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Laguerre polynomials and the inverse Laplace transform using discrete data

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Abstract

We consider the problem of finding a function defined on $(0,\infty)$ from a countable set of values of its Laplace transform. The problem is severely ill-posed. We shall use the expansion of the function in a series of Laguerre polynomials to convert the problem in an analytic interpolation problem. Then, using the coefficients of Lagrange polynomials we shall construct a stable approximation solution. Error estimate is given. Numerical results are produced.

Keywords: Inverse Laplace transform; Laguerre polynomials; Lagrange polynomials; Ill-posed problem; Regularization

1. Introduction

Let $L^2_\rho(0,\infty)$ be the space of real Lebesgue measurable functions defined on $(0,\infty)$ such that

$$\|f\|^2_{L^2_\rho} \equiv \int_0^\infty \left|f(x)\right|^2 e^{-x}\,dx < \infty.$$

This is a Hilbert space corresponding to the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx.$$

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We consider the problem of recovering a function $f \in L^2_\rho(0,\infty)$ satisfying the equations

$$\mathcal{L}f(p_j) \equiv \int_0^\infty e^{-p_j x} f(x)\,dx = \mu_j \tag{DIL}$$

where $p_j \in (0,\infty)$, $j = 1,2,3,\ldots.$

Generally, we have the classical problem of finding a function $f(x)$ from its given image $g(p)$ satisfying

$$\mathcal{L}f(p) \equiv \int_0^\infty e^{-px} f(x)\,dx = g(p), \tag{1}$$

where p is in a subset ω of the complex plane. We note that $\mathcal{L}f(p)$ is usually an analytic function on a half plane $\{\operatorname{Re} p > \alpha\}$ for an appropriate real number α. Frequently, the image of a Laplace transform is known only on a subset ω of the right half plane $\{\operatorname{Re} p > \alpha\}$. Depending on the set ω, we shall have appropriate methods to construct the function f from the values in the set

$$\{\mathcal{L}f(p)\colon\ p \in \omega\}.$$

Hence, there are no universal methods of inversion of the Laplace transform.

If the data $g(p)$ is given as a function on a line $(-i\infty + a, +i\infty + a)$ (i.e., $\omega = \{p\colon\ p = a + iy,\ y \in \mathbf{R}\}$) on the complex plane then we can use the Bromwich inversion formula [26, p. 67] to find the function $f(x)$.

If $\omega \subset \{p \in \mathbf{R}\colon\ p > 0\}$ then we have the problem of real inverse Laplace transform. The right-hand side is known only on $(0,\infty)$ or a subset of $(0,\infty)$. In this case, the use of the Bromwich formula is therefore not feasible. The literature on the subject is impressed in both theoretical and computational aspects (see, e.g., [2,3,10,16,18,22]). In fact, if the data $g(p)$ is given exactly then, by the analyticity of g, we have many inversion formulas (see, e.g., [3,7,8, 20,21,23]). In [3], the author approximate the function f by

$$f(t) \cong \sum_{k=0}^{N} b_k(a) d^k\big(e^x g\big(e^x\big)\big)/dx^k$$

where $b_k(a)$ are calculated and tabulated regularization coefficients and g is the given Laplace transform of f. Another method is developed by Saitoh and his group [4,5,20,21], where the function f is approximated by integrals having the form

$$u_N(t) = \int_0^\infty g(s)e^{-st} P_N(st)\,ds, \quad N = 1,2,\ldots,$$

where P_N is known (see [5]). Using the Saitoh formula, we can get directly error estimates.

However, in the case of inexact data, we have a severely trouble by the ill-posedness of the problem. In fact, a solution corresponding to the inexact data do not exist if the data is non-smooth, and in the case of existence, these do not depend continuously on the given data (that are represented by the right-hand side of the equalities). Hence, a regularization method is in order. In [7], the authors used the Tikhonov method to regularize the problem. In fact, in this method, we can approximate u_0 by functions u_β satisfying

$$\beta u_\beta + \mathcal{L}^*\mathcal{L}u_\beta = \mathcal{L}^* g, \quad \beta > 0.$$

Since $\mathcal{L}$ is self-adjoint (cf. [7]), the latter equation can be written as

$$\beta u_\beta + \int_0^\infty \frac{u_\beta(s)}{s+t}\,ds = \int_0^\infty e^{-st} g(s)\,ds.$$

The latter problem is well-posed.

Although the inverse Laplace transform has a rich literature, the papers devoted to the problem with discrete data are scarce. In fact, from the analyticity of $\mathcal{L}f(p)$, if $\mathcal{L}f(p)$ is known on a countable subset of $\omega \subset \{\operatorname{Re} p > \alpha\}$

accumulating at a point then $\mathcal{L}f(p)$ is known on the whole $\{\operatorname{Re} p > \alpha\}$. Hence, generally, a set of discrete data is enough for constructing an approximation function of f. It is a moment problem. In [15], the authors presented some theorems on the stabilization of the inverse Laplace transform. The Laplace image is measured at N points to within some error ϵ. This is achieved by proving parallel stabilization results for a related Hausdorff moment problem. For a construction of an approximate solution of (DIL), we note that the sequence of functions $(e^{-p_j x})$ is (algebraically) linear independent and moreover the vector space generated by the latter sequence is dense in $L^2(0,\infty)$. The method of truncated expansion as presented in [6, Section 2.1] is applicable and we refer the reader to this reference for full details. In [11,13], the authors convert (DIL) into a moment problem of finding a function in $L^2(0,1)$ and, then, they use Muntz polynomials to construct an approximation for f.

Now, in the present paper, we shall convert (DIL) to an analytic interpolation problem on the Hardy space of the unit disk. After that, we shall use Laguerre polynomials and coefficients of Lagrange polynomials to construct the function f. An approximation corresponding to the non-exact data and error estimate will be given.

The remainder of the paper divided into two sections. In Section 2, we convert our problem into an interpolation one and give a uniqueness result. In Section 3, we shall give two regularization results in the cases of exact data and non-exact data. Numerical comparisons with exact solution are given in the last section.

2. A uniqueness result

In this paper we shall use Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right).$$

We note that $\{L_n\}$ is a sequence of orthonormal polynomials on $L^2_\rho(0,\infty)$. We note that (see, e.g., [1], [9, p. 67])

$$\exp\left(\frac{xz}{z-1}\right)(1-z)^{-1} = \sum_{n=0}^{\infty} L_n(x)z^n.$$

Hence, if we have the expansion

$$f(x) = \sum_{n-0}^{\infty} a_n L_n(x)$$

then

$$\int_0^\infty f(x)\exp\left(\frac{xz}{z-1}\right)(1-z)^{-1}e^{-x}\,dx = \sum_{n=0}^{\infty} a_n z^n.$$

It follows that

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^\infty f(x)\exp\left(\frac{x}{z-1}\right)(1-z)^{-1}\,dx.$$

Put $\Phi f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\alpha_j = 1 - 1/p_j$, one has

$$\Phi f(\alpha_j) = p_j \mu_j,$$

i.e., we have an interpolation problem of finding an analytic function Φf in the Hardy space $H^2(U)$. Here, we denote by U the unit disk of the complex plane and by $H^2(U)$ the Hardy space. In fact, we recall that $H^2(U)$ is the space of all functions ϕ analytic in U and if, $\phi \in H^2(U)$ has the expansion $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ then

$$\|\phi\|^2_{H^2(U)} = \sum_{k=0}^{\infty} |a_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\phi\left(e^{i\theta}\right)\right|^2 d\theta.$$

We can verify directly that the linear operator Φ is an isometry from L^2_ρ onto $H^2(U)$. In fact, we have

Lemma 1. *Let* $f \in L^2_\rho(0,\infty)$. *Then* $\mathcal{L}f(z)$ *is analytic on* $\{z \in \mathbf{C} \mid \operatorname{Re} z > 1/2\}$. *If we have an expansion*

$$f = \sum_{n=0}^{\infty} a_n L_n$$

then one has $\Phi f \in H^2(U)$ *and*

$$\|\Phi f\|^2_{H^2(U)} = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2_{L^2_\rho(0,\infty)}.$$

Moreover, if we have in addition that $\sqrt{x} f' \in L^2_\rho$ *then*

$$\sum_{n=0}^{\infty} n|a_n|^2 \leqslant \|\sqrt{x} f'\|^2_{L^2_\rho}.$$

Proof. Putting $F_z(t) = e^{-zt} f(t)$, we have $F_z \in L^2(0,\infty)$ for every $\operatorname{Re} z > 1/2$. Hence $\mathcal{L}f(z) = \int_0^\infty F_z(t)\,dt$ is analytic for $\operatorname{Re} z > 1/2$. From the definitions of $L^2_\rho(0,\infty)$ and $H^2(U)$, we have the isometry equality. Now we prove the second inequalities. We first consider the case f', f'' in the space

$$B = \bigl\{g \text{ Lebesgue measurable on } (0,\infty) \bigm| \sqrt{x} g \in L^2_\rho(0,\infty)\bigr\}.$$

We have the expansion

$$f = \sum_{n=0}^{\infty} a_n L_n$$

where $a_n = \langle f, L_n\rangle$.

The function $y = L_n$ satisfies the following equation (see [17])

$$xy'' + (1-x)y' + ny = 0$$

which gives

$$\bigl(xe^{-x} y'\bigr)' + nye^{-x} = 0.$$

It follows that

$$\begin{aligned} na_n &= \int_0^\infty f(x) n L_n(x) e^{-x}\,dx = -\int_0^\infty f(x)\bigl(xe^{-x} L_n'(x)\bigr)'\,dx = \int_0^\infty f'(x) x e^{-x} L_n'(x)\,dx \\ &= -\int_0^\infty \bigl(f'(x) x e^{-x}\bigr)' L_n(x)\,dx = -\int_0^\infty \bigl(xf''(x) + f'(x) - xf'(x)\bigr) L_n(x) e^{-x}\,dx = -\langle xf'' + f' - xf', L_n\rangle. \end{aligned}$$

Since L_n is an orthonormal basis, we have the Fourier expansion

$$xf'' + f' - xf' = \sum_{n=0}^{\infty} (-na_n) L_n.$$

Using the Parseval equality we have

$$\langle xf'' + f' - xf', f\rangle = \sum_{n=0}^{\infty} (-na_n) a_n.$$

It can be rewritten as

$$\int_0^\infty \bigl(xe^{-x} f'(x)\bigr)' f(x)\,dx = -\sum_{n=0}^{\infty} na_n^2.$$

Integrating by parts, we get

$$\int_0^\infty xe^{-x}\left|f'(x)\right|^2\,dx=\sum_{n=0}^\infty na_n^2.$$

Now, for $f'\in B$ we choose (f_k) such that $f_k', f_k''\in B$ for every $k=1,2,\dots$ and $\sqrt{x}f_k'$ (respectively f_k) $\to\sqrt{x}f'$ (respectively f) in L^2_ρ as $k\to\infty$. Assume that

$$f_k=\sum_{n=0}^\infty a_{kn}L_n.$$

Then we have

$$\int_0^\infty xe^{-x}\left|f_k'(x)\right|^2\,dx=\sum_{n=0}^\infty na_{kn}^2.$$

The latter equality involves for every N

$$\sum_{n=0}^N na_{kn}^2\leqslant\left\|\sqrt{x}f_k'\right\|^2_{L^2_\rho(0,\infty)}. \tag{2}$$

Since $f_k\to f$ in L^2_ρ as $k\to\infty$ we have that $a_{kn}\to a_n$ as $k\to\infty$, for each n. On the other hand, we have $\sqrt{x}f_k'\to\sqrt{x}f'$ in L^2_ρ as $k\to\infty$. Therefore, letting $k\to\infty$ in (2) we get

$$\sum_{n=0}^N na_n^2\leqslant\|\sqrt{x}f'\|^2_{L^2_\rho(0,\infty)}.$$

Letting $N\to\infty$ in the latter inequality, we get the desired inequality. □

Using Lemma 1, one has a uniqueness result

Theorem 1. *Let $p_j>1/2$ for every $j=1,2,\dots$. If*

$$\sum_{p_j>1}\frac{1}{p_j}+\sum_{1/2<p_j<1}\frac{2p_j-1}{p_j}=\infty$$

then Problem (DIL) *has at most one solution in $L^2_\rho(0,\infty)$.*

Proof. Let $f_1,f_2\in L^2_\rho(0,\infty)$ be two solutions of (DIL). Putting $g=f_1-f_2$ then $g\in L^2_\rho(0,\infty)$ and $\mathcal{L}g(p_j)=0$. It follows that $\Phi g(1-1/p_j)=0$, $j=1,2,\dots$. It follows that $\alpha_j=1-1/p_j$ are zeros of Φg. We have $\Phi g\in H^2(U)$ and

$$\sum_{j=1}^\infty\bigl(1-|\alpha_j|\bigr)=\sum_{p_j>1}\frac{1}{p_j}+\sum_{1/2<p_j<1}\frac{2p_j-1}{p_j}=\infty.$$

Hence we get $\Phi g\equiv 0$ (see, e.g., [19, p. 308]). It follows that $g\equiv 0$. This completes the proof of Theorem 1. □

3. Regularization and error estimates

In the section, we assume that (p_j) is a bounded sequence, $p_j\neq p_k$ for every $j\neq k$. Without loss of generality, we shall assume that $\rho=1$ is an accumulation point of p_j. In fact, if p_j has an accumulation point $\rho_0>1$ then, by putting

$\tilde{f}(x) = e^{-(\rho_0-1)x} f(x)$ and $p'_j = p_j - \rho_0 + 1$, we can transform the problem to the one of finding $\tilde{f} \in L^2_\rho(0,\infty)$ such that

$$\int_0^\infty e^{-p'_j x} \tilde{f}(x)\,dx = \mu_j, \quad j = 1, 2, \ldots,$$

in which p'_j has the accumulation point $\rho = 1$. In fact, in Theorem 2 below, we shall assume that $|1 - \frac{1}{p_j}| \leqslant \sigma$ for every $j = 1, 2, \ldots$, where σ is a given number.

We denote by $\ell_k^{(m)}(\nu)$ the coefficient of z^k in the expansion of the Lagrange polynomial $L_m(\nu)$ ($\nu = (\nu_1, \ldots, \nu_m)$) of degree (at most) $m-1$ satisfying

$$L_m(\nu)(z_k) = \nu_k, \quad 1 \leqslant k \leqslant m,$$

where $z_k = \alpha_k$. If ϕ is an analytic function on U, we also denote

$$L_m(\phi) = L_m\big(\phi(z_1), \ldots, \phi(z_m)\big).$$

We define

$$L_m^\theta(\nu)(z) = \sum_{0 \leqslant k \leqslant \theta(m-1)} \ell_k^{(m)}(\nu) z^k.$$

The polynomial $L_m^\theta(\nu)$ is called a truncated Lagrange polynomial (see also [25]). For every $g \in L^2_\rho(0,\infty)$, we put

$$T_n g = \big(p_1 \mathcal{L}g(p_1), \ldots, p_n \mathcal{L}g(p_n)\big),$$
$$T g = \big(p_1 \mathcal{L}g(p_1), \ldots, p_n \mathcal{L}g(p_n), \ldots\big) \in \ell^\infty.$$

Here, we recall that $\alpha_n = 1 - 1/p_n$. We shall approximate the function f by

$$F_m = \Phi^{-1} L_m^\theta(T_m f) = \sum_{0 \leqslant k \leqslant \theta(m-1)} \ell_k^{(m)}(T_m f) L_k.$$

We shall prove that F_m is an approximation of f. Before stating and proving the main results, some remarks are in order.

We first recall the concept of regularization. Let f be an exact solution of (DIL), we recall that a sequence of linear operator $A_n : \ell^\infty \to L^2_\rho(0,\infty)$ is a regularization sequence (or a regularizer) of Problem (DIL) if (A_n) satisfies two following conditions (see, e.g., [14, p. 25])

(R1) For each n, A_n is bounded,
(R2) $\lim_{n\to\infty} \|A_n(Tf) - f\| = 0$.

The number "n" is called the *regularization parameter*. As a consequence of (R1), (R2), we can get

(R3) For $\epsilon > 0$, there exists the functions $n(\epsilon)$ and $\delta(\epsilon)$ such that $\lim_{\epsilon\to 0} n(\epsilon) = \infty$, $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$ and that

$$\big\|A_{n(\epsilon)}(\mu) - f\big\| \leqslant \delta(\epsilon)$$

for every $\mu \in \ell^\infty$ such that $\|\mu - Tf\|_\infty < \epsilon$.

In the present paper, the operator A_n is $\Phi^{-1} L_m^\theta$. The number ϵ is the error between the exact data Tf and the measured data μ. For a given error ϵ, there are infinitely many ways of choosing the regularization parameter $n(\epsilon)$. In the present paper, we give an explicit form of $n(\epsilon)$.

Next, in our paper, we have the interpolation problem of reconstruction the analytic function $\phi = \Phi f \in H^2(U)$ from a sequence of its values $(\phi(\alpha_n))$. As known, the convergence of $L_m(\phi)$ to ϕ depends heavily on the properties of the points (α_n). The Kalmár–Walsh theorem (see, e.g., [12, p. 65]) shows that $L_m(\phi) \to \phi$ for every ϕ in $C(\overline{U})$ for all ϕ analytic in a *neighborhood* of $\overline{U}$ if and only if (α_n) is uniformly distributed in $\overline{U}$, i.e.,

$$\lim_{m\to\infty} \sqrt[m]{\max_{|z|\leqslant 1} \big|(z-\alpha_1)\cdots(z-\alpha_m)\big|} = 1.$$

The Fejer points and the Fekete points are the sequences of points satisfying the latter condition (see [12, p. 67]). The Kalmár–Walsh fails if $C(\overline{U})$ is replaced by $H^2(U)$ (see [25] for a counterexample). Hence, the Lagrange polynomial cannot use to reconstruct ϕ. In [12], we proved a theorem similar to the Kalmár–Walsh theorem for the case of $H^2(U)$. In fact, the Lagrange polynomials will convergence if we "cut off" some terms of the Lagrange polynomial. Especially, in [12] and the present paper, the points (α_n) are, in general, not uniformly distributed.

In Theorem 2, we shall verify the condition (R2). More precisely, we have

Theorem 2. *Let $\sigma \in (0, 1/3)$, let $f \in L^2_\rho(0,\infty)$ and let $p_j > 1/2$ for $j = 1, 2, \dots$ satisfy*

$$\left|1 - \frac{1}{p_j}\right| \leqslant \sigma.$$

Put θ_0 be the unique solution of the equation (unknown x)

$$\frac{2\sigma^{1-x}}{1-\sigma} = 1.$$

Then for $\theta \in (0, \theta_0)$, one has

$$\|f - F_m\|^2_{L^2_\rho} \to 0 \quad \text{as } m \to \infty.$$

If, we assume in addition that $\sqrt{x} f' \in L^2_\rho(0,\infty)$ then

$$\|f - F_m\|^2_{L^2_\rho} \leqslant (1 + m\theta)^2 \|f\|^2_{L^2_\rho} \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \frac{1}{m\theta} \|\sqrt{x} f'\|^2_{L^2_\rho(0,\infty)}.$$

Proof. We have in view of Lemma 1

$$\|f - F_m\|^2_{L^2_\rho} = \sum_{0 \leqslant k \leqslant \theta(m-1)} \left|\delta_k^{(m)}\right|^2 + \sum_{k > \theta(m-1)} |a_k|^2 \tag{3}$$

where $\delta_k^{(m)} = a_k - \ell_k^{(m)}(T_m f)$. We shall give an estimate for $\delta_k^{(m)}$. In fact, we have

$$\left\|\Phi f - L_m(T_m f)\right\|^2_{H^2(U)} = \sum_{k=0}^{m-1} \left|\delta_k^{(m)}\right|^2 + \sum_{k=m}^{\infty} |a_k|^2.$$

On the other hand, the Hermite representation (see, e.g., [12, p. 59], [24]) gives

$$\Phi f(z) - L_m(T_m f)(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\omega_m(z)(\Phi f)(\zeta)\, d\zeta}{\omega_m(\zeta)(\zeta - z)}$$

where $\omega_m(z) = (z - \alpha_1)\dots(z - \alpha_m)$. Now, if we denote by $\sigma_{-1}^{(m)} = \sigma_{-2}^{(m)} = \cdots = 0$ and

$$\sigma_0^{(m)} = 1,$$
$$\sigma_r^{(m)} = \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant m} \alpha_{j_1} \dots \alpha_{j_r} \quad (1 \leqslant r \leqslant m),$$
$$\beta_s^{(m)} = \frac{1}{2\pi i} \int_{\partial U} \frac{\Phi f(\zeta)\, d\zeta}{\zeta^{s+1} \omega_m(\zeta)}$$

then we can write in view of the Hermite representation

$$\Phi f(z) - L_m(T_m f)(z) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)} \right) z^k.$$

From the latter representation, one gets

$$\delta_k^{(m)} = \sum_{r=0}^{k} (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)}, \quad 0 \leqslant k \leqslant m - 1.$$

Now, by direct computation, one has

$$\left|\beta_s^{(m)}\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|\Phi f(e^{i\theta})|}{|\omega_m(e^{i\theta})|}\,d\theta.$$

But one has

$$\left|\omega_m\left(e^{i\theta}\right)\right| \geqslant \left(\left|e^{i\theta}\right| - |\alpha_1|\right)\ldots\left(\left|e^{i\theta}\right| - |\alpha_m|\right) \geqslant (1-\sigma)^m.$$

Hence

$$\left|\beta_s^{(m)}\right| \leqslant \frac{1}{2\pi(1-\sigma)^m}\int_0^{2\pi} \left|\Phi f\left(e^{i\theta}\right)\right| d\theta \leqslant \|\Phi f\|_{H^2(U)}(1-\sigma)^{-m}.$$

We also have

$$\left|\sigma_{m-r}^{(m)}\right| \leqslant \sigma^{m-r} C_m^r \leqslant \sigma^{m-k} 2^m,$$

where $C_m^k = \frac{m!}{k!(m-k)!}$. Hence, we have

$$\left|\delta_k^{(m)}\right| \leqslant (1+m\theta)\|f\|_{L^2_\rho}\left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^m.$$

From the latter inequality, one has in view of (3)

$$\|f - F_m\|_{L^2_\rho}^2 \leqslant (1+m\theta)^2\|f\|_{L^2_\rho}^2\left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \sum_{k\geqslant m\theta}^{\infty} |a_k|^2.$$

For $\theta \in (0, \theta_0)$, one has

$$0 < \frac{2\sigma^{1-\theta}}{1-\sigma} < \frac{2\sigma^{1-\theta_0}}{1-\sigma} = 1.$$

Hence, we have

$$\lim_{m\to\infty} \|f - F_m\|_{L^2_\rho}^2 = 0$$

as desired, since, on the one hand, we have the comparison between an exponential with base $b < 1$ and a power function and, in the other hand, the remain of a convergent series $\sum_{k=0}^{\infty} |a_k|^2$.

Now if $\sqrt{x} f' \in L^2_\rho(0,\infty)$ then one has since $\frac{k}{m\theta} > 1$ and from Lemma 1

$$\sum_{k>m\theta}^{\infty} |a_k|^2 \leqslant \frac{1}{m\theta}\sum_{k=0}^{\infty} k|a_k|^2 \leqslant \frac{1}{m\theta}\|\sqrt{x} f'\|_{L^2_\rho}^2.$$

This completes the proof of Theorem 2. □

Now, we consider the case of non-exact data. In Theorem 3, we shall consider the condition (R3) of the definition of the regularization. Put

$$D_m = \max_{1\leqslant n\leqslant m}\left(\max_{|z|\leqslant R}\left|\frac{\omega_m(z)}{(z-\alpha_n)\omega_m'(\alpha_n)}\right|\right).$$

Let $\psi : [0,\infty) \to \mathbf{R}$ be an increasing function satisfying

$$\psi(m) \geqslant m D_m, \quad m = 1, 2, \ldots,$$

and

$$m(\epsilon) = \left[\psi^{-1}\left(\epsilon^{-3/4}\right)\right] - 1$$

where $[x]$ is the greatest integer $\leqslant x$.

Theorem 3. *Let $\sigma \in (0, 1/3)$, let $f, \sqrt{x} f' \in L^2_\rho(0, \infty)$ and let $p_j > 1/2$ for $j = 1, 2, \ldots$ satisfy*

$$\left|1 - \frac{1}{p_j}\right| \leqslant \sigma.$$

Put θ_0 be the unique solution of the equation (unknown x)

$$\frac{2\sigma^{1-x}}{1-\sigma} = 1.$$

Let $\epsilon > 0$ and let (μ_j^ϵ) be a measured data of $(\mathcal{L} f(p_j))$ satisfying

$$\sup_j \left| p_j \left(\mathcal{L} f(p_j) - \mu_j^\epsilon\right)\right| < \epsilon.$$

Then for $\theta \in (0, \theta_0)$, one has

$$\left\| f - \Phi^{-1} L^\theta_{m(\epsilon)}\left(\nu^\epsilon\right)\right\|^2_{L^2_\rho} \leqslant 2\left(1 + m(\epsilon)\theta\right)^2 \|f\|^2_{L^2_\rho} \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m(\epsilon)} + \frac{2}{m(\epsilon)\theta} \|\sqrt{x} f'\|^2_{L^2_\rho} + 2\epsilon^{1/2},$$

where $\nu_j^\epsilon = p_j \mu_j^\epsilon$ for $j = 1, 2, \ldots$.

Proof. We note that

$$L_m(T_m f)(z) - L_m\left(\nu^\epsilon\right)(z) = \sum_{j=1}^m \left(p_j \mu_j - \nu_j^\epsilon\right) \frac{\omega_m(z)}{(z - \alpha_j)\omega'_m(\alpha_j)}.$$

It follows that

$$\left\| L_m(T_m f) - L_m\left(\nu^\epsilon\right)\right\|_\infty \leqslant \epsilon m D_m.$$

Hence

$$\left\| L^\theta_m(T_m f) - L^\theta_m\left(\nu^\epsilon\right)\right\|_{H^2(U)} \leqslant \left\| L_m(T_m f) - L_m\left(\nu^\epsilon\right)\right\|_\infty \leqslant \epsilon m D_m.$$

It follows by the isometry property of Φ

$$\begin{aligned}\left\| f - \Phi^{-1} L^\theta_m\left(\nu^\epsilon\right)\right\|^2_{L^2_\rho} &\leqslant 2\|f - F_m\|^2_{L^2_\rho} + 2\left\|\Phi^{-1} L^\theta_m(T_m f) - \Phi^{-1} L^\theta_m\left(\nu^\epsilon\right)\right\|^2_{L^2_\rho} \\ &\leqslant 2(1 + m\theta)^2 \|f\|^2_{L^2_\rho} \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \frac{2}{m\theta} \|\sqrt{x} f'\|^2_{L^2_\rho} + 2\epsilon^2 m^2 D_m^2.\end{aligned}$$

By choosing $m = m(\epsilon)$ we get the desired result. □

4. Numerical results

We present some results of numerical comparison between the function $f(x)$ given in $L^2_p(0, \infty)$ and its approximated form F_m as it is stated in Theorem 2.

First consider the function $f(x) = e^{-x}$ and its expansion in Laguerre series

$$e^{-x} = \sum_{n \geqslant 0} \frac{1}{2^{n+1}} L_n(x). \tag{4}$$

So in the Hardy space $H^2(U)$, we have to interpolate the analytic function

$$\Phi f(x) = \sum_{n \geqslant 0} \frac{1}{2^{n+1}} x^n = \frac{1}{2 - x} \tag{5}$$

by the Lagrange polynomial $L_m(T_m f)$, interpolation defined by

$$L_m(T_m f)\left(1 - \frac{1}{p_i}\right) = p_i \int_0^\infty e^{-p_i x} e^{-x}\, dx = \frac{p_i}{p_i + 1} \tag{6}$$

where $p_i \to 1$ as $i \to \infty$.

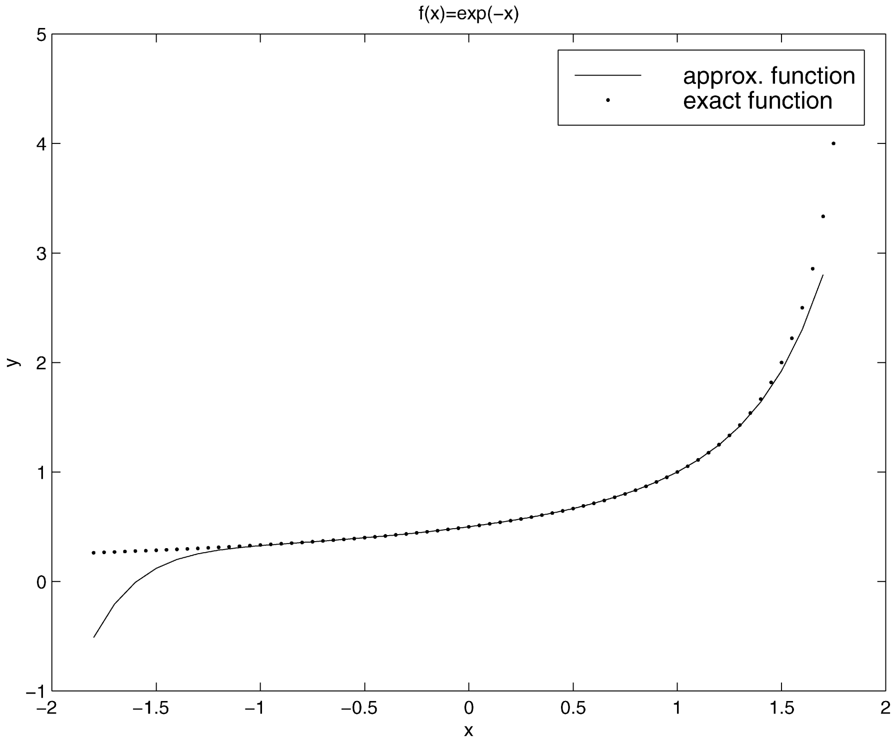
f(x)=exp(−x)
approx. function
exact function
y
x
5
4
3
2
1
0
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Fig. 1.

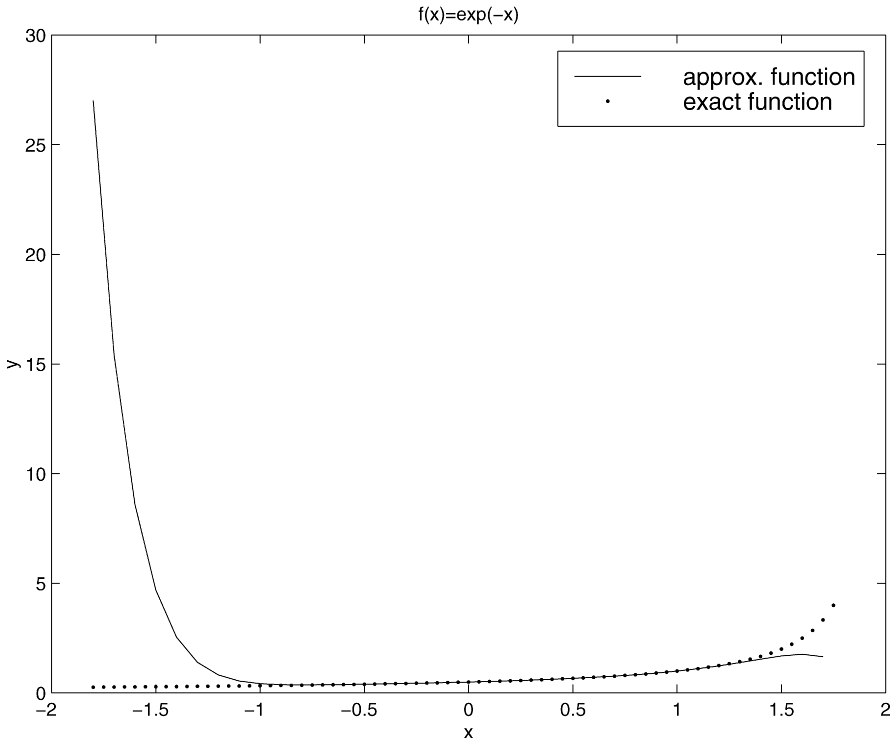
f(x)=exp(−x)
approx. function
exact function
y
x
30
25
20
15
10
5
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Fig. 2.

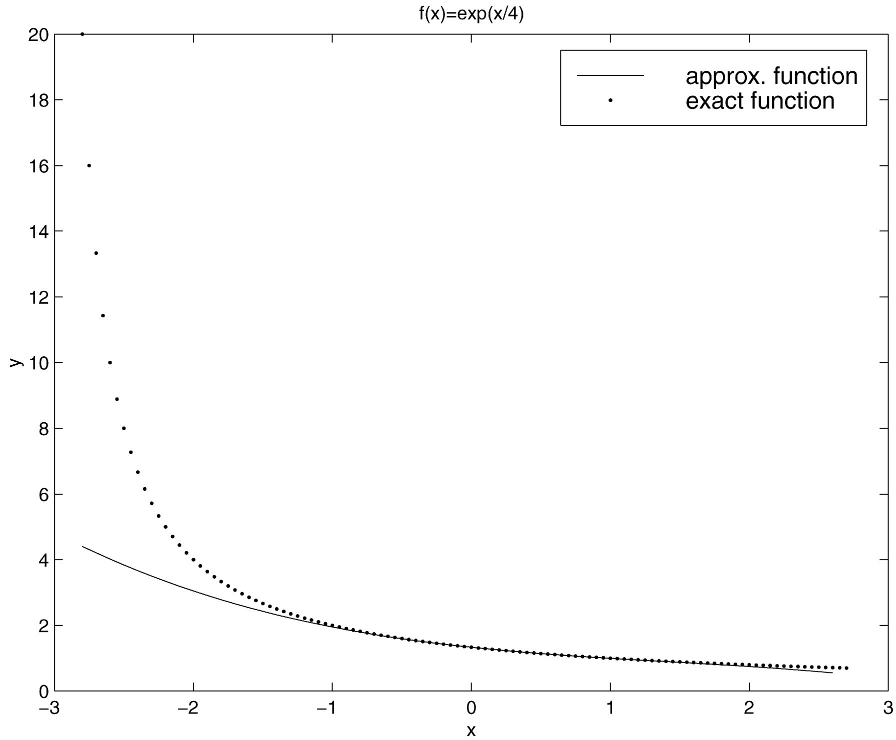
f(x)=exp(x/4)
approx. function
exact function
y
x
20
18
16
14
12
10
8
6
4
2
0
−3
−2
−1
0
1
2
3

Fig. 3.

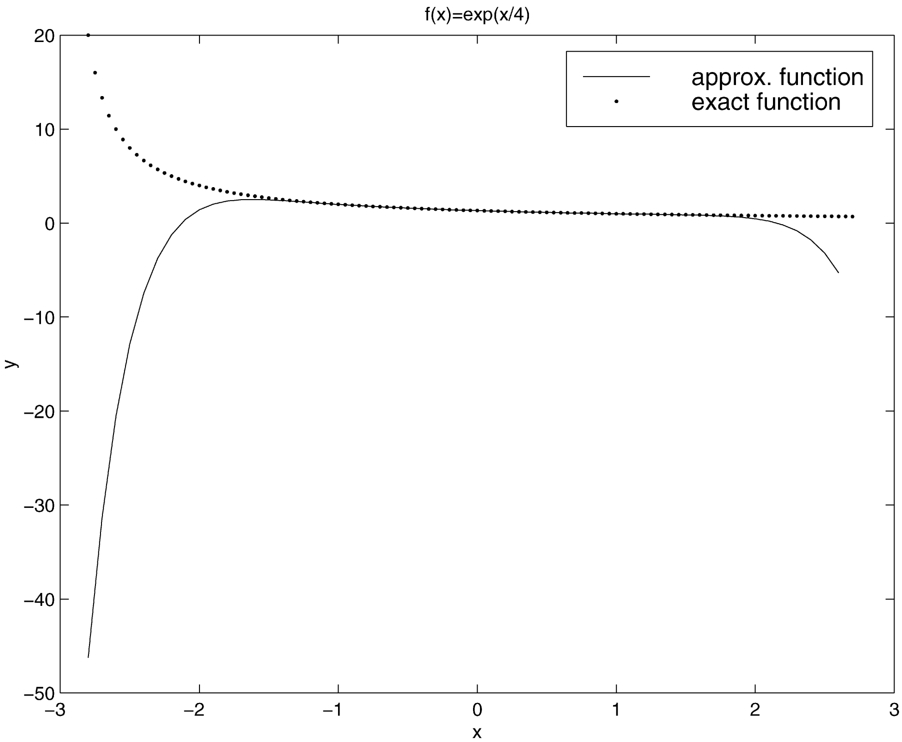
f(x)=exp(x/4)
approx. function
exact function
y
x
20
10
0
−10
−20
−30
−40
−50
−3
−2
−1
0
1
2
3

Fig. 4.

On the interval $(-1.8,+1.8)$ we have drawn in Fig. 1 the curves e^{-x} and its approximation $L_m(T_m f)(x)$ for $m=10$. If $m=12$ there is divergence for our interpolation (Fig. 2) outside the interval $(-1,+1)$.

In our 2nd example we have chosen the function

$$f(x)=e^{x/4}=\frac{4}{3}\sum_{n\geqslant 0}\left(\frac{-1}{3}\right)^n L_n(x). \tag{7}$$

In the Hardy space the function

$$\Phi f(x)=\frac{4}{3}\sum_{n\geqslant 0}\left(\frac{-x}{3}\right)^n=\frac{4}{3+x}$$

is approximated by the Lagrange polynomial $L_m(T_m f)$ at the points $(1-\frac{1}{p_i},\frac{-4p_i}{1-4p_i})$, $p_i\to 1$ as $i\to\infty$.

Figure 3 (respectively Fig. 4) shows the quite good convergence (respectively divergence) on the interval $(-2.8,2.8)$ with $m=4$ (respectively $m=11$).

In both cases we have chosen $\theta_0=0.29$ with $\sigma=0.25$ (θ_0 given by $\frac{2\sigma^{1-\theta_0}}{1-\sigma}=1$, $0<\sigma<\frac{1}{3}$). So in the 2nd case the truncated Lagrange polynomial is almost verified since $11\times 0.29\sim 3.2$.

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