circ m} = \big(h_1\big(f^{\circ m}(w_1)\big), \dots, h_1\big(f^{\circ m}(w_p)\big)\big).$$

From the above discussion, we see that

$$\deg\big(f^{\circ m}; \kappa(\mathfrak{X})\big) \leqslant L \max\big(\big|f^{\circ m} w_1\big|_{\mathcal{A},red}, \dots, \big|f^{\circ m} w_p\big|_{\mathcal{A},red}\big).$$

Therefore, by the definitions of algebraic entropy and spectral radius:

$$\begin{aligned} \mathrm{e}_{\mathrm{alg}}(f) &= \limsup_{n\to\infty} \frac{1}{n} \log\big(\deg\big(f^{\circ p}; \kappa(\mathfrak{X})\big)\big) \\ &\leqslant \limsup_{n\to\infty} \frac{1}{n} \log\big(\max\big(\big|f^{\circ m} w_1\big|_{\mathcal{A},red}, \dots, \big|f^{\circ m} w_p\big|_{\mathcal{A},red}\big)\big) \leqslant \rho(f). \qquad \square \end{aligned}$$

5.2. The lower bound

Proposition 5.2 *(The lower bound). In the notation of Theorem* 1*:*

$$\mathrm{e}_{\mathrm{alg}}(f) \geqslant \rho(f).$$

The proof of the lower bound is more involved than the proof of the upper bound. We begin by recalling some necessary material.

5.3. Bruhat–Tits trees

Let K be a non-Archimedean complete field of characteristic 0, equipped with a valuation ν. Let $\mathcal{O}_K$ be the ring of integers, $\mathcal{M}_K$ the maximal ideal of $\mathcal{O}_K$, $k = \frac{\mathcal{O}_K}{\mathcal{M}_K}$ its residue field. Let $q = |k|$ be the number of elements of k, and let p be its characteristic. For an algebraic group G defined over K, let G_K be the subgroup of K points. G_K has a natural action on a simplicial complex called the *Bruhat–Tits building of* G_K. This building plays an analogous role to the symmetric space in the Archimedean case. We will only need to use this theory for $SL_2(K)$, in which case the building is a regular tree. All of the information that we use can be found in [9]. Recall that a *lattice* in K^2 is an $\mathcal{O}_K$ submodule of the form $\mathcal{O}_K v \oplus \mathcal{O}_K w$, with $v, w \in K^2$ linearly independent. We always denote by L_0 the so called standard lattice: $L_0 = \mathcal{O}_K\binom{1}{0} \oplus \mathcal{O}_K\binom{0}{1}$. Lattices L and L' are called *homothetic* if $\exists x \in K$ such that $L = xL'$. Homothety is an equivalence relation, and we denote the equivalence class of the lattice L by $[L]$. We say that two homothety classes $[L]$ and $[L']$ are *incident* if there are representatives L_1, L_2 of $[L]$ and L' of $[L']$ such that

$$L_2 <_p L' <_p L_1$$

where the symbol $<_p$ is read: is a subgroup of index p in. It is a simple exercise to check that incidence is a symmetric relation.

We are now ready to define the Bruhat–Tits building (which we denote by $\mathcal{T}_K$) for $SL_2(K)$. $\mathcal{T}_K$ is a graph with a vertex for each homothety class of lattices and two vertices connected by an edge if the corresponding homothety classes are incident. $SL_2(K)$ acts on $\mathcal{T}_K$ by simplicial automorphisms. We summarize the properties of this graph and the $SL_2(K)$ action on it that we need.

Proposition 5.3.

1. *$\mathcal{T}_K$ is a $\frac{q^2-1}{q-1}$ regular tree.*
2. *Given $A \in SL_2(K)$, its translation length is given by $-2\max(\nu(\mathrm{tr}(A)), 0)$.*
3. *The action of $SL_2(K)$ is transitive.*
4. *$Stab_{SL_2(K)}([L_0]) = SL_2(\mathcal{O}_K)$, $Stab_{SL_2(K)}(A[L_0]) = ASL_2(\mathcal{O}_K)A^{-1}$.*
5. *The set of connected components of $T_K/[L_0]$ (i.e. the set of neighbors of $[L_0]$) can be identified with $\mathcal{P}(k^2)$, so that the action of $SL_2(\mathcal{O}_K)$ on this set of components is conjugate to its action on $\mathcal{P}(k^2)$ (by taking conjugates, this statement can be made for each vertex of $\mathcal{T}_K$).*
6. *The axis of a diagonal matrix passes through $[L_0]$.*

We now state and prove a technical lemma for bounding algebraic entropy from below. The following two lemmas set up the conditions for using this technical lemma.

Lemma 5.4. *Let K be a field of characteristic 0 equipped an absolute value $|.|_\nu$, let V be an affine variety defined over K, let f be an endomorphism of V which is defined over K, and let $y \in K[V_K]$. If there exists $P_0 \in V_k$ with the following properties:*

1. *$\exists\epsilon > 0$ such that $\forall n$: $|y(f^{\circ n}(P_0))|_\nu > \epsilon$,*
2. *$\limsup_{n\to\infty} \frac{1}{n}\log(\log(|y(f^{\circ n}(P_0))|_\nu)) = l$,*

then $l \leqslant \mathrm{e}_{\mathrm{alg}}(f)$.

Proof. Choose an ordered subset $Y = \{y_1, \dots, y_N\} \subset K[V_K]$ such that Y generates $K[V_K]$ and $y_1 = y$. The set Y defines an embedding $V_K \hookrightarrow K^N$. For the remainder of the proof we will ignore the difference between points in V_K and their image under this embedding.

Given $P \in K^N$, let $P^{(i)}$ denote its i-th coefficient and $|P|_\nu = \max_i |P^{(i)}|_\nu$. Note that for $P \in V_K$, one has that $P^{(1)} = y(P)$.

Extend the endomorphism f to an endomorphism of $\mathbb{A}^N$ of degree d, which we also call f. In coordinates we can write f as a vector of polynomials with coefficients in K.

The function $\frac{|f(P)|_\nu}{|P|_\nu^d}$ is bounded on the set $\{P \in K^N\colon |P|_\nu \geqslant \epsilon\}$. Therefore, $\exists C$ such that

$$\left|f(P_0)\right|_\nu \leqslant C|P_0|_\nu^d.$$

Suppose first that $d \geqslant 2$. Iterating f we get:

$$\left|f^{\circ n}(P_0)\right|_v \leqslant C^{1+d+\cdots+d^{n-1}}|P_0|_v^{d^n} = C^{\frac{d^n-1}{d-1}}|P_0|_v^{d^n}.$$

Taking logarithms we get:

$$\log\left(\left|f^{\circ n}(P_0)\right|_v\right) \leqslant \frac{d^n-1}{d-1}\log(C) + d^n\log\left(|P_0|_v\right) = d^n\left[\log\left(|P_0|_v\right) + \frac{1-\frac{1}{d^n}}{d-1}\log(C)\right].$$

Taking logarithms once again, and manipulating further, we get:

$$\frac{1}{n}\log\left(\log\left(\left|f^{\circ n}(P_0)\right|_v\right)\right) \leqslant \log(d) + \frac{1}{n}\log\left[\log\left(|P_0|_v\right) + \frac{1-\frac{1}{d^n}}{d-1}\log(C)\right].$$

Therefore, $\exists D > 0$ such that

$$\frac{1}{n}\log\left(\log\left(\left|f^{\circ n}(P_0)\right|_v\right)\right) \leqslant \log(d) + \frac{D}{n}.$$

Now, since

$$\left|y\left(f^{\circ n}(P_0)\right)\right|_v = \left|f^{\circ n}(P_0)^{(1)}\right|_v \leqslant \left|f^{\circ n}(P_0)\right|_v,$$

then

$$l = \limsup_{n\to\infty}\frac{1}{n}\log\left(\log\left(\left|y\left(f^{\circ n}(P_0)\right)\right|_v\right)\right) \leqslant \limsup_{n\to\infty}\log(d) + \frac{D}{n} = \log(d).$$

Given an integer q, we have that

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\log\left(\left|y\left(f^{\circ qn}(P_0)\right)\right|_v\right)\right) = \limsup_{n\to\infty}\frac{q}{qn}\log\left(\log\left(\left|y\left(f^{\circ qn}(P_0)\right)\right|_v\right)\right) = ql.$$

Replacing f by $f^{\circ q}$ in the above discussion, we have that for any q and any extension of $f^{\circ q}$ to an endomorphism of K^N (which we also denote by $f^{\circ q}$):

$$ql \leqslant \log\left(\deg\left(f^{\circ q}\right)\right).$$

By dividing by q and using the definition of degree relative to a subvariety we have that

$$l \leqslant \frac{1}{q}\log\left(\deg\left(f^{\circ}; V\right)\right).$$

Taking the limits we get:

$$l \leqslant \mathrm{e}_{\mathrm{alg}}(f).$$

Now assume that $d = 1$. In this case, we get that

$$\left|f^{\circ n}(P_0)\right|_v \leqslant C^n|P_0|_v.$$

Taking logarithms twice and dividing by n we get:

$$\frac{1}{n}\log\bigl(\log\bigl(\bigl|f^{\circ n}(P_0)\bigr|_\nu\bigr)\bigr) \leqslant \frac{\log n}{n} + \frac{1}{n}\log\bigl[\log C + \log|P_0|_\nu\bigr].$$

Taking limits, we get $l \leqslant 0$. Since algebraic entropy is always non-negative, then we have $l \leqslant \mathrm{e}_{\mathrm{alg}}(f)$, as required. □

Lemma 5.5. *Let $F_n = \langle \mathcal{S}\rangle = \langle x_1, \dots, x_n\rangle$ be a free group on n generators. Suppose that α is an action of F_n on the 2d-regular tree $\mathcal{T}_{2d}$ satisfying the following conditions:*

1. *$\alpha(x_1), \dots, \alpha(x_n)$ are all hyperbolic with translation distance t.*
2. *There exists a unique vertex v_0 such that $\{v_0\} = \mathcal{L}_i \cap \mathcal{L}_j$ for any i, j, where $\mathcal{L}_i$ is the axis of $\alpha(x_i)$.*

Then given $w \in F_n$, the translation length of $\alpha(w)$ is $t|w|_{\mathcal{S},red}$.

Proof. Notice that since every hyperbolic automorphism with translation distance t is a power of an automorphism with the same axis and translation length 1, it is enough to prove the lemma for $t = 1$. Furthermore, by adding hyperbolic automorphisms we can assume $d = n$. In this case, we have that α is conjugate to the action of F_d on the Cayley graph of F_d associated to the generating set $\{x_1, \dots, x_d\}$, where v_0 corresponds to the identity element. □

Lemma 5.6. *There exists a valuation ν on $\mathbb{Q}$ (resp. $\mathbb{Q}[i]$), and a representation $\Psi : \pi \to SL_2(\mathbb{Q})$ (resp. $SU_2(\mathbb{Q}[i])$) such that the induced action of π on $\mathcal{T}_\nu$, the Bruhat–Tits tree associated to $SL_2(\mathbb{Q}_\nu)$ (resp. $SL_2(\mathbb{Q}[i]_\nu)$) satisfies the conditions of Lemma 5.5.*

Proof. Let $n = 2g$ and let $\pi = \langle x_1 \dots x_{2g}\rangle$. We separate into two cases.

The SL_2 case. Let p be any sufficiently large prime (just how large it needs to be will be clear from the construction). Let ν be the p-adic valuation and let $\mathcal{T}_p$ be Bruhat–Tits tree for $SL_2(\mathbb{Q}_p)$. Let $[L_0]$ be the homothety class of the standard lattice. Let $D = \bigl(\begin{smallmatrix} \frac{1}{p} & 0 \\ 0 & p\end{smallmatrix}\bigr)$. Then D is hyperbolic, and its axis passes through $[L_0]$. The segment connecting $[L_0]$ to $D[L_0]$ passes through the neighbor of $[L_0]$ corresponding to the point $\bigl(\begin{smallmatrix}1\\0\end{smallmatrix}\bigr)$ in $\mathbb{P}^2(\mathbb{F}_p)$. The segment connecting $[L_0]$ to $D^{-1}[L_0]$ passes through the neighbor of $[L_0]$ corresponding to the point $\bigl(\begin{smallmatrix}0\\1\end{smallmatrix}\bigr)$. Let S be any element of $SL_2(\mathbb{Z})$ for which the set

$$\left\{\begin{pmatrix}0\\1\end{pmatrix}, S\begin{pmatrix}0\\1\end{pmatrix}, \dots, S^{n-1}\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, S\begin{pmatrix}1\\0\end{pmatrix}, \dots, S^{n-1}\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

projects to a set of $2n$ different points in $\mathbb{QP}^2$. If p is chosen to be sufficiently high, this set will project to $2n$ different points in $\mathbb{P}^2(\mathbb{F}_p)$. Note that $S[L_0] = [L_0]$.

For $i = 1, \dots, n$ define:

$$\Psi(x_i) = S^{(i-1)} D S^{-(i-1)}.$$

We have that $\Psi(x_1), \dots, \Psi(x_n)$ act hyperbolically on $\mathcal{T}_p$, they all have the same translation length, and the intersection of any two of their axes is precisely $[L_0]$. The first two assertions follow from the fact that every $\Psi(x_i)$ is conjugate to D. The third assertion follows from the fact that the axis of $\Psi(x_i)$ is $S^{i-1}\mathcal{L}$, where $\mathcal{L}$ is the axis of D, and by construction these are n lines that intersect only at $[L_0]$.

The $G = SU(2)$ case. The construction is almost identical to the previous case. We let $K = \mathbb{Q}[i]$. Let p be a sufficiently large prime number. There exist integers a, b such that $a^2 + b^2 = p$. There are two primes of $\mathcal{O}_K$ that lie above p, these are $(a + bi)$, $(a - bi)$. Let $\mathfrak{p} = (a + bi)$. Let ν be the ν-adic valuation, and let $\mathcal{T}_\mathfrak{p}$ be the Bruhat–Tits tree for $SL_2(K_\nu)$. Once again, let $[L_0]$ be the homothety class of the standard lattice.

Let $D = \begin{pmatrix} \frac{(a+bi)^2}{p} & 0 \\ 0 & \frac{(a-bi)^2}{p} \end{pmatrix}$. Then D is hyperbolic with axis passing through $[L_0]$. If we take S to be any element of the $\mathbb{Q}$-points of $SU(2)$ that is not of finite order, and whose elements have denominators that are coprime to p then the construction from the previous case may be applied verbatim to this case. $\square$

Proof of Proposition 5.2. First the sake of simplicity, we first assume that $G = SL_2(\mathbb{R})$, or $SU(2)$.

Fix $\epsilon \geqslant 0$. Choose a generating set $\mathcal{S} = \{x_1, \dots, x_{2g}\}$ of π for which:

$$\left|\limsup_{n\to\infty} \frac{1}{n} \log\left(\left|f^{\circ n} x_1\right|_{\mathcal{S},red}\right) - \rho(f)\right| \leqslant \epsilon.$$

Choose a representation Ψ as in Lemma 5.6, and let ν and $\mathcal{T}_\nu$ be as in the construction of Ψ. Let $|.|_\nu$ be the absolute value associated to ν, i.e. $|.|_\nu = p^{-\nu(.)}$. Let $\psi \in \mathfrak{X}$ be the character of Ψ. The character ψ is a $\mathbb{Q}$ or $\mathbb{Q}[i]$ point of $\mathfrak{X}$.

By Lemma 5.5, for any m one has that the translation length of $f^{\circ m}(x_1)$ on $\mathcal{T}_\nu$ is equal to $|f^{\circ m}(x_1)|_{\mathcal{S},red}$. By Proposition 5.3 part 2 we get:

$$\left|f^{\circ m}(x_1)\right|_{\mathcal{S},red} = -2\nu\left(\mathrm{tr}_{f^{\circ m}(x_1)}(\psi)\right).$$

We now wish to apply Lemma 5.4. In order to set up the notation of the lemma, let $V = \mathfrak{X}$, $K = \mathbb{Q}$ or $K = \mathbb{Q}[i]$ (depending on which part of Lemma 5.6 we used), $|.|_\nu$ be the norm defined above, $P_0 = \psi$, $y = \mathrm{tr}_{x_1}$, $f = f$.

Since word length is always positive, we have that $\nu(\mathrm{tr}_{f^{\circ m}(x_1)}(\psi)) < 0$, and thus in the notation of Lemma 5.4: $|y(f^{\circ m}(P_0))|_\nu \geqslant 1$.

Due to our choice of x_1, we have that:

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log\left(\log\left(\left|y\left(f^{\circ n}(P_0)\right)\right|_\nu\right)\right) &= \limsup_{n\to\infty} \frac{1}{n} \log\left(\log\left(p^{-\nu(\mathrm{tr}_{f^{\circ n}x_1}(\psi))}\right)\right) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log\left(\frac{1}{2}\left|f^{\circ n}(x_1)\right|_{\mathcal{S},red} + p\right) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log\left(\left|f^{\circ n}(x_1)\right|_{\mathcal{S},red}\right) \geqslant \rho(f) - \epsilon.
\end{aligned}$$

Thus, by Lemma 5.4 we have that $\rho(f) - \epsilon \leqslant \mathrm{e}_{\mathrm{alg}}(f)$. Since ϵ was chosen arbitrarily, we get that $\rho(f) \leqslant \mathrm{e}_{\mathrm{alg}}(f)$.

Now, suppose G is one of the groups GL_N, SL_N, SU_N, or Sp_{2N}. Each of these groups contains a copy of $G = SL_2(\mathbb{R})$, or $SU(2)$ embedded in the top right corner. If we take the representation Ψ to have image in this copy, and take $y = \mathrm{tr}_{x_1} - (N-2)$, then the proof proceeds exactly as above.

If $G = SO(3)$, notice that $SO(3)$ is double covered by $SU(2)$, and that any $SO(3)$ representation can be lifted to an $SU(2)$ representation where the trace of each element is multiplied by ± 1. Thus, the proof carries over to the $SO(3)$ case. For $G = SO(N)$, $N \geqslant 4$ and $G = O(N)(N \geqslant 3)$, notice that these groups contain $SO(3)$ embedded as 3×3 diagonal matrices, and proceed by the same method. □

Proof of Theorem 1. Theorem 1 is a direct consequence of Propositions 5.1 and 5.2. □

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