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Parsimonious edge coloring

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Abstract

In a graph G of maximum degree Δ, let γ denote the largest fraction of edges that can be Δ-edge-colored. This paper investigates lower bounds for γ together with infinite families of graphs in which γ is bounded away from 1. For instance, if G is cubic, then $\gamma \geqslant \frac{13}{15}$; and there exists an infinite family of 3-connected cubic graphs in which $\gamma \leqslant \frac{25}{27}$.

1. Introduction

The earliest equivalent formulation of the Four Color Theorem (not just a restriction) was proposed by Tait in 1880 [1]:

> "The edges of a polyhedron, which has trihedral summits only, can be divided into three groups, one from each group ending in each summit."

In modern language, the edges of a cubic, planar, bridgeless graph can be 3-colored. None of these hypotheses can be weakened. Tait gave an example to show that bridgelessness is necessary. Petersen's graph witnesses the necessity of planarity. Subdividing one edge of a K_4 yields a graph with maximum degree 3 that cannot be 3-edge-colored. On the positive side, it is straightforward to see that any graph with maximum degree 3 can be 4-edge-colored. This follows from Brooks' Theorem as well as being a special case of Vizing's Theorem.

Frequently, in vertex coloring problems, there exist r-chromatic graphs with the property that any r-coloring has all the color classes of the same size. For example, the icosahedron is a K_4-free planar graph with 12 vertices and independence number 3. Thus in any 4-coloring each color class consists of three vertices. In contrast, sometimes one or several colors do not need to be used very much. For example Dirac

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has shown that in 7-coloring toroidal graphs the seventh color need be used at most once [3]. Recently Thomassen established a considerable strengthening: all but three of the vertices of any toroidal graph can be 5-colored — more generally, all but 2^{14g+6} of the vertices of any graph embedded on a surface of genus g can be 5-colored [10]. This paper investigates the extent to which almost all of the edges of a graph can be colored by all but one of the necessary colors.

Our notation is mostly standard. G will be a simple graph with $V = V(G)$ vertices, $E = E(G)$ edges, and $b = b(G)$ bridges. The maximum degree of a vertex in G is denoted by $\Delta = \Delta(G)$ and the minimum number of colors needed to color the edges of G so that incident edges receive different colors is denoted by $\chi' = \chi'(G)$. It is a familiar result of Vizing that $\Delta \leqslant \chi' \leqslant \Delta + 1$ [11]. If $\chi' = \Delta$, G is said to be class 1, otherwise G is class 2. We are interested in the maximal number of edges in a class 1 subgraph of a graph G. For convenience let

$$c = c(G) = \max\{E(H)\colon H \text{ is a subgraph of } G \text{ and } H \text{ is class } 1\},$$

$$u = u(G) = E(G) - c(G) \quad \text{and} \quad \gamma = \gamma(G) = \frac{c(G)}{E(G)}.$$

For example $\gamma(C_{2k+1}) = 2k/(2k+1)$. The Four Color Theorem is equivalent to the statement: If G is a cubic, planar, bridgeless graph, then $\gamma(G) = 1$. For an arbitrary graph with $\Delta = 3$, selecting the three most popular classes of edges in a 4-edge-coloring suffices to show that $\gamma \geqslant \frac{3}{4}$. Our goal is to improve this bound. In the next section we consider arbitrary cubic graphs. Subsequently we look at graphs with maximum degree 3, bridgeless planar graphs with maximum degree 3, cubic planar graphs with bridges, and finally graphs that are regular of degree 4.

2. Cubic graphs

It is not a priori obvious that any improvement in the $\frac{3}{4}$ bound is possible; however, previous results from the literature do have such implications. Staton (and independently Locke) showed that if G is a cubic graph ($\neq K_4$), then G contains a bipartite subgraph containing at least $\frac{7}{9}$ of the edges of G [7, 9]. Since bipartite graphs and even ordered complete graphs are class 1, $\gamma(G) \geqslant \frac{7}{9}$. Hopkins and Staton showed that if G is a triangle free, cubic graph, then G contains a bipartite subgraph with at least $\frac{4}{5}$ of the edges of G [6]. Bondy and Locke obtained the same result weakening the hypothesis of cubic to $\Delta \leqslant 3$ [2]. It turns out that the triangle free assumption is not an obstacle. Thus if G is cubic, $\gamma(G) \geqslant \frac{4}{5}$. How far can this improvement go? Not surprisingly, Petersen's graph with $\gamma = \frac{13}{15}$ supplies the extremal example.

Theorem 1. *If G is cubic, then $\gamma(G) \geqslant \frac{13}{15}$.*

Proof. We may assume G is connected. The proof will be by induction on $b(G)$. To obtain the base case we will apply Petersen's theorem that cubic graphs with fewer than three bridges contain a perfect matching. But first we need some control on the appearance of triangles in G. Suppose that vertices r, s, and t form a K_3 in G and are adjacent to the distinct vertices x, y, and z respectively. Let G' be obtained from G by contracting r, s, and t to a new vertex v. Fix a 3-coloring of (some of) the edges of G' and transfer the coloring to G. Give the color assigned (v,x) $\{$respectively (v,y), $(v,z)\}$ to (r,x) $\{$respectively (s,y), $(t,z)\}$. The edges of the original K_3 can be easily colored. It follows that $u(G) = u(G')$ and thus $\gamma(G) > \gamma(G')$. Consequently we may assume that G contains no triangle with three distinct neighbors. Thus if a triangle occurs, then it occurs as one of a pair of triangles which share a common edge. The advantage of this condition is that a perfect matching in such a G must contain one edge from every triangle in G. Now we apply Petersen's theorem. If G has $b \leqslant 2$, then G contains a perfect matching, say M. $G - M$ is a 2-regular graph, which by the preceding argument contains no triangle. Thus at least $\frac{4}{5}E(G - M)$ of the edges of $G - M$ can be 2-colored. Thus

$$\gamma(G) \geqslant |M| + \tfrac{4}{5}E(G-M) = \tfrac{1}{2}V + \tfrac{4}{5}V = \tfrac{13}{15}E.$$

Now we assume that G is a cubic graph with b ($\geqslant 3$) bridges, say $e_1 = (x_1, y_1)$, $e_2 = (x_2, y_2), \ldots, e_b = (x_b, y_b)$. We construct $T(G)$, a variation on the block cut-point tree, by contracting every edge of G that is not a bridge. The vertices of $T(G)$ correspond to either the blocks of G or those vertices of G that are incident with three bridges, and the edges of $T(G)$ are just $e_1, \ldots, e_b$. Suppose x_1 and x_2 are distinct leaves in $T(G)$. In G, x_1 and x_2 represent blocks each incident with exactly one bridge.

Case 1: y_1 and y_2 are distinct and not adjacent. Take G and delete e_1 and e_2. Form G' by adding the edge (x_1, x_2) and G'' by adding the edge (y_1, y_2). Both G' and G'' have fewer bridges than G. Thus $c(G') \geqslant \frac{13}{15}E(G')$ and $c(G'') \geqslant \frac{13}{15}E(G'')$. Any 3-coloring of (some of) the edges of G' and G'' transfers immediately to a 3-coloring of the corresponding edges of G. There are colors available for both edges e_1 and e_2 whether or not (x_1, x_2) and (y_1, y_2) are colored. Thus

$$c(G) \geqslant c(G') + c(G'') \geqslant \tfrac{13}{15}(E(G') + E(G'')) = \tfrac{13}{15}E(G).$$

Case 2: No matter which leaves of $T(G)$ are chosen, say x_1 and x_2, the vertices y_1 and y_2 are either identical or adjacent. It is immediate that $T(G)$ has at most two non-leaf vertices. If $T(G)$ has exactly one non-leaf vertex, then $T(G) = K_{1,b}$. Since all of the y_j's are pairwise adjacent and G is cubic, $b \leqslant 3$. But $b \leqslant 2$ was covered in the base of the induction. Thus we may assume that G has exactly three bridges (x_1, y_1), (x_2, y_2), and (x_3, y_3). Furthermore, either y_1, y_2, y_3 forms a K_3 or $y_1 = y_2 = y_3$. We can reduce the former case to the latter as was done previously. Thus we assume that G consists of three blocks B_1, B_2, and B_3 each joined by its bridge to a single vertex y. Create a new graph $G_{1,2}$ consisting of $B_1 \cup B_2$ together with the bridge (x_1, x_2). Similarly create $G_{1,3}$ and $G_{2,3}$. Each of these graphs has one bridge. Thus $c(G_{i,j}) \geqslant \frac{13}{15}E(G_{i,j})$. By permuting the colors of the bridges, if necessary, and by choosing the edge coloring of each block that colors most edges we see that

$$\begin{aligned} c(G) &\geqslant \tfrac{1}{2}(c(G_{1,2}) + c(G_{1,3}) + c(G_{2,3})) \\ &\geqslant \tfrac{13}{15}\tfrac{1}{2}(E(G_{1,2}) + E(G_{1,3}) + E(G_{2,3})) \\ &\geqslant \tfrac{13}{15}E(G). \end{aligned}$$

If $T(G)$ has exactly two adjacent non-leaf vertices, say y_1 and y_2, then G consists of four end blocks, say B_1 and B_2 joined to y_1 and B_3 and B_4 joined to y_2. Create $G_{1,3}$ (resp. $G_{2,4}$) by joining blocks B_1 and B_3 (resp. B_2 and B_4) by an edge. Since each of these graphs has exactly one bridge $c(G_{1,3}) \geqslant \frac{13}{15}E(G_{1,3})$ and $c(G_{2,4}) \geqslant \frac{13}{15}E(G_{2,4})$. By permuting colors we can color the bridge in each of these graphs differently and extend the coloring to G leaving a color available for the edge (y_1, y_2). Thus $c(G) \geqslant \frac{13}{15}(E(G) - 1) + 1$.

Petersen's graph shows that the above theorem is, in one sense, best possible; however, we do not know an infinite family of graphs in which γ equals (or even approaches) $\frac{13}{15}$. The smallest we can make γ in an infinite family of 3-connected graphs is the following: Let P^- denote the Petersen graph with one vertex deleted. Take a 3-connected cubic graph with $2t$ vertices. Replace each vertex by a copy of P^-. The three edges that were incident with a given vertex in the original graph are now incident with the three vertices of degree two in the copy of P^- (in any order). The resulting graph has $18t$ vertices and consequently $27t$ edges. Since in any 3-edge-coloring of P^- at least one edge remains uncolored, we know that $u \geqslant 2t$, thus $\gamma \leqslant \frac{25}{27}$.

3. Graphs with $\Delta = 3$

If one takes any cubic graph and subdivides one edge, a simple parity argument shows that the resulting graph cannot be 3-edge-colored. The smallest such instance is subdividing an edge in K_4. This graph which we call a *gadget* has $\gamma = \frac{6}{7}$. Our goal in this section is to establish a lower bound on γ for graphs with $\Delta = 3$. Suppose G is such a graph. We can first delete any vertex of degree 1 or contract any path of vertices of degree 2 to a single vertex of degree 2. An edge coloring of the resulting graph will easily extend to an edge coloring of the original without increasing u. Thus we may assume that G is a graph with V_2 vertices of degree 2 (no two of which are adjacent) and V_3 vertices of degree 3. Now we construct G' from G by attaching a gadget using a bridge to every vertex of degree 2. G' is cubic and $V(G') = V_3 + 6V_2$. By Theorem 1 $u(G') \leqslant \frac{2}{15}E(G')$. Since there must be at least one uncolored edge in every gadget,

$$u(G) \leqslant u(G') - V_2 \leqslant \tfrac{2}{15}\tfrac{3}{2}(V_3 + 6V_2) - V_2 = \tfrac{1}{5}(V_3 + V_2).$$

Thus

$$c(G) \geqslant \tfrac{1}{2}(3V_3 + 2V_2) - \tfrac{1}{5}(V_3 + V_2) = \tfrac{13}{15}E - \tfrac{1}{15}V_2. \tag{1}$$

If V_2 is small this bound is satisfactory. If V_2 is large we offer the following alternative. G contains $2V_2$ edges that are incident with vertices of degree 2. Thus there are $\frac{3}{2}V_3 - V_2$ edges that join two vertices of degree 3. Construct G'' by inserting a vertex of degree 2 in each such edge. Since G'' is biparite, $u(G'') = 0$. Now $u(G) \leqslant u(G'') + \frac{3}{2}V_3 - V_2$. Thus

$$c(G) \geqslant 2V_2. \tag{2}$$

Combining (1) and (2) yields $c(G) \geqslant \frac{26}{31}E$. We have proved the following theorem.

Theorem 2. *If $\Delta(G) = 3$, then $\gamma \geqslant \frac{26}{31}$.*

If G is planar, then the preceding inequality can be improved. Suppose G is an embedded plane graph with $\Delta(G) = 3$. We may assume no two vertices of degree 2 are incident with the same face: if they were, we just join them with an edge. Euler's formula together with these assumptions implies that $V_2 \leqslant 1 + \frac{1}{4}V_3$. From this we easily obtain:

$$V_2 \leqslant \frac{6+E}{7}.$$

This combines with (1) to show the following result.

Theorem 3. *If G is bridgeless, planar and $\Delta(G) = 3$, then $c(G) \geqslant \frac{6}{7}E - \frac{2}{35}$.*

It is straightforward to construct a bridgeless planar graph with $\Delta = 3$ and $V_2 = 1 + \frac{1}{4}V_3$. However, we know of no such graph in which $u > 1$. Thus though we prove that γ is asymptotically at least $\frac{6}{7}$, we believe that the correct lower bound for γ is $1 - 1/(V + \frac{1}{3})$. This has previously been noticed by Grötzsch [8] and independently by Hoffman et al. [5]. We formalize

Conjecture. If G is planar, bridgeless with $\Delta = 3$ and $V_2 \geqslant 2$, then $\gamma = 1$.

4. Cubic planar graphs with bridges

If we delete every bridge from a cubic planar graph the components that contain edges fall into the category of graphs considered in the preceding section. In such a component u might equal 0, 1, or for all we know as much as $\frac{1}{7}E$. Thus appealing to results of the last section is unlikely to provide precise information. However we can construct a graph with a large value of u by having lots of bridges.

Begin with a 3-1 tree, say T, a tree in which every vertex has degree three or 1. If L denotes the number of leaves in such a tree, then $L = \frac{1}{2}V + 1$. Replace each leaf with a gadget attaching the bridge so that the resulting graph is cubic. This graph has

$u = L$ and $E = \frac{9}{2}V(T) + 6$. It is a straightforward count to see that

$$\gamma = \frac{8}{9} - \frac{2}{27V(T) + 36} = \frac{8}{9} - \frac{2}{9V}.$$

We suspect that this is a best possible lower bound for cubic planar graphs.

5. Four-regular graphs

Since the $\frac{13}{15}$ result was obtained using a perfect matching together with a partial edge coloring of the remaining cycles, it is natural to try to inductively establish bounds on γ for 4-regular graphs. Indeed this can be done though not without restriction. We expect that bounds on γ for 4-regular graphs will depend on connectivity. If G is a 4-regular, 4-connected graph with an even number of vertices, then G necessarily contains a perfect matching, say M. This together with a partial edge coloring of the cubic graph $G - M$ shows that

$$c(G) \geqslant \tfrac{1}{2}V + \tfrac{13}{15}\tfrac{3}{2}V = \tfrac{9}{10}E.$$

If G is a 4-regular, 4-connected graph with an odd number of vertices, then G contains a matching with $\frac{1}{2}(V-1)$ edges, say M. $G - M$ will be regular of degree 3 with the exception of one vertex of degree four, say x. Let e denote an edge incident with x. Set $G' = G - M - e$. We can apply Eq. (1) to G' with $V_2(G') = 1$ to obtain $c(G') \geqslant \frac{13}{15}E(G') - \frac{1}{15}$. A partial edge coloring of G' together with M shows that

$$c(G) \geqslant c(G') + \tfrac{1}{2}(V-1) \geqslant \tfrac{1}{2}(V-1) + \tfrac{13}{15}\tfrac{1}{2}(3V-1) - \tfrac{1}{15} = \tfrac{9}{10}E - 1.$$

Theorem 4. *If G is a 4-regular, 4-connected graph and $\varepsilon \equiv V$ (mod 2), then $\gamma(G) \geqslant \frac{9}{10} - \varepsilon/E$.*

K_5 shows that this bound can be achieved. We now supply an infinite family of 4-connected, 4-regular graphs in which γ is bounded away from 1. First double every edge in the standard perfect matching in the Petersen graph.

Next replace every vertex in this multigraph with a copy of $K_{3,4}$. Use the edges of the multi-Petersen graph to construct a matching that meets every vertex of degree 3. The resulting graph, say H, is well known as the class 2, Meredith graph [4]. Let H^- denote H with one vertex deleted. It is straightforward to check that H^- is also a class 2 graph. Now take any 4-connected, 4-regular graph with t vertices. Replace each vertex with a copy of H^-. The four edges that were incident with a given vertex in the original graph are now incident with the four vertices of degree three in the copy of H^- (in any order). The resulting graph is 4-regular, 4-connected, and contains $69t$ vertices and $138t$ edges. Since in any coloring of H^- at least one edge remains uncolored, we know that $u \geqslant t$, whence $\gamma \leqslant \frac{137}{138}$.

In contrast with the preceding construction we do not know of any planar, 4-regular, 4-connected graph, in which $u > 2$.

On the other hand if we only insist on 2-connectivity we can make γ considerably smaller. Let H be any 4-regular graph with an odd number of vertices. It is immediate that $u(H) \geqslant 2$. Let H^* denote the graph obtained from H by removing an edge. Create an r-cycle of copies of H^* joining two vertices of degree 3 in consecutive copies. If H contains t vertices, then the resulting graph has $V = rt$ and $E = 2rt$. Since $u \geqslant r$, $\gamma \leqslant (2t - 1)/2t$. If $H = K_5$, then $\gamma = \frac{9}{10}$. If we wish to impose planarity, H will need to have $t \geqslant 7$, whence $\gamma = \frac{13}{14}$.

Note added in proof. Dan Archdeacon points out that M. Rosenfeld has considered the smallest class 1 cubic graph homomorphic to a given graph. [*On Tait coloring of cubic graphs*, in: Combinatorial Structures and their Applications, Gordon and Breach, NY, 1970, 373–376.]

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