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# Submanifolds, isoperimetric inequalities and optimal transportation

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**Abstract**

The aim of this paper is to prove isoperimetric inequalities on submanifolds of the Euclidean space using mass transportation methods. We obtain a sharp "weighted isoperimetric inequality" and a nonsharp classical inequality similar to the one obtained in Michael and Simon (1973) [19]. The proof relies on the description of a solution of the problem of Monge when the initial measure is supported in a submanifold and the final one supported in a linear subspace of the same dimension.


**Résumé**

Le but de cet article est de démonter des inégalités isopérimétriques sur les sous-variétés de l'espace euclidien en utilisant des méthodes de transport optimal de mesures. On obtient ainsi une "inégalité isopérimétrique à poids" avec constante optimale et une inégalité classique similaire à celle obtenue dans Michael et Simon (1973) [19]. La preuve repose sur la description d'une solution du problème de Monge entre une mesure initiale supportée par une sous-variété et une mesure finale supportée par un sous-espace de même dimension.


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## 0. Introduction

The classical isoperimetric inequality of the Euclidean space states that, for any regular domain $\Omega \subset \mathbb{R}^n$,

$$n\omega_n^{\frac{1}{n}} \operatorname{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \operatorname{vol}(\partial\Omega)$$

with equality if and only if $\Omega$ is a ball ($\omega_n$ being the volume of the unit ball). This inequality admits a lot of generalization to other geometries (cf. [20] for a classical survey, and [22] for a more recent one), and on the other hand, a natural question is to find geometries that share the Euclidean isoperimetric inequality. One of the class of Riemannian manifolds expected to satisfy this inequality is the class of minimal submanifolds in Euclidean spaces, and more generally in Cartan–Hadamard manifolds.

In this setting, the existence of a positive isoperimetric constant was proved by W.K. Allard (cf. [1]) and by J.H. Michael and L.M. Simon (cf. [19]). In the more general setting of arbitrary submanifolds we have the following inequality (cf. [19], Theorem 2.1): *there exists a positive constant $C_n$, depending only on $n$, such that for any domain $\Omega$ in an $n$-dimensional submanifold of $\mathbb{R}^{n+k}$*

$$C_n \operatorname{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + n \int_{\Omega} |H| \, dv_M$$

*where $H$ is the mean curvature vector of $M$.*

This result was then extended to submanifolds in Cartan–Hadamard manifolds (cf. [16] and [6]), but the question of the optimal constant for this inequality is still an open problem, even for minimal surfaces in $\mathbb{R}^3$ (cf. [8,10,11] for partial results, and [9] for a survey on this question).

A way to prove the Euclidean isoperimetric inequality is to construct a map, with fine geometric properties, which push forward the uniform measure on $\Omega$ to the uniform measure on the unit ball: this has to be seen as a way to compare the domain $\Omega$ to the model domain satisfying the equality case. This approach was first used by M. Gromov using a map constructed by Knothe (cf. for example [7] for the proof), and in the sequel we shall refer to such a mapping as a “Knothe map”.

More recently, D. Cordero-Erausquin, B. Nazaret and C. Villani observed that the solution of an optimal transportation problem between the two measures could be used as a “Knothe map” (cf. [12]): a theorem by Y. Brenier [2] states that, if $\mu$ is a probability measure on $\mathbb{R}^n$ that do not give mass to small sets (i.e. sets with Hausdorff dimension less than or equal to $n-1$) then, for any probability measure $\nu$, there exists a convex function whose gradient push forward $\mu$ on $\nu$. This approach was also used in [14] to get isoperimetric type inequalities in space form.

In the case of an $n$-dimensional submanifold of $\mathbb{R}^{n+k}$ we would like to compare the uniform measure on $\Omega$ with the model measure which is the uniform one on the unit ball of $n$-dimensional subspace of $\mathbb{R}^{n+k}$; however, we are precisely in the case where Brenier's theorem does not hold as the first measure is supported in a small set. The goal of this paper is to deal with the two following questions: considering two measures in $\mathbb{R}^{n+k}$ supported in submanifold and in a linear subspace of the same dimension, what are the solutions of the optimal transportation problem? Do these solutions have fine geometric properties to give isoperimetric inequalities on the submanifold?

In the first section we recall the main results which will be used in the remainder of the paper: the equivalence between isoperimetric and Sobolev inequalities, existence and properties of the solution of the optimal transportation problem in Euclidean space, and differentiability properties of convex functions.

In the second section we describe solutions of the mass transportation problem between a measure supported in a submanifold and a measure supported in a linear subspace. It is shown in particular that orthogonal projections play a natural role in this problem.

The third section is devoted to the proof of the main theorem: using the optimal map we can compare the uniform measure on a domain in a submanifold with the model measure. We get the following sharp "weighted isoperimetric inequality" (cf. Theorem 3.1):

**Theorem.** *Let $i : M^n \to \mathbb{R}^{n+k}$ be an isometric immersion, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$. For any regular domain $\Omega \subset M$ we have*

$$n\omega_n^{\frac{1}{n}}\left(\int_\Omega J_E^{\frac{1}{n-1}}\,dv_M\right)^{\frac{n-1}{n}} \leqslant \mathrm{vol}(\partial\Omega) + n\int_\Omega |H|\,dv_M,$$

*where $H$ is the mean curvature vector of the immersion, and $J_E$ is the absolute value of the Jacobian determinant of the orthogonal projection from $M$ to $E$. This inequality is sharp, as we have equality when $\Omega$ is a geodesic ball in $E$.*

*The Sobolev counterpart of this inequality is*

$$n\omega_n^{\frac{1}{n}}\left(\int_M J_E^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\,dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\,dv_M + n\int_M |H||u|\,dv_M$$

*for any function $u \in C_c^\infty(M)$.*

We also obtain in this section a classical isoperimetric inequality (i.e. of the form $C\,\mathrm{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \mathrm{vol}(\partial\Omega) + n\int_\Omega |H|\,dv_M$) with a constant which is not sharp but improve by far the constants given in [19] and [16] (cf. Theorem 3.2 and the remark thereafter).

The fourth section is devoted to the study of certain warped products on which our method still applies and gives weighted Sobolev inequalities.

## 1. Preliminaries

### *1.1. Isoperimetric and Sobolev inequalities*

It is a well-known fact (due to Federer and Fleming, cf. for example [7] for a proof) that, on Riemannian manifolds, the isoperimetric inequality is equivalent to the $L^1$ Sobolev inequality: $C\,\mathrm{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \mathrm{vol}(\partial\Omega)$ for any domain $\Omega \subset M$ if and only if $C(\int_M |u|^{\frac{n}{n-1}})^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|$ for any $u \in C_c^\infty(M)$ (with the same constant in both inequalities). This equivalence still holds true for the (weighted) isoperimetric inequalities with the extra curvature term we are considering in this paper.

In the sequel, we shall prove the Sobolev statement of the inequalities. By density of the smooth functions, the Sobolev inequality still holds for functions in Sobolev spaces, and since $|\nabla u| = |\nabla|u||$ almost everywhere, it is sufficient to consider nonnegative smooth functions.

As was observed in [12], the $L^p$ Sobolev inequalities on $\mathbb{R}^n$ can also be obtained using the mass transportation method. In fact, they obtain a nice duality principle, and if

$$S_{n,p} = \inf\left\{ \frac{\|\nabla u\|_p}{\|u\|_{\frac{np}{n-p}}} \mid u \in C_c^\infty\left(\mathbb{R}^n\right) \right\}$$

is the $L^p$ Sobolev constant of $\mathbb{R}^n$, then $S_{n,p}$ can also be obtained as the following supremum over smooth functions (cf. [12, Theorem 2]):

$$S_{n,p} = \frac{n(n-p)}{p(n-1)} \sup\left\{ \frac{\int |v|^{\frac{p(n-1)}{n-p}}}{\left(\int |y|^{\frac{p}{p-1}} |v(y)|^{\frac{np}{n-p}}\, dy\right)^{\frac{p-1}{p}}} \mid v \in C_c^\infty\left(\mathbb{R}^n\right),\ \|v\|_{\frac{np}{n-p}} = 1 \right\}.$$

As our method to get the Sobolev inequalities is derived from the one used in [12], this characterization of $S_{n,p}$ will appear naturally.

### *1.2. Mass transportation problems*

Consider two Polish spaces $X_1$ and $X_2$, and a "cost function" $c : X_1 \times X_2 \to \mathbb{R}$. Given two probability measures $\mu$ and $\nu$ on $X_1$ and $X_2$ respectively, the cost of a map $T : X_1 \to X_2$ which push forward $\mu$ on $\nu$ is $J(T) = \int_{X_1} c(x, Tx)\, d\mu$. The problem of Monge consists in finding a map whose cost is the infimum of the costs of all maps pushing forward $\mu$ on $\nu$.

The problem of Monge may have no solution, and it is useful to consider a relaxed form: the Monge–Kantorovich problem. We now consider transference plans between $\mu$ and $\nu$, that is probability measures $\rho$ on $X_1 \times X_2$ whose marginals are $\pi^1_\# \rho = \mu$ and $\pi^2_\# \rho = \nu$ (where $\pi^i$ is the projection on $X_i$). The cost of a transference plan $\rho$ is $J(\rho) = \int_{X_1 \times X_2} c(x_1, x_2)\, d\rho(x_1, x_2)$, and an optimal transference plan (i.e. a solution of Monge–Kantorovich problem) is a transference plan whose cost is the infimum of the costs of all transference plan between $\mu$ and $\nu$.

In particular, if a map $T : X_1 \to X_2$ push forward $\mu$ on $\nu$, then it gives rise to a transference plan $\rho = (Id \times T)_\# \mu$ whose support in $X_1 \times X_2$ is $\mathrm{Spt}(\rho) = \{(x, Tx) \mid x \in \mathrm{Spt}(\mu)\}$; if an optimal transference plan is of this form, then the map $T$ is a solution of the problem of Monge.

The properties of optimal maps and transference plans depend on the properties of the Polish spaces $X_1$ and $X_2$ and on the cost functions; the main reference on this subject is [24]. In the sequel we shall work with the "quadratic cost": $X = Y$ and $c(x, y) = d(x, y)^2$ where $d$ is the distance on $X$. The main result we shall use on optimal transportation is the following theorem due to Y. Brenier (cf. [24] for a proof):

**Theorem 1.1.** *If $\mu$ and $\nu$ are probability measures on $\mathbb{R}^n$ which do not charge small sets* (*i.e. sets with Hausdorff dimension less than or equal to $n-1$*), *then there exists a unique optimal transference plan $\rho$ between $\mu$ and $\nu$.*

*Moreover, $\rho = (Id \times T)_\# \mu$, where $T : \mathrm{Spt}(\mu) \to \mathrm{Spt}(\nu)$ is the gradient of a convex function.*

The optimality of a transference plan is related to the c-cyclical monotonicity of its support (cf. [24]). It is not true in general that a transference plan is optimal if and only if its support is c-cyclically monotone, but in our setting, as the cost function is continuous, we have the following criterion (cf. [21, Theorem B]):

**Theorem 1.2.** *A transference plan $\rho \in P(X \times Y)$ is optimal if and only if for every finite family $(x_1, y_1), \dots, (x_n, y_n)$ of points of* $\mathrm{Spt}(\rho)$ *and for any permutation $s \in \mathcal{S}_n$ we have*

$$\sum_{i=1}^{n} d^2(x_i, y_i) \leqslant \sum_{i=1}^{n} d^2(x_i, y_{s(i)}).$$

For more results on the relations between optimality of transference plans and c-cyclical monotonicity of their supports, cf. [21].

### 1.3. Restriction of convex functions to submanifolds

Considering an isometric immersion $i : M^n \to N^{n+k}$, we shall note $\mathcal{A}_x$ its second fundamental form at $x$, dans $H_x = \frac{1}{n} \sum \mathcal{A}_x(e_i, e_i)$ its mean curvature vector, where the sum is taken over an orthonormal basis of $T_x M$.

In the sequel we shall note $\nabla$ and $D^2$ (resp. $\bar{\nabla}$ and $\bar{D}^2$) the gradient and the Hessian on $M$ (resp. on $N$).

In particular, the second fundamental form appears when writing the Hessian of the restriction of a function to the submanifold in term of the Hessian of the function on the ambient manifold. Let $F : N \to \mathbb{R}$ be a smooth function and let $f = F_{|M}$ be its restriction to $M$. For all $x \in M$ and all $\xi, \eta \in T_x M$ we have

$$D^2 f(x)(\xi, \eta) = \bar{D}^2 F(x)(\xi, \eta) + \big\langle (\bar{\nabla} F)_x, \mathcal{A}_x(\xi, \eta) \big\rangle.$$

As a consequence, we get the Laplacian of $f$:

$$\Delta f(x) = \mathrm{tr}\big(\bar{D}^2 F(x)_{|T_x M}\big) + n\big\langle (\bar{\nabla} F)_x, H_x \big\rangle. \tag{1.1}$$

The solution of the problem of Monge is given by the gradient of a convex function, however, there is no reason for this function to be smooth; so we have to get a formula similar to Eq. (1.1) for the Laplacian in the sense of distribution.

Let $\bar{V} : \mathbb{R}^{n+k} \to \mathbb{R}$ be a convex function. It is well known that $\bar{V}$ is locally Lipschitz, and therefore differentiable almost everywhere. Moreover, its Hessian in the sense of distribution is a Radon measure, and, almost everywhere, $\bar{V}$ has second derivative given by the absolutely continuous part of this measure with respect to Lebesgue measure (cf. for example [13]). This second derivative is known as the Hessian in the sense of Aleksandrov, and will be noted $\bar{D}_A^2 \bar{V}$ in the sequel.

Considering an isometric immersion $i : M^n \to \mathbb{R}^{n+k}$ and a convex function $\bar{V} : \mathbb{R}^{n+k} \to \mathbb{R}$, we shall prove that Eq. (1.1) holds "in Aleksandrov sense". In fact, we only need to consider the following particular case: let $E \subset \mathbb{R}^{n+k}$ be an $n$-dimensional linear subspace, let $p$ be the orthogonal projection on $E$, let $V : E \to \mathbb{R}$ be a convex function, and let $\bar{V} = V \circ p$; the function $\bar{V}$ is convex and invariant in the directions of $E^{\perp}$. In this context, we have the following proposition:

**Proposition 1.3.** *Let $V$ and $\bar{V}$ be as above, and suppose that $|\nabla V| \leqslant C$ on $E$. For any bounded domain $\Omega \subset M$, the restriction $V_\Omega : \Omega \to \mathbb{R}$ of $\bar{V}$ to $\Omega$ has the following properties*:

i. *$V_\Omega$ is Lipschitz and $|\nabla V_\Omega| \leqslant C$;*
ii. *there exist $h \in L^2(\Omega)$ and a nonnegative Radon measure $\nu$ such that, in the sense of distribution, $\Delta_{\mathcal{D}'} V_\Omega = \nu + h$ where $h$ and $\nu$ have the following properties*:
   - *for all $\varphi \in C_c^\infty(\Omega)$, $|\int_\Omega \varphi h| \leqslant nC \int_\Omega |\varphi||H|$;*
   - *if $D \subset \Omega$ is a domain such that the orthogonal projection $p : D \to E$ is a local diffeomorphism, then $h = n\langle H, \bar\nabla \bar V\rangle$ a.e. in $D$ and the Lebesgue decomposition of $\nu$ reads $\nu = g\,dv_M + \nu_s$ with $g(x) = \operatorname{tr}(\bar D_A^2 \bar V(x)_{|T_xM})$ for a.a. $x \in D$, and $\nu_s$ singular with respect to $dv_M$.*

**Proof.** As $|\nabla V| \leqslant C$, the function $\bar V$ is $C$-Lipschitz and for any $x$, $y$ in $\Omega$ we have $|V_\Omega(x) - V_\Omega(y)| \leqslant C|x-y| \leqslant C d_M(x,y)$, where $d_M$ is the distance in $M$. Therefore, $V_\Omega$ is $C$-Lipschitz on $\Omega$ and, by Rademacher's theorem, differentiable almost everywhere with $|\nabla V_\Omega| \leqslant C$.

To prove ii we follow [13]. Let $V_\varepsilon = \rho_\varepsilon * V$, where $\rho_\varepsilon$ is a mollifier on $E$; $\bar V_\varepsilon = V_\varepsilon \circ p$ is a smooth convex function on $\mathbb{R}^{n+k}$, and we note $V_{\Omega,\varepsilon}$ its restriction to $\Omega$. Moreover, we have $\nabla V_\varepsilon = \rho_\varepsilon * \nabla V$ on $E$, and $|\bar\nabla \bar V_\varepsilon| \leqslant C$.

By formula (1.1) and integration by part on $\Omega$ we have

$$\int_\Omega V_{\Omega,\varepsilon}\Delta\varphi - n\int_\Omega \varphi\langle H, \bar\nabla \bar V_\varepsilon\rangle = \int_\Omega \varphi \operatorname{tr}\bigl(\bar D^2 \bar V_{\varepsilon\,|TM}\bigr) \tag{1.2}$$

for all $\varphi \in C_c^\infty(\Omega)$. As $|\bar\nabla \bar V_\varepsilon| \leqslant C$, the functions $\langle H, \bar\nabla \bar V_\varepsilon\rangle$ are uniformly bounded in $L^2(\Omega)$ and, by weak compacity, there exists $h \in L^2(\Omega)$ and a sequence $\varepsilon_j \to 0$ such that $n\int_M \varphi\langle H, \bar\nabla \bar V_{\varepsilon_j}\rangle \to \int_M \varphi h$ for all $\varphi \in C_c^\infty(\Omega)$. Moreover, as $n|\int_M \varphi\langle H, \bar\nabla \bar V_{\varepsilon_j}\rangle| \leqslant nC\int_\Omega |\varphi||H|$ for all $j$, we also have $|\int_\Omega \varphi h| \leqslant nC\int_\Omega |\varphi||H|$.

Since $\bar V_\varepsilon$ is convex, passing to the limit in Eq. (1.2) gives

$$\int_\Omega V_\Omega \Delta\varphi - \int_\Omega \varphi h \geqslant 0,$$

and by Riesz representation theorem, there exists a nonnegative Radon measure $\nu$ on $\Omega$ such that, for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega V_\Omega \Delta\varphi - \int_\Omega \varphi h = \int_\Omega \varphi\, d\nu,$$

which implies that, in the sense of distribution, $\Delta_{\mathcal{D}'} V_\Omega = \nu + h$.

Let $D \subset \Omega$ be a domain such that $p : D \to E$ is a local diffeomorphism; in particular a.a. points of $D$ are Lebesgues points of $\bar\nabla \bar V$ and $\bar V$ is twice differentiable a.e. in $D$. We have that $\bar\nabla \bar V_{\varepsilon_j} \to \bar\nabla \bar V$ a.e. in $D$, and, by the dominated convergence theorem, $\int_D \varphi\langle H, \bar\nabla \bar V_{\varepsilon_j}\rangle \to \int_D \varphi\langle H, \bar\nabla \bar V\rangle$ for all $\varphi \in C_c^\infty(D)$; this implies that $h = n\langle H, \bar\nabla \bar V\rangle$ a.e. in $D$.

As the last point we want to prove is of local nature, we can assume that $p : D \to E$ is a diffeomorphism. For any $z \in E^\perp$, let $D_z = \{y + z \mid y \in D\}$, and note $V_D$ and $V_{D_z}$ the restrictions of $\bar V$ to $D$ and $D_z$ respectively. The set $\bar D = \{y + z \mid y \in D,\ z \in E^\perp\}$ is open in $\mathbb{R}^{n+k}$. Considering the diffeomorphism $\Phi : D \times E^\perp \to \bar D$ defined by $\Phi(y,z) = y + z$, we can write the Lebesgue measure $\lambda_{n+k}$ on $\bar D$ in term of the Riemannian measure $dv_M$ on $D$ and the Lebesgue measure $\lambda_k$

on $E^\perp$: $\lambda_{n+k} = J(y)\,dv_M\,\lambda_k$, where $J$ is the absolute value of the Jacobian determinant of $p$ (in particular, $J$ is smooth and positive). For any function $F$ on $\bar{D}$ we have

$$\int_{\bar{D}} F(x)\,dx = \int_D \int_{E^\perp} F(y+z)J(y)\,dv_M(y)\,dz. \tag{1.3}$$

Considering now the smooth functions $\bar{V}_\varepsilon$, we note $V_{D,\varepsilon}$ its restriction to $D$. Using that $\bar{V}_\varepsilon$ is invariant in the directions of $E^\perp$ we have, for any function $\bar{\varphi} \in C_c^\infty(\bar{D})$,

$$\begin{aligned}
\int_{\bar{D}} \operatorname{tr}\big(\bar{D}^2 \bar{V}_{\varepsilon\,|T_x D_z}\big)\frac{\bar{\varphi}(x)}{J(y)}\,dx &= \int_D \int_{E^\perp} \operatorname{tr}\big(\bar{D}^2 \bar{V}_{\varepsilon\,|T_{y+z} D_z}\big)\bar{\varphi}(y+z)\,dz\,dv_M(y)\\
&= \int_D \operatorname{tr}\big(\bar{D}^2 \bar{V}_{\varepsilon\,|T_y D}\big)\int_{E^\perp} \bar{\varphi}(y+z)\,dz\,dv_M(y)\\
&= \int_D \big(\Delta V_{D,\varepsilon} - n\langle H, \bar{\nabla}\bar{V}_\varepsilon\rangle\big)\bigg(\int_{E^\perp} \bar{\varphi}(y+z)\,dz\bigg)\,dv_M(y),
\end{aligned}$$

where, for $x \in \bar{D}$, $y$ and $z$ are the points in $D$ and $E^\perp$ defined by $x = y + z$.

Let $\varphi \in C_c^\infty(D)$, let $\rho \in C_c^\infty(E^\perp)$ be such that $\int_{E^\perp}\rho = 1$, and let $\bar{\varphi}$ be defined by $\bar{\varphi}(y+z) = \varphi(y)\rho(z)$. We get

$$\begin{aligned}
\int_{\bar{D}} \operatorname{tr}\big(\bar{D}^2 \bar{V}_{\varepsilon\,|T_x D_z}\big)\frac{\bar{\varphi}(x)}{J(y)}\,dx &= \int_D \big(\Delta V_{D,\varepsilon} - n\langle H, \bar{\nabla}\bar{V}_\varepsilon\rangle\big)\varphi\,dv_M\\
&= \int_D \big(V_{D,\varepsilon}\Delta\varphi - n\varphi\langle H, \bar{\nabla}\bar{V}_\varepsilon\rangle\big)\,dv_M,
\end{aligned}$$

and letting $\varepsilon$ tend to 0 gives

$$\int_{\bar{D}} \operatorname{tr}\big(\bar{D}^2_{\mathcal{D}'} \bar{V}_{|T_x D_z}\big)\frac{\bar{\varphi}(x)}{J(y)}\,dx = \int_D \big(V_D\Delta\varphi - n\varphi\langle H, \bar{\nabla}\bar{V}\rangle\big)\,dv_M. \tag{1.4}$$

As $\bar{V}$ is a convex function on $\mathbb{R}^{n+k}$, $\operatorname{tr}(\bar{D}^2_{\mathcal{D}'} \bar{V}_{|T_x D_z})$ is a Radon measure of the form

$$\operatorname{tr}\big(\bar{D}^2_{\mathcal{D}'} \bar{V}_{|T_x D_z}\big) = \operatorname{tr}\big(\bar{D}^2_A \bar{V}_{|T_x D_z}\big)\lambda_{n+k} + \bar{\mu}_s$$

with $\bar{\mu}_s$ a singular measure. Moreover, the invariance of $\bar{V}$ in the directions of $E^\perp$ implies that $\operatorname{tr}(\bar{D}^2_A \bar{V}_{|T_x D_z})$ is also invariant, and $\bar{\mu}_s = \mu_s \otimes \lambda_k$ with $\mu_s$ a singular measure on $E$. Finally, using (1.3), equality (1.4) becomes

$$\int_D \big(V_D\Delta\varphi - n\varphi\langle H, \bar\nabla \bar V\rangle\big)\,dv_M = \int_D \operatorname{tr}\big(\big(\bar D_A^2 \bar V\big)_{|T_y D}\big)\varphi(y)\,dv_M(y) + \int_{p(D)} \frac{\varphi(p^{-1}(u))}{J(p^{-1}(u))}\,d\mu_s(u)$$
$$= \int_D \operatorname{tr}\big(\big(\bar D_A^2 \bar V\big)_{|T_y D}\big)\varphi(y)\,dv_M(y) + \int_D \frac{\varphi}{J}\,d\big(p^{-1}\big)_{\#}\mu_s,$$

and we get

$$\Delta_{\mathcal{D}'} V_D = \big(\operatorname{tr}\big(\bar D_A^2 \bar V_{|TD}\big) + n\varphi\langle H, \bar\nabla \bar V\rangle\big)\,dv_M + \frac{1}{J}\big(p^{-1}\big)_{\#}\mu_s. \qquad \square$$

**Remark 1.4.** Denote by $V_M$ the restriction of $\bar V$ to $M$. As a consequence of the above proposition, we have that the Laplacian of $V_M$ in the sense of distributions is a Radon measure; in the sequel we shall note $\Delta_A V_M$ the density of its regular part in the Lebesgue decomposition with respect to $dv_M$.

In particular, if $D \subset M$ is a bounded domain such that $p: D \to E$ is a local diffeomorphism, then $\bar\nabla \bar V$ is well defined a.e. on $D$ and we have

$$\Delta_A V_M = \operatorname{tr}\big(\bar D_A^2 \bar V_{|TM}\big) + n\langle H, \bar\nabla \bar V\rangle.$$

This has to be seen has the generalization of formula (1.1) to nonsmooth convex functions which are invariant in the directions of $E^\perp$.

## 2. Optimal transportation and orthogonal projection on a subspace

### 2.1. The general case

As a direct consequence of Theorem 1.2, we have that projections (if well defined) are optimal transportations. Consider a Polish space $X$ and a closed subset $C \subset X$ on which the projection $p: X \to C$ is well defined: for all $x \in X$ the function $d(x,\cdot): C \to \mathbb{R}$ admits a unique minimum, $p(x)$ being, by definition, the point where this minimum is achieved. For any measure $\mu \in P(X)$, $\rho = (Id \times p)_{\#}\mu$ is a transference plan between the measures $\mu$ and $\nu = p_{\#}\mu$. Applying Theorem 1.2, it is easy to see that this transference plan is optimal: consider $(x_1, y_1), \dots, (x_n, y_n)$ in the support of $\rho$, for all $1 \leqslant i \leqslant n$ we have $y_i = p(x_i)$ so that for any permutation $s \in \mathcal{S}_n$ and any $1 \leqslant i \leqslant n$ we get $d^2(x_i, y_i) \leqslant d^2(x_i, y_{s(i)})$, which implies that $\rho$ is optimal. A particular case is when $C$ is a linear subspace of $\mathbb{R}^n$, $p$ being the orthogonal projection on $C$.

In the sequel we consider the product of three Polish spaces $X_i$, $i = 1, 2, 3$, and we note $\pi^{ij}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ the projection (i.e. $\pi^{ij}(x_1, x_2, x_3) = (x_i, x_j)$).

**Definition 2.1** *(Gluing of transference plans).* Consider three measures $\mu_i \in P(X_i)$, $i = 1, 2, 3$, and two transference plans $\rho_{12} \in P(X_1 \times X_2)$ between $\mu_1$ and $\mu_2$, and $\rho_{23} \in P(X_2 \times X_3)$ between $\mu_2$ and $\mu_3$.

A gluing of $\rho_{12}$ and $\rho_{23}$ is a probability measure $\Gamma \in P(X_1 \times X_2 \times X_3)$ whose marginals on $X_1 \times X_2$ and $X_2 \times X_3$ are $\rho_{12}$ and $\rho_{23}$ respectively.

As soon as the second marginal of the first transference plan equals the first marginal of the second one, gluing of transference plans always exist (cf. the “gluing lemma” in [25]), and they can be seen as a way of composing transference plans: with the notation of Definition 2.1, we have that $\pi^{13}_{\#}\Gamma$ is a transference plan between $\mu_1$ and $\mu_3$.

This is well illustrated by the particular case where $\mu_2 = F^1_{\#}\mu_1$ and $\mu_3 = F^2_{\#}\mu_2$. Consider the transference plans $\rho_{12} = (Id \times F^1)_{\#}\mu_1$ and $\rho_{23} = (Id \times F^2)_{\#}\mu_2$. Suppose $\Gamma$ is a gluing of $\rho_{12}$ and $\rho_{23}$. For any $(x_1, x_2, x_3) \in \mathrm{Spt}(\Gamma)$, we have $(x_1, x_2) \in \mathrm{Spt}(\rho_{12})$ and $(x_2, x_3) \in \mathrm{Spt}(\rho_{23})$, so we get $x_2 = F^1(x_1)$ and $x_3 = F^2(x_2)$. From this we can conclude that $\Gamma = (Id \times F^1 \times F^2 \circ F^1)_{\#}\mu_1$, and that $\pi^{13}_{\#}\Gamma = (Id \times F^2 \circ F^1)_{\#}\mu_1$ which is the transference plan associated to the map $F^2 \circ F^1$: the gluing of transference plans extends the composition of maps.

In general, there is no reason for $\pi^{13}_{\#}\Gamma$ to be optimal, even if $\rho_{12}$ and $\rho_{23}$ are optimal, however, in the setting of projections on a linear subspace, we have the following result:

**Theorem 2.2.** *Let $E$ be a linear subspace of $\mathbb{R}^n$, and let $p_E$ denote the orthogonal projection on $E$. Consider two probability measures $\mu \in P(\mathbb{R}^n)$ and $\nu \in P(E)$, the optimal transference plans $\rho = (Id \times p_E)_{\#}\mu$ between $\mu$ and $(p_E)_{\#}\mu$, and an optimal transference plan $\sigma$ between $(p_E)_{\#}\mu$ and $\nu$.*

*If $\mathrm{Spt}((p_E)_{\#}\mu)$ is compact, then, for any gluing $\Gamma$ of $\rho$ and $\sigma$, $\pi^{13}_{\#}\Gamma$ is an optimal transference plan between $\mu$ and $\nu$.*

For the proof we shall use the following lemma:

**Lemma 2.3.** *Let $X$ and $Y$ be Polish spaces and let $\rho \in P(X \times Y)$ be a transference plan between two measures $\mu \in P(X)$ and $\nu \in P(Y)$.*

*If $\mathrm{Spt}(\nu)$ is compact, then for all $x \in \mathrm{Spt}(\mu)$ there exists $y \in \mathrm{Spt}(\nu)$ such that $(x, y) \in \mathrm{Spt}(\rho)$.*

**Proof.** First, it is easy to see that $\mathrm{Spt}(\rho) \subset \mathrm{Spt}(\mu) \times \mathrm{Spt}(\nu)$. Now, suppose $x \in \mathrm{Spt}(\mu)$; for any $\varepsilon > 0$, we have $0 < \mu(B_x(\varepsilon)) = \rho(B_x(\varepsilon) \times Y)$, and there exists $(x_\varepsilon, y_\varepsilon) \in \mathrm{Spt}(\rho) \cap (B_x(\varepsilon) \times Y)$.

In particular, we have $x_\varepsilon \in B_x(\varepsilon)$ and $y_\varepsilon \in \mathrm{Spt}(\nu)$ which are compact. Therefore, there exists $y \in \mathrm{Spt}(\nu)$ and a sequence $(x_k, y_k)_{k\in\mathbb{N}}$ of points in $\mathrm{Spt}(\rho)$ tending to $(x, y)$. As $\mathrm{Spt}(\rho)$ is closed, we have $(x, y) \in \mathrm{Spt}(\rho)$ which concludes the proof. $\square$

**Remark 2.4.** The previous lemma is false without the compactness of $\mathrm{Spt}(\nu)$.

**Proof of Theorem 2.2.** Consider $n$ points $(x_1, z_1), \ldots, (x_n, z_n)$ in $\mathrm{Spt}(\pi^{13}_{\#}\Gamma)$. By Lemma 2.3, there exist points $y_1, \ldots, y_n$ in $\mathrm{Spt}((p_E)_{\#}\mu)$ such that $(x_i, y_i, z_i) \in \mathrm{Spt}(\Gamma)$ for all $i$. Moreover, as $(x_i, y_i) \in \mathrm{Spt}(\rho)$, we have that $y_i = p_E(x_i)$, $(x_i, p_E(x_i), z_i) \in \mathrm{Spt}(\Gamma)$, and $(p_E(x_i), z_i) \in \mathrm{Spt}(\sigma)$ for all $i$.

Let $s \in \mathcal{S}_n$. Using Pythagoras's formula we have

$$\sum_1^n d^2(x_i, z_i) = \sum_1^n d^2\big(x_i, p_E(x_i)\big) + \sum_1^n d^2\big(p_E(x_i), z_i\big).$$

As $\sigma$ is an optimal transference plan, using Theorem 1.2 we get

$$\sum_1^n d^2(x_i, z_i) \leqslant \sum_1^n d^2\big(x_i, p_E(x_i)\big) + \sum_1^n d^2\big(p_E(x_i), z_{s(i)}\big).$$

Using Pythagoras's formula once again we have

$$\sum_1^n d^2(x_i, z_i) \leqslant \sum_1^n d^2(x_i, z_{s(i)}).$$

This implies the optimality of $\pi_\#^{1,3}\Gamma$ by Theorem 1.2. $\square$

As soon as we are working with the square of the distance in the Euclidean space, it is not surprising that Pythagoras's formula naturally appears, and it has an other consequence on the geometry of Wasserstein space: if $E_1$ and $E_2$ are two orthogonal subspaces of $\mathbb{R}^n$, then for any measures $\mu_1$ and $\mu_2$ supported in $E_1$ and $E_2$ respectively, all the transference plan between $\mu_1$ and $\mu_2$ are optimal. If $\rho$ is a transference plan between $\mu_1$ and $\mu_2$, then $\mathrm{Spt}(\rho) \subset E_1 \times E_2$ and its cost satisfies

$$\begin{aligned} J(\rho) &= \int\limits_{E_1 \times E_2} |x_1 - x_2|^2 \, d\rho(x_1, x_2) \\ &= \int\limits_{E_1 \times E_2} \big(|x_1|^2 + |x_2|^2\big) \, d\rho(x_1, x_2) \\ &= \int\limits_{E_1} |x_1|^2 \, d\mu_1(x_1) + \int\limits_{E_2} |x_2|^2 \, d\mu_2(x_2). \end{aligned}$$

Therefore, all the transference plan have the same cost, and they all are optimal.

For example, let $D \subset \mathbb{R}^2$ be the unit disc and let $\mu$ be the normalized Lebesgue measure on $D$. Consider the two inclusions $i_k : \mathbb{R}^2 \to \mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$, $k = 1, 2$, defined by $i_1(x) = (x, 0)$ and $i_2(x) = (0, x)$, and the two measures $\mu_1 = (i_1)_\#\mu$ and $\mu_2 = (i_2)_\#\mu$. For any $t \in \mathbb{R}$ the map $F_t : \mathrm{Spt}(\mu_1) \to \mathrm{Spt}(\mu_2)$ defined by $F_t(x, 0) = (0, \mathrm{e}^{it}x)$ push forward $\mu_1$ on $\mu_2$ and, because of the preceding remark, gives rise to an optimal transference plan.

Now, using displacement interpolation (cf. for example [18, §2]), each of these optimal transference plans gives rise to a geodesic in the Wasserstein space $P_2(\mathbb{R}^4)$, and we constructed a continuous family of geodesics in $P_2(\mathbb{R}^4)$ with common end points and having the same length.

It is easy to find such a phenomenon in the Wasserstein space of a Riemannian manifold with positive curvature: considering for example the Dirac masses on the north and south pole of the sphere, each geodesic between the poles gives rise to a geodesic in the Wasserstein space (the map $x \mapsto \delta_x$ is an isometric embedding between the manifold and its Wasserstein space). However, our example is of different nature as there is a unique geodesic between any two points in $\mathbb{R}^n$.

This situation is of "positive curvature" nature: on a Riemannian manifold, such a situation implies that the end points are conjugate points along the geodesics, and therefore implies the presence of positive sectional curvature. Therefore, although the Euclidean space has vanishing curvature, its Wasserstein has positive curvature in some sense; this remark has to be compared with J. Lott's curvature calculations on the spaces of measures with $C^\infty$ densities with respect to the Lebesgue measure (cf. [17, Corollary 1]).

### 2.2. The case of measures supported in a submanifold

In the sequel we want to use solutions to the problem of Monge to compare measures supported in a submanifold with measures supported in a linear subspace. By the previous theorem, it is natural to consider the push forward of the first measure by the orthogonal projection on the linear subspace, and to use a solution of the problem of Monge in the linear subspace. A sufficient condition for such a solution to exist, is that the pushed measure does not give mass to small sets of the linear subspace.

Consider an isometric immersion $i : M^n \to \mathbb{R}^{n+k}$, and let $E$ be a linear subspace of $\mathbb{R}^{n+k}$. We shall note $P : \mathbb{R}^{n+k} \to E$ the orthogonal projection on $E$, $p = P_{|M}$ its restriction to $M$, and $\mathcal{C} = \{x \in M \mid T_x p : T_x M \to E \text{ is not onto}\}$ the critical set of $p$. In particular, $\mathcal{C}$ is a closed subset of $M$.

**Proposition 2.5.** *Let $i : M^n \to \mathbb{R}^{n+k}$ be an isometric immersion, let $E$ be a linear subspace of $\mathbb{R}^{n+k}$ with $\dim(E) \leqslant n$, and let $p : M \to E$ be the orthogonal projection on $E$.*

*For any nonnegative function $f$ on $M$ vanishing on $\mathcal{C}$, the measure $\mu = f\, dv_M$ is such that $p_{\#}\mu$ is absolutely continuous with respect to the Lebesgue measure of $E$.*

**Proof.** Let $A \subset E$ be a Borelian subset such that $p_{\#}\mu(A) > 0$. As $\mu(p^{-1}(A)) > 0$, there exists $x \in p^{-1}(A)$ such that $f(x) > 0$, and a neighborhood $U$ of $x$ such that $p_{|U}$ is a submersion and $\mu(U \cap p^{-1}(A)) > 0$. Since $p_{|U}$ is a submersion we have $\lambda(p(U \cap p^{-1}(A))) > 0$ which implies that $\lambda(A) > 0$. $\square$

As a consequence, we have the following result on the existence of a solution for the problem of Monge between $\mu$ and any measure on $E$:

**Corollary 2.6.** *For any nonnegative function $f$ with compact support on $M$ and vanishing on $\mathcal{C}$, and for any measure $\nu$ on $E$, the problem of Monge between the measures $\mu = f\, dv_M$ and $\nu$ admits a solution $T : M \to E$.*

*Moreover, there exists a convex function $V$ on $\mathbb{R}^{n+k}$ such that $T$ is the restriction to $M$ of the gradient of $V$.*

**Proof.** Using the proposition above and Brenier's theorem, the problem of Monge between $p_{\#}\mu$ and $\nu$ has a solution $S = \nabla W$ in $E$, where $W$ is a convex function on $E$.

By Theorem 2.2, $T = S \circ p = \nabla W \circ p = \nabla(W \circ p)$ is a solution to the problem of Monge between $\mu$ and $\nu$, and $V = W \circ p$ is the desired convex function on $\mathbb{R}^{n+k}$. $\square$

**Remark 2.7.** Although the result above looks like Brenier's theorem, there are some differences. In particular, even if $\nu$ does not give mass to small sets in $E$, the problem of Monge between $\nu$ and $\mu$ could have no solution as the projection $p : M \to E$ may not be one to one.

Let us now consider the case where $\dim(E) = \dim(M) = n$, and assume that the measure $\mu = f\, dv_M$ has compact support (with $f$ still vanishing on $\mathcal{C}$). In the sequel we shall note $J_E(x)$ the absolute value of the Jacobian determinant of $p$ at $x$ (i.e. $J_E(x) = |\det(T_x p)|$, where the determinant is taken in orthonormal basis of $T_x M$ and $E$).

If $y \in E$ is such that $p^{-1}(y) \cap \mathrm{Spt}(\mu)$ is not finite, then, by the compactness of $\mathrm{Spt}(\mu)$, $y$ must be a critical value of $p$. As a consequence of Morse–Sard's theorem (cf. for example [15]), we have that $p^{-1}(y) \cap \mathrm{Spt}(\mu)$ is finite for almost all $y \in E$ with respect to Lebesgue measure $\lambda$.

Using this fact, we have $p_{\#}\mu = F\lambda$ where

$$F(y) = \sum_{x \in p^{-1}(y) \cap \mathrm{Spt}(\mu)} \frac{f(x)}{J_E(x)} \tag{2.1}$$

is well defined for almost all $y \in E$.

In the sequel we shall need a regularity result for the solution of the problem of Monge; it is given by the following proposition:

**Proposition 2.8.** *If $f \in C_c^\infty(M \setminus \mathcal{C})$ and $g \in C^\infty(\bar{D})$ where $D$ is a smooth convex domain in $E$, then there exists a smooth convex function $W$ on $E$ such that $\nabla(W \circ p)$ is a solution to the problem of Monge between $\mu = f\, dv_M$ and $\nu = g\lambda$.*

**Proof.** The smoothness of $W$ will be a consequence of Caffarelli's regularity theory for solutions of the problem of Monge (cf. [3–5]). In order to use this theory, we just have to prove that the density $F$ of $p_{\#}\mu$ with respect to Lebesgue measure belongs to $C_c^\infty(E)$.

As $\mathrm{Spt}(\mu)$ is compact, so is $\mathrm{Spt}(p_{\#}\mu)$. Let $y \in \mathrm{Spt}(p_{\#}\mu)$, $p^{-1}(y) \cap \mathrm{Spt}(\mu)$ is finite, and for each $x \in p^{-1}(y) \cap \mathrm{Spt}(\mu)$ there exists a neighborhood $U_x$ of $x$ such that $p : U_x \to p(U_x)$ is a diffeomorphism. Moreover we can assume that for all $x$, $p(U_x) = B_\varepsilon(y)$.

Since $\mathrm{Spt}(\mu) \setminus \bigcup_x U_x$ is compact in $\mathbb{R}^{n+k}$, there exist $0 < \alpha \leqslant \varepsilon$ such that the cylinder $B_\alpha(y) + E^\perp$ does not intersect $\mathrm{Spt}(\mu) \setminus \bigcup_x U_x$. Therefore, on $B_\alpha(y)$, $F$ is a sum of smooth functions, and $F$ is smooth on $E$. $\square$

## 3. Isoperimetric inequalities for submanifolds of the Euclidean space

In this section we consider an isometric immersion $i : M^n \to \mathbb{R}^{n+k}$, and a linear subspace $E \subset \mathbb{R}^{n+k}$ of dimension $n$.

For any $n$-plane $F \subset \mathbb{R}^{n+k}$, let $K_E(F) = |\det(q)|$ where $q : F \to E$ is the orthogonal projection from $F$ to $E$ and $\det(q)$ is taken in orthonormal basis of $F$ and $E$.

In particular, if $p : M \to E$ denotes the orthogonal projection on $E$, and $J_E(x) = |\det(T_x p)|$, we have $J_E(x) = K_E(T_x M)$.

### 3.1. A weighted isoperimetric inequality

**Theorem 3.1.** *Let $i : M^n \to \mathbb{R}^{n+k}$ be an isometric immersion, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$. For any regular domain $\Omega \subset M$ we have*

$$n\omega_n^{\frac{1}{n}} \left( \int_\Omega J_E^{\frac{1}{n-1}}\, dv_M \right)^{\frac{n-1}{n}} \leqslant \mathrm{vol}(\partial\Omega) + n \int_\Omega |H|\, dv_M.$$

*The Sobolev counterpart of this inequality is*

$$n\omega_n^{\frac{1}{n}}\left(\int_M J_E^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\,dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\,dv_M + n\int_M |H||u|\,dv_M$$

*for any function $u \in C_c^\infty(M)$.*

*These inequalities are sharp.*

**Proof.** Let $u \in C_c^\infty(M)$ be a nonnegative function and let $f = \frac{J_E^{\frac{1}{n-1}} u^{\frac{n}{n-1}}}{c_E(u)}$, where $c_E(u) = \int_M J_E^{\frac{1}{n-1}} u^{\frac{n}{n-1}}\,dv_M$. The function $f$ vanishes on $\mathcal{C}$, therefore, the measure $\mu = f\,dv_M$ is such that $p_\#\mu$ is absolutely continuous with respect to Lebesgue measure on $E$ with a density $F$ given by the formula (2.1).

Using Brenier's theorem, there exists a convex function $V$ such that $\nabla V$ is the solution of the problem of Monge in $E$ between $p_\#\mu$ and $\frac{\chi_{B_E}}{\omega_n}\,dz$, where $B_E$ is the unit ball in $E$. Moreover, by Brenier's theorem, we have that $\nabla V(\mathrm{Spt}(p_\#\mu)) \subset B_E$, so that $|\nabla V| \leqslant 1$ on $\mathrm{Spt}(p_\#\mu)$, and we can assume that $V$ is finite on $E$. In fact, if this is not the case, just replace $V$ by

$$W(x) = \sup\bigl\{a(x) \bigm| a \text{ affine function, } |\nabla a| \leqslant 1,\ a \leqslant V \text{ on } \mathrm{Spt}(p_\#\mu)\bigr\}.$$

This function is convex on $E$ with $|\nabla W| \leqslant 1$, and $W = V$ on $\mathrm{Spt}(p_\#\mu)$ so that $\nabla W$ push forward $p_\#\mu$ on $\frac{\chi_{B_E}}{\omega_n}\,dz$. In the sequel we shall assume that $V$ is finite on the whole of $E$.

Let $\bar{V}$ denotes the extension of $V$ to $\mathbb{R}^{n+k}$ (that is $\bar{V} = V \circ p$), and $V_M$ denotes the restriction of $\bar{V}$ to $M$. The singular set of $\bar{V}$ (i.e. the set where $\bar{V}$ is not twice differentiable) is the preimage by $p$ of the singular set of $V$, and since $p$ is a local diffeomorphism on $\mathrm{Spt}(\mu)$, $\bar{V}$ and $V_M$ are twice differentiable almost everywhere in $\mathrm{Spt}(\mu)$.

Consider now the change of variable $z = \nabla V(y)$ in $E$. As in [12], using a remark due to McCann, this change of variable gives $\omega_n F(y) = |\det(D_A^2 V(y))|$, and by (2.1) we get

$$\omega_n \frac{f(x)}{J_E(x)} \leqslant \omega_n F\bigl(p(x)\bigr) = \bigl|\det\bigl(D_A^2 V\bigl(p(x)\bigr)\bigr)\bigr|$$

for almost all $x$ in the support of $\mu$.

From the definition of the function $\bar{V}$, we have that its Hessian is given by $\bar{D}_A^2 \bar{V}(x)(\xi,\eta) = D_A^2 V(p(x))(P(\xi), P(\eta))$ for a.a. points $x \in \mathbb{R}^{n+k}$ and any vectors $\xi$ and $\eta$, where $P$ is the orthogonal projection on $E$. As the orthogonal projection on $E$ is also the tangent map of $p$, it follows that, for a.a. $x \in \mathrm{Spt}(\mu)$,

$$\det\bigl(\bar{D}_A^2 \bar{V}(x)_{|T_x M}\bigr) = J_E^2(x)\det\bigl(D_A^2 V\bigl(p(x)\bigr)\bigr),$$

from which we deduce

$$\omega_n J_E(x) f(x) \leqslant \bigl|\det\bigl(\bar{D}_A^2 \bar{V}(x)_{|T_x M}\bigr)\bigr|.$$

As the restriction of a nonnegative matrix is still nonnegative, the arithmetic–geometric inequality gives

$$n\omega_n^{\frac{1}{n}} J_E(x)^{\frac{1}{n}} f(x)^{\frac{1}{n}} \leqslant \operatorname{tr}\bigl(\bar{D}_A^2 \bar{V}(x)_{|T_x M}\bigr). \tag{3.1}$$

As $f$ vanishes on $\mathcal{C}$, Proposition 1.3 and Remark 1.4 imply that a.e. in $\operatorname{Spt}(\mu)$

$$n\omega_n^{\frac{1}{n}} J_E(x)^{\frac{1}{n}} f(x)^{\frac{1}{n}} \leqslant \Delta_A V_M - n\langle H, \bar{\nabla}\bar{V}\rangle, \tag{3.2}$$

where $H$ is the mean curvature vector of $M$.

Multiplication by $u$ of the previous inequality gives

$$\frac{n\omega_n^{\frac{1}{n}}}{c_E(u)^{\frac{1}{n}}} J_E^{\frac{1}{n-1}} u^{\frac{n}{n-1}} \leqslant u\Delta_A V_M - nu\langle H, \bar{\nabla}\bar{V}\rangle. \tag{3.3}$$

By Proposition 1.3 we have that $\Delta_{\mathcal{D}'} V_\Omega = \nu + h$ with $\nu$ a nonnegative Radon measure. Using Remark 1.4 and the Lebesgue decomposition $\nu = \nu_{ac} + \nu_s$, we get

$$\int_{M\setminus\mathcal{C}} u\Delta_A V_M - nu\langle H, \bar{\nabla}\bar{V}\rangle = \int_{M\setminus\mathcal{C}} u\, d\nu_{ac},$$

and since $\nu$ and $u$ are nonnegative we obtain

$$\begin{aligned}
\int_{M\setminus\mathcal{C}} \bigl(u\Delta_A V_M - nu\langle H, \bar{\nabla}\bar{V}\rangle\bigr)\, dv_M &\leqslant \int_{M\setminus\mathcal{C}} u\, d\nu \\
&\leqslant \int_M u\, d\nu \\
&\leqslant \int_M u\Delta_{\mathcal{D}'} V_M\, dv_M \\
&\quad - \int_M uh\, dv_M &(3.4)\\
&\leqslant -\int_M \langle \nabla u, \nabla V_M\rangle\, dv_M \\
&\quad + n\int_M u|H|\, dv_M. &(3.5)
\end{aligned}$$

As $|\bar{V}| \leqslant 1$, we also have $|\nabla V_M| \leqslant 1$ on $M$, and, since the left-hand side of Eq. (3.3) vanishes on $\mathcal{C}$, integrating this equation on $M \setminus \mathcal{C}$ gives the desired Sobolev inequality:

$$n\omega_n^{\frac{1}{n}}\left(\int_M J_E^{\frac{1}{n-1}} u^{\frac{n}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\, dv_M + n\int_M u|H|\, dv_M.$$

The isoperimetric companion of this Sobolev inequality is

$$n\omega_n^{\frac{1}{n}}\left(\int_\Omega J_E^{\frac{1}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + n\int_\Omega |H|\, dv_M,$$

and this inequality is sharp as we have equality if $M = E$ and $\Omega$ is a ball. $\square$

*3.2. The classical isoperimetric inequality*

To get the usual isoperimetric inequality (without any weight), we can perform an integration on the Grassmannian of $n$-planes in $\mathbb{R}^{n+k}$.

Let $F$ be an $n$-plane in $\mathbb{R}^{n+k}$, and let

$$\alpha_{n,k} = \frac{1}{\operatorname{Vol}(G_{n,n+k})}\int_{G_{n,n+k}} K_E(F)^{\frac{1}{n}}\, dE,$$

where the integration is taken for the Haar measure of $G_{n,n+k}$. Using the homogeneity of $G_{n,n+k}$ and the invariance of the Haar measure, it is easy to see that $\alpha_{n,k}$ does not depend on the choice of $F$.

**Theorem 3.2.** *Let $i : M^n \to \mathbb{R}^{n+k}$ be an isometric immersion, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$. For any regular domain $\Omega \subset M$ we have*

$$n\omega_n^{\frac{1}{n}}\alpha_{n,k}\operatorname{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + n\int_\Omega |H|\, dv_M.$$

*The Sobolev counterpart of this inequality is*

$$n\omega_n^{\frac{1}{n}}\alpha_{n,k}\left(\int_M |u|^{\frac{n}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\, dv_M + n\int_M |H||u|\, dv_M$$

*for any function $u \in C_c^\infty(M)$.*

**Proof.** Choose $a > 0$, and let $f = \frac{J_E^a u^{\frac{n}{n-1}}}{c_{E,a}(u)}$, where $c_{E,a}(u) = \int_M J_E^a u^{\frac{n}{n-1}}$. Following the previous proof, Eq. (3.3) becomes

$$\frac{n\omega_n^{\frac{1}{n}}}{c(u)^{\frac{1}{n}}} J_E^{\frac{a+1}{n}} u^{\frac{n}{n-1}} \leqslant u\Delta_A V_M - nu\langle H, \bar{\nabla}\bar{V}\rangle$$

a.e. in $M \setminus \mathcal{C}$, where we also used that $c_{E,a}(u) \leqslant c(u) = \int_M u^{\frac{n}{n-1}}$. Integrating on $M \setminus \mathcal{C}$, using inequality (3.5) and letting $a \to 0$ gives

$$\frac{n\omega_n^{\frac{1}{n}}}{c(u)^{\frac{1}{n}}} \int_M J_E^{\frac{1}{n}} u^{\frac{n}{n-1}}\, dv_M \leqslant \int_M |\nabla u|\, dv_M + n \int_M u|H|\, dv_M.$$

As $J_E(x) = K_E(T_x M)$, integrating on $G_{n,n+k}$ with respect to $E$ we get

$$n\omega_n^{\frac{1}{n}} \alpha_{n,k} \left( \int_M u^{\frac{n}{n-1}}\, dv_M \right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\, dv_M + n \int_M u|H|\, dv_M.$$

The isoperimetric companion of this Sobolev inequality is

$$n\omega_n^{\frac{1}{n}} \alpha_{n,k} \operatorname{Vol}(\Omega)^{\frac{n-1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + n \int_\Omega |H|\, dv_M$$

for any regular domain $\Omega \subset M$. $\square$

The isoperimetric inequality obtained in this theorem is not the expected one, as $\alpha_{n,k} < 1$. However, we have that $\lim_{n\to\infty} \alpha_{n,1} = 1$, so that this inequality is not far from being sharp for hypersurfaces of high dimension.

To compute the limit, note that $\alpha_{n,1} = \frac{1}{\operatorname{vol}(S^n)} \int_{S^n} |\langle \eta, \xi \rangle|^{\frac{1}{n}}\, dv_{S^n}(\xi)$, for a given $\eta \in S^n$. Taking normal coordinates on $S^n$ centered at $\eta$ we get

$$\alpha_{n,1} = \frac{\operatorname{vol}(S^{n-1})}{\operatorname{vol}(S^n)} \int_0^\pi |\cos r|^{\frac{1}{n}} \sin^{n-1} r\, dr = \frac{\int_0^\pi |\cos r|^{\frac{1}{n}} \sin^{n-1} r\, dr}{\int_0^\pi \sin^{n-1} r\, dr}.$$

Using that $|\cos r| \geqslant \cos(\frac{\pi}{2} - \frac{1}{n}) \chi_{[0, \frac{\pi}{2} - \frac{1}{n}] \cup [\frac{\pi}{2} + \frac{1}{n}, \pi]}$, we have

$$\begin{aligned}
\alpha_{n,1} &\geqslant \frac{\cos^{\frac{1}{n}}(\frac{\pi}{2} - \frac{1}{n})(\int_0^\pi \sin^{n-1} r\, dr - \int_{\frac{\pi}{2}-\frac{1}{n}}^{\frac{\pi}{2}+\frac{1}{n}} \sin^{n-1} r\, dr)}{\int_0^\pi \sin^{n-1} r\, dr} \\
&\geqslant \frac{\cos^{\frac{1}{n}}(\frac{\pi}{2} - \frac{1}{n})(\int_0^\pi \sin^{n-1} r\, dr - \frac{1}{2n})}{\int_0^\pi \sin^{n-1} r\, dr}.
\end{aligned}$$

As Wallis' integral satisfies $\int_0^\pi \sin^{n-1} r\, dr \sim_\infty \sqrt{\frac{2\pi}{n-1}}$, this lower bound tends to 1 when $n$ tends to infinity.

This shows that our result improves the constant of this kind of isoperimetric inequalities for submanifolds. In fact, the constants given in [19] and [16] are of the form $n\omega_n^{\frac{1}{n}} \beta_n$ with $\beta_n$ tending to 0 when the dimension tends to infinity.

Using ideas of L.M. Simon, P.M. Topping obtained the inequality $2\pi \operatorname{Vol}(\Omega) \leqslant (\operatorname{vol}(\partial\Omega) + 2\int_\Omega |H|)^2$ for any surface in $\mathbb{R}^{2+k}$ (cf. [23, Appendix A]). A simple calculation proves that this inequality is better than the one we get by our method. Note that for minimal surfaces in $\mathbb{R}^3$, A. Ros and A. Stone obtained the inequality $2\pi\sqrt{2}\,\mathrm{Vol}(\Omega) \leqslant \mathrm{vol}(\partial\Omega)^2$ (cf. [9, §10.1 for a proof]).

### 3.3. Transference plans "moving with the point"

In the preceding section, we do not get the expected isoperimetric inequality because the Jacobian of the projection on $E$, which is less than or equal to one, naturally appear. To avoid this problem, the idea would be to use at each point of $M$ the projection on the tangent space $T_xM$, and hence to use a family of transportations "moving with the point".

To illustrate this point, let us consider the case of hypersurfaces. Let $i : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion, and let $u \in C_c^\infty(M)$ be a nonnegative function.

Choose a nondecreasing smooth function $\varphi$ on $\mathbb{R}_+$ such that $\varphi$ vanishes in a neighborhood of 0, $0 \leqslant \varphi \leqslant 1$, and $\varphi(1) = 1$.

For each $\xi \in S^n$, we consider the orthogonal projection $p_\xi : M \to \xi^\perp$, $J_\xi$ the determinant of its Jacobian, and we note $f_\xi = \frac{\varphi(J_\xi)u^{\frac{n}{n-1}}}{c_\xi(u)}$, where $c_\xi(u) = \int_M \varphi(J_\xi)u^{\frac{n}{n-1}}$.

Considering the optimal transportations $T_\xi : M \to \xi^\perp$ which push forward the measure $f_\xi\, dv_M$ on $M$ to the normalized Lebesgue measure of the unit ball of $\xi^\perp$, we can define the following map

$$\Phi : \begin{cases} M \times S^n \to \mathbb{R}^{n+1}, \\ (x, \xi) \mapsto T_\xi(x). \end{cases}$$

Using the Gauss map $g$ of $M$, we define

$$X : \begin{cases} M \to \mathbb{R}^{n+1}, \\ x \mapsto \Phi(x, g(x)). \end{cases}$$

As $X_x \in g(x)^\perp$ for each $x \in M$, $X$ is just a vector field on $M$, and the question is: can we use this vector field as a "Knothe map" to prove some Sobolev inequality on $M$?

For each $\xi \in S^n$, the optimal transportation $T_\xi$ is the gradient of a convex function $\bar{V}_\xi$ which is the extension to $\mathbb{R}^{n+1}$ of a convex function in $\xi^\perp$. By Proposition 2.8, the function $\bar{V}_\xi$ is smooth.

In the sequel we shall note $T^x_{(x,\xi)}\Phi : T_xM \to \mathbb{R}^{n+1}$ (resp. $T^\xi_{(x,\xi)}\Phi : \xi^\perp \to \mathbb{R}^{n+1}$) the tangent map to $\Phi$ with respect to the first (resp. to the second) variable.

As the derivative of the Gauss map is given by the shape operator, for a vector $e \in T_xM$ we have, for any $x \in M$,

$$(e.X)(x) = (e.\bar{\nabla}\bar{V}_\xi)_{|\xi=g(x)} - T^\xi_{(x,g(x))}\Phi.S_x(e)$$

where $S_x$ is the shape operator of $M$ at $x$. And making the sum over an orthonormal basis of $T_xM$ we get

$$\mathrm{div}(X)(x) = \mathrm{tr}\big(\bar{D}^2\bar{V}_\xi(x)_{|T_xM}\big)_{|\xi=g(x)} - \mathrm{tr}\big(T^\xi_{(x,g(x))}\Phi \circ S_x\big). \tag{3.6}$$

From this expression for $\operatorname{div}(X)$ we can deduce the following proposition:

**Proposition 3.3.** *Let $i : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. For any regular domain $\Omega \subset M$ we have*

$$n\omega_n^{\frac{1}{n}} \operatorname{Vol}(\Omega)^{1-\frac{1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + \int_\Omega \left|\operatorname{tr}\left(T^{\xi}_{(x,g(x))}\Phi \circ S_x\right)\right| dv_M(x).$$

*The Sobolev counterpart of this inequality is*

$$n\omega_n^{\frac{1}{n}} \left(\int_M |u|^{\frac{n}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\, dv_M + \int_M |u| \left|\operatorname{tr}\left(T^{\xi}_{(x,g(x))}\Phi \circ S_x\right)\right| dv_M(x)$$

*for any function $u \in C_c^\infty(M)$.*

*These inequalities are sharp.*

**Proof.** Following the proof of Theorem 3.1, for any $\xi \in S^n$ and any $x \in \operatorname{Spt}(f_\xi)$, Eq. (3.1) gives

$$n\omega_n^{\frac{1}{n}} J_\xi(x)^{\frac{1}{n}} f_\xi(x)^{\frac{1}{n}} \leqslant \operatorname{tr}\left(\bar{D}^2 \bar{V}_\xi(x)_{|T_x M}\right),$$

with the usual Hessian, $\bar{V}_\xi$ being smooth. Using the fact that $J_{g(x)}(x) = 1$ and $c_\xi(u) \leqslant \int_M u^{\frac{n}{n-1}} = c(u)$ we get

$$\begin{aligned} n\omega_n^{\frac{1}{n}} \frac{u(x)^{\frac{1}{n-1}}}{c(u)^{\frac{1}{n}}} &\leqslant \operatorname{tr}\left(\bar{D}^2 \bar{V}_\xi(x)_{|T_x M}\right)_{|\xi = g(x)} \\ &\leqslant \operatorname{div}(X)(x) + \operatorname{tr}\left(T^{\xi}_{(x,g(x))}\Phi \circ S_x\right). \end{aligned}$$

Multiplying by $u$, integrating by part, and using that $|X_x| \leqslant 1$ for any $x \in M$ we obtain

$$n\omega_n^{\frac{1}{n}} \left(\int_M u^{\frac{n}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \leqslant \int_M |\nabla u|\, dv_M + \int_M u \operatorname{tr}\left(T^{\xi}_{(x,g(x))}\Phi \circ S_x\right) dv_M.$$

The isoperimetric counterpart of this Sobolev inequality is

$$n\omega_n^{\frac{1}{n}} \operatorname{Vol}(\Omega)^{1-\frac{1}{n}} \leqslant \operatorname{vol}(\partial\Omega) + \int_M \operatorname{tr}\left(T^{\xi}_{(x,g(x))}\Phi \circ S_x\right) dv_M,$$

and this inequality is sharp as we have equality for any geodesic ball lying in any hyperplane of $\mathbb{R}^{n+1}$. $\square$

Note that the result of the previous proposition is not so far from that of Theorem 3.1, as the third term involves the shape operator whose trace is the mean curvature. The remaining problem is to deal with the derivative of the transports map with respect to the parameter $\xi$.

### 3.4. A weighted $L^p$ Sobolev inequality

In [12] the authors also obtained the sharp $L^p$ Sobolev inequalities on $\mathbb{R}^n$ in a similar way, using a different target measure (cf. [12, Theorem 2]). In our setting, we get weighted Sobolev inequalities, with weights involving a negative power of $J_E$. For this weight to be finite almost everywhere, we shall assume that the critical set $\mathcal{C}$ of the projection is negligible in $M$.

**Theorem 3.4.** *Let $i: M^n \to \mathbb{R}^{n+k}$ be an isometric immersion, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$ such that the critical set of the orthogonal projection from $M$ to $E$ is negligible.*

*For any $1 < p < n$, and for any function $u \in C_c^\infty(M)$ we have*

$$S_{n,p}\left(\int_M J_E^{\frac{1}{n-1}} |u|^{\frac{np}{n-p}}\, dv_M\right)^{\frac{n-p}{np}} \leqslant \int_M J_E^{-\frac{p-1}{n-1}} |\nabla u|^p\, dv_M + \frac{n(n-p)}{p(n-1)} \int_M J_E^{-\frac{p-1}{n-1}} |H||u|\, dv_M$$

*where $S_{n,p}$ is the $L^p$ Sobolev constant of $\mathbb{R}^n$. This inequality is sharp.*

**Proof.** Let $f = \frac{J_E^{\frac{1}{n-1}} u^{\frac{np}{n-p}}}{c_E(u)}$, where $c_E(u) = \int_M J_E^{\frac{1}{n-1}} u^{\frac{np}{n-p}}$. As $f$ vanishes on $\mathcal{C}$, we have $p_\# \mu = F(y)\, dy$ with $F$ given by formula (2.1). We follow the proof of Theorem 3.1, except that $\nabla V$ is the solution of the problem of Monge between the measures $p_\# \mu$ and $G(z)\, dz$, where the function $G \in C_c^\infty(E)$ will be made precise later.

Using the change of variable formula between $F(y)\, dy$ and $G(z)\, dz$, the relation between $f$ and $G$ becomes

$$\begin{aligned} \frac{f(x)}{J_E(x)} &\leqslant F\big(p(x)\big) \\ &\leqslant G\big(\nabla V\big(p(x)\big)\big)\big|\det\big(D_A^2 V\big(p(x)\big)\big)\big| \\ &\leqslant G\big(\bar\nabla \bar V(x)\big)\big|\det\big(D_A^2 V\big(p(x)\big)\big)\big|. \end{aligned}$$

From this point, we follow the steps of the proof of Theorem 3.1: by the arithmetic–geometric inequality and Proposition 1.3, Eq. (3.2) becomes

$$J_E(x)^{\frac{1}{n}} G\big(\nabla \bar V(x)\big)^{-\frac{1}{n}} \leqslant \frac{1}{n} f(x)^{-\frac{1}{n}} \Delta_A V_M - f(x)^{-\frac{1}{n}} \langle H, \bar\nabla \bar V\rangle.$$

As $\mathcal{C}$ is negligible, this inequality occurs a.e. in the support of $u$. Multiplying both parts by $J_E(x)^{-\frac{1}{n}} f(x)$ and integrating on $M$ we get

$$\int_M G\big(\nabla \bar V(x)\big)^{-\frac{1}{n}} f(x)\, dv_M \leqslant \frac{1}{n c_E(u)^{\frac{n-1}{n}}} \int_M u^{\frac{p(n-1)}{n-p}} \Delta_A V_M - \frac{1}{c_E(u)^{\frac{n-1}{n}}} \int_M u^{\frac{p(n-1)}{n-p}} \langle H, \bar\nabla \bar V\rangle.$$

As the map $\bar\nabla \bar V : M \to E$ push the measure $\mu = f\, dv_M$ on $G(y)\, dy$, the left-hand side of this inequality reads $\int_E G(y)^{\frac{n-1}{n}}\, dy$. On the right-hand side, we use Proposition 1.3 to compare $\Delta_A V_M$

and $\Delta_{\mathcal{D}'} V_M$, as in Eq. (3.4) where since $\mathcal{C}$ is negligible, $h = n\langle H, \bar{\nabla}\bar{V}\rangle$ a.e. Then, integration by part gives

$$\int_E G(y)^{\frac{n-1}{n}}\,dy \leqslant -\frac{p(n-1)}{n(n-p)c_E(u)^{\frac{n-1}{n}}}\int_M u^{\frac{n(p-1)}{n-p}}\langle\nabla u, \nabla V_M\rangle - \frac{1}{c_E(u)^{\frac{n-1}{n}}}\int_M u^{\frac{p(n-1)}{n-p}}\langle H, \bar{\nabla}\bar{V}\rangle.$$

If $q = \frac{p}{p-1}$ is the dual exponent to $p$, using Hölder inequality and $|\nabla V_M| \leqslant |\bar{\nabla}\bar{V}|$ we get

$$\begin{aligned}
&\frac{n(n-p)}{p(n-1)}\int_E G(y)^{\frac{n-1}{n}}\,dy \\
&\quad\leqslant \frac{1}{c_E(u)^{\frac{n-1}{n}}}\left(\int_M J_E^{\frac{1}{n-1}} u^{\frac{np}{n-p}}|\bar{\nabla}\bar{V}|^q\right)^{\frac{1}{q}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|\nabla u|^p\right)^{\frac{1}{p}} \\
&\qquad+\frac{n(n-p)}{p(n-1)c_E(u)^{\frac{n-1}{n}}}\left(\int_M J_E^{\frac{1}{n-1}} u^{\frac{np}{n-p}}|\bar{\nabla}\bar{V}|^q\right)^{\frac{1}{q}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|u|^p|H|^p\right)^{\frac{1}{p}},
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{n(n-p)}{p(n-1)}\int_E G(y)^{\frac{n-1}{n}}\,dy &\leqslant \frac{1}{c_E(u)^{\frac{n-p}{np}}}\left(\int_M f|\bar{\nabla}\bar{V}|^q\right)^{\frac{1}{q}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|\nabla u|^p\right)^{\frac{1}{p}} \\
&\quad+\frac{n(n-p)}{p(n-1)c_E(u)^{\frac{n-p}{np}}}\left(\int_M f|\bar{\nabla}\bar{V}|^q\right)^{\frac{1}{q}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|u|^p|H|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Using once again that $\bar{\nabla}\bar{V}$ push $\mu$ on $G(z)\,dz$ we obtain

$$\begin{aligned}
\frac{n(n-p)}{p(n-1)}\frac{\int_E G(y)^{\frac{n-1}{n}}\,dy}{(\int_E |y|^q G(y)\,dy)^{\frac{1}{q}}} &\leqslant \frac{1}{c_E(u)^{\frac{n-p}{np}}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|\nabla u|^p\right)^{\frac{1}{p}} \\
&\quad+\frac{n(n-p)}{p(n-1)c_E(u)^{\frac{n-p}{np}}}\left(\int_M J_E^{-\frac{p-1}{n-1}}|u|^p|H|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Taking $G = v^{\frac{np}{n-p}}$ where $\|v\|_{\frac{np}{n-p}} = 1$, the supremum over all function $v$ of the left-hand side is the Sobolev constant of $\mathbb{R}^n$ (cf. the characterization of $S_{n,p}$ given in Section 1). Thus we have

$$S_{n,p}\left(\int_M J_E^{\frac{1}{n-1}}|u|^{\frac{np}{n-p}}\,dv_M\right)^{\frac{n-p}{np}} \leqslant \int_M J_E^{-\frac{p-1}{n-1}}|\nabla u|^p\,dv_M + \frac{n(n-p)}{p(n-1)}\int_M J_E^{-\frac{p-1}{n-1}}|H||u|\,dv_M.$$

Moreover, this inequality is sharp because it is just the Euclidean $L^p$ Sobolev inequality of $\mathbb{R}^n$ when $M = E$. $\square$

## 4. Inequalities for submanifolds in warped products

In the previous section the main tools where the projection on a subspace (seen as Euclidean $n$ space) and the use of optimal transport in this subspace. As soon as we have these tools on a manifold, we can expect Sobolev inequalities for its submanifolds.

A typical example is hyperbolic space, where horospheres are isometric to Euclidean space and where the projections on them are well defined. In fact, hyperbolic space is a particular case of warped product for which we can use optimal transportation to get weighted Sobolev inequalities on their submanifolds.

### *4.1. Warped products*

Consider a warped product $N = \mathbb{R} \times \mathbb{R}^{n+k}$ (with $k \geqslant 0$) endowed with the metric $g_N = dt^2 + w(t)^2\, dy^2$ where $w$ is a smooth function, and $dy^2$ is the Euclidean metric on $\mathbb{R}^{n+k}$. In the sequel we shall note $(t, y)$ a point in $N$ where $y = (y_1, \dots, y_{n+k}) \in \mathbb{R}^{n+k}$.

Let $E$ be an $n$-linear subspace of $\mathbb{R}^{n+k}$; we can assume, without loss of generality, that $E$ is the subspace spanned by the first $n$ vectors of the canonical basis of $\mathbb{R}^{n+k}$. We denote by $p : N \to E$ the projection on $E$: $p(t, y) = (y_1, \dots, y_n)$. In the sequel we assume that $E$ is endowed with the Euclidean metric, and we have that, if $\xi \in T_{(t,y)}N$ belongs to the subspace spanned by $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n})$ then $|T_{(t,y)}p.\xi| = \frac{1}{w(t)}|\xi|$.

Let $V : E \to \mathbb{R}$ be a function on $E$, and let $\bar{V}$ be its extension to $N$ defined by $\bar{V}(t, y) = V(y_1, \dots, y_n)$. By a standard computation we have $w(t)|\bar{\nabla}\bar{V}(t, y)| = |\nabla V(p(t, y))|$ and

$$\bar{D}^2\bar{V} = -2\frac{w'}{w}\sum_{i=1}^{n}\frac{\partial V}{\partial y_i}\,dy_i\,dt + \sum_{i,j=1}^{n}\frac{\partial^2 V}{\partial y_i \partial y_j}\,dy_i\,dy_j. \tag{4.1}$$

The main difference with the Euclidean case, is that, with the terms coming from the Hessian of $V$, we get extra terms coming from the extrinsic curvature of $\{t\} \times \mathbb{R}^{n+k}$ in $N$.

Consider now an isometric immersion $i : M^n \to N$, where $M$ is an $n$-dimensional manifold, and let $\tau : M \to \mathbb{R}$ be the restriction to $M$ of the first coordinate function on $N$.

For $x \in M$, let $J_E(x) = |\det(q)|$, where $q$ is the orthogonal projection (in $T_xN$) from $T_xM$ to the subspace spanned by $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n})$. If we still note $p : M \to E$ the restriction of the projection $p$ to the submanifold $M$, for each $x \in M$ the absolute value of the Jacobian determinant of $p$ at $x$ is $\frac{1}{w(\tau(x))^n} J_E(x)$. The critical set of $p$ is $\mathcal{C} = \{x \in M \mid J_E(x) = 0\}$.

Considering a convex function $V$ on $E$, we have that the symmetric two form

$$\mathcal{B} = \bar{D}^2\bar{V} + 2\frac{w'}{w}\sum_{i=1}^{n}\frac{\partial V}{\partial y_i}\,dy_i\,dt$$

is nonnegative, and for any $x \in T_xM$ we have

$$\operatorname{tr}(\mathcal{B}_{|T_xM}) = \operatorname{tr}\big(\bar{D}^2\bar{V}_{|T_xM}\big) + 2\frac{w'}{w}\langle\nabla\tau, \nabla V_M\rangle. \tag{4.2}$$

In the sequel we will use a nonsmooth convex function $V$ on $E$. Using its second derivatives (well defined almost everywhere) and Eq. (4.1), we define the Hessian of $\bar{V}$ in the sense of Aleksandrov:

$$\bar{D}_A^2\bar{V} = -2\frac{w'}{w}\sum_{i=1}^{n}\frac{\partial V}{\partial y_i}\,dy_i\,dt + \sum_{i,j=1}^{n}\frac{\partial^2 V}{\partial y_i\partial y_j}\,dy_i\,dy_j. \tag{4.3}$$

Moreover we can mimic the proof of Proposition 1.3, the main point for doing this being that the Riemannian measure of $N$ is a product measure which can be written using the measure on $M$ and the Jacobian determinant of $p$. Let $V_M$ be the restriction of $\bar{V}$ to $M$, using Riesz theorem together with Eqs. (1.1) and (4.2) we have that $\Delta_{\mathcal{D}'}V_M - n\langle H, \bar{\nabla}\bar{V}\rangle + 2\frac{w'}{w}\langle\nabla\tau, \nabla V_M\rangle$ is a nonnegative Radon measure $\nu$. Therefore, $\Delta_{\mathcal{D}'}V_M$ is also a Radon measure and, if we note $\Delta_A V_M$ the density of its absolutely continuous part with respect to $dv_M$, we have

$$\Delta_A V_M = \operatorname{tr}\bigl(\bar{D}_A^2\bar{V}(x)_{|T_xM}\bigr) + n\langle H, \bar{\nabla}\bar{V}\rangle$$

on any domain $D \subset M$ on which $p$ is a local diffeomorphism.

The other consequence of the nonnegativity of $\nu$ is that, mimicking the arguments leading to inequality (3.5), we have

$$\begin{aligned}\int_{M\setminus\mathcal{C}} \varphi\Delta_A V_M - n\varphi\langle H, \bar{\nabla}\bar{V}\rangle + 2\varphi\frac{w'}{w}\langle\nabla\tau, \nabla V_M\rangle \leqslant{}& -\int_M \langle\nabla\varphi, \nabla V_M\rangle\,dv_M \\ &+ n\int_M \varphi\frac{|H|}{w} + 2\varphi\frac{w'}{w}\langle\nabla\tau, \nabla V_M\rangle\end{aligned} \tag{4.4}$$

for any nonnegative $\varphi \in C_c^\infty(M)$.

**Remark 4.1.** When taking $w(t) = \mathrm{e}^t$, the manifold $N$ is isometric to hyperbolic space $\mathbb{H}^{n+k+1}$. In this case, the first coordinate function is a Buseman function centered at some point at infinity, the submanifolds $\{t\}\times\mathbb{R}^{n+k}$ are horospheres, and the metric $g_N = dt^2 + \mathrm{e}^{2t}\,dy^2$ is the hyperbolic metric read in horospherical coordinates.

### *4.2. Weighted isoperimetric inequality*

Using the notation above we get the following result:

**Theorem 4.2.** *Let $i : M^n \to \mathbb{R}\times\mathbb{R}^{n+k}$ be an isometric immersion where $\mathbb{R}\times\mathbb{R}^{n+k}$ is endowed with the metric $dt^2 + w(t)^2\,dy^2$, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$. For any regular domain $\Omega \subset M$ we have*

$$n\omega_n^{\frac{1}{n}}\left(\int_\Omega \bigl(w(\tau)^n J_E\bigr)^{\frac{1}{n-1}}\,dv_M\right)^{\frac{n-1}{n}} \leqslant \int_{\partial\Omega} w(\tau)\,dv_{\partial\Omega} + n\int_\Omega w(\tau)|H|\,dv_M.$$

*The Sobolev counterpart of this inequality is*

$$n\omega_n^{\frac{1}{n}}\left(\int\limits_M \left(w(\tau)^n J_E\right)^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\,dv_M\right)^{\frac{n-1}{n}} \leqslant \int\limits_M w(\tau)|\nabla u|\,dv_M + n\int\limits_M w(\tau)|H||u|\,dv_M$$

*for any function $u \in C_c^\infty(M)$.*

**Proof.** Let $f = \frac{(w(\tau)^n J_E)^{\frac{1}{n-1}} u^{\frac{n}{n-1}}}{c_E(u)}$, where $u \in C_c^\infty(M)$ is a nonnegative function, and $c_E(u) = \int_M (w(\tau)^n J_E)^{\frac{1}{n-1}} u^{\frac{n}{n-1}}\,dv_M$.

Following the proof of Theorem 3.1, there exists a convex function $V$ on $E$ such that $\nabla V$ is the solution of the problem of Monge between $p_\#\mu$ and $\frac{\chi_{B_E}}{\omega_n}\,dz$.

As $f$ vanishes on the critical set $\mathcal{C}$, the measure $p_\#\mu$ is absolutely continuous with respect to the Lebesgue measure on $E$ and its density reads

$$F(y) = \sum_{x\in p^{-1}(y)\cap \mathrm{Spt}(\mu)} \frac{w(\tau(x))^n f(x)}{J_E(x)}.$$

Also $V$ may not be smooth, we can use derivatives in the sense of Aleksandrov and, by a change of variable in $E$, we get

$$\omega_n \frac{w(\tau(x))^n f(x)}{J_E(x)} \leqslant \omega_n F\big(p(x)\big) = \det\big(D_A^2 V\big(p(x)\big)\big) \tag{4.5}$$

for a.a. $x \in \mathrm{Spt}(\mu)$. Let $\mathcal{B} = \bar{D}_A^2 \bar{V} + 2\frac{w'}{w}\sum_{i=1}^n \frac{\partial V}{\partial y_i}\,dy_i\,dt$; for any unitary vector $\xi \in T_x M$, using Eq. (4.3), we have $\mathcal{B}(\xi,\xi) = D_A^2 V(p(x))(T_x p.\xi, T_x p.\xi)$. Therefore we get

$$\frac{J_E(x)^2}{w(\tau(x))^{2n}} \det\big(D_A^2 V\big(p(x)\big)\big) = \det(\mathcal{B}_{|T_x M}).$$

It follows from Eq. (4.3) that, as $V$ is convex, $\mathcal{B}_{|T_x M}$ is nonnegative and, with the geometric–arithmetic inequality, the inequality (4.5) becomes

$$n\omega_n^{\frac{1}{n}} \frac{J_E^{\frac{1}{n}}}{w(\tau)} f^{\frac{1}{n}} \leqslant \mathrm{tr}(\mathcal{B}_{|T_x M}), \tag{4.6}$$

from which we have

$$n\omega_n^{\frac{1}{n}} \frac{J_E^{\frac{1}{n}}}{w(\tau)} f^{\frac{1}{n}} \leqslant \Delta_A V_M - n\langle H, \bar{\nabla}\bar{V}\rangle + 2\frac{w'(\tau)}{w(\tau)}\langle \nabla\tau, \nabla V_M\rangle, \tag{4.7}$$

and multiplying by $u(x)w(\tau(x))^2$ gives

$$\frac{n\omega_n^{\frac{1}{n}}}{c_E(u)^{\frac{1}{n}}}\big(w(\tau)^n J_E\big)^{\frac{1}{n-1}} u^{\frac{n}{n-1}} \leqslant uw(\tau)^2\Delta_A V_M - nuw(\tau)^2\langle H, \bar{\nabla}\bar{V}\rangle + 2uw(\tau)w'(\tau)\langle\nabla\tau, \nabla V_M\rangle.$$

Integrating this inequality on $M \setminus \mathcal{C}$ and using inequality (4.4) we get

$$\frac{n\omega_n^{\frac{1}{n}}}{c_E(u)^{\frac{1}{n}}} \int_M \left(w(\tau)^n J_E\right)^{\frac{1}{n-1}} u^{\frac{n}{n-1}}\, dv_M \leqslant -\int_M w(\tau)^2 \langle \nabla u, \nabla V_M \rangle + n \int_M u w(\tau)|H|,$$

and since $w(\tau)|\nabla V_M| \leqslant w(\tau)|\bar{\nabla} \bar{V}| \leqslant 1$ we obtain the desired inequality:

$$\frac{n\omega_n^{\frac{1}{n}}}{c_E(u)^{\frac{1}{n}}} \int_M \left(w(\tau)^n J_E\right)^{\frac{1}{n-1}} u^{\frac{n}{n-1}}\, dv_M \leqslant \int_M w(\tau)|\nabla u| + n \int_M u w(\tau)|H|. \qquad \square$$

*4.3. Weighted $L^p$ Sobolev inequalities*

As for the Euclidean submanifolds, we can also prove weighted $L^p$ Sobolev inequalities.

**Theorem 4.3.** *Let $i : M^n \to \mathbb{R} \times \mathbb{R}^{n+k}$ be an isometric immersion where $\mathbb{R} \times \mathbb{R}^{n+k}$ is endowed with the metric $dt^2 + w(t)^2 dy^2$, and let $E$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$ such that the critical set of the projection on $E$ is negligible in $M$. For any $1 < p < n$, and for any function $u \in C_c^\infty(M)$ we have*

$$S_{n,p}\left(\int_M \left(w(\tau)^n J_E\right)^{\frac{1}{n-1}} |u|^{\frac{np}{n-p}}\, dv_M\right)^{\frac{n-p}{np}} \leqslant \int_M J_E^{-\frac{p-1}{n-1}} w(\tau)^{\frac{n-p}{n-1}} |\nabla u|^p\, dv_M + \frac{n(n-p)}{p(n-1)} \int_M J_E^{-\frac{p-1}{n-1}} w(\tau)^{\frac{n-p}{n-1}} |H|^p |u|^p\, dv_M.$$

**Sketch of proof.** Let us start with a function $u \in C_c^\infty(M)$ and the measure $\mu = f\, dv_m$ where $f = \frac{(w(\tau)^n J_E)^{\frac{1}{n-1}} u^{\frac{np}{n-p}}}{c_E(u)}$, with $c_E(u) = \int_M (w(\tau)^n J_E)^{\frac{1}{n-1}} u^{\frac{np}{n-p}}$.

Then we just have to follow step by step the proof of Theorem 3.4, using the tools of the proof of Theorem 4.2 to handle the different terms coming from the metric of $N$. $\square$

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