On Two Classes of Matrices with Positive Diagonal Solutions to the Lyapunov Equation

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ABSTRACT

We characterize real indecomposable quasi-Jacobi matrices of class $\mathscr{D}$, i.e., those which satisfy the Lyapunov equation $PA + A'P = -Q$ with P diagonal and both P and Q positive definite. The subclass $\mathscr{D}_2$ (of class $\mathscr{D}$) when also Q is diagonal is also characterized in the case of general indecomposable real matrices.

I. INTRODUCTION

In some recent studies on the stability of nonlinear systems [11, 17], the importance of a class $\mathscr{D}$ of real $n \times n$ matrices defined as $\mathscr{D} = \{A = (a_{ij}): PA + A'P = -Q,\ Q\ n \times n$ positive definite and $P\ n \times n$ positive definite and diagonal$\}$, is apparent (in [6], the class $\mathscr{D}$ was called "Volterra-Lyapunov stable"). Also recently, Barker, Berman, and Plemmons [1] have derived necessary conditions for the class $\mathscr{D}$, however; these conditions are not easy to use as a test. On the other hand, Khalil [12] proposed on optimization-based numerical test to decide whether a given matrix is of class $\mathscr{D}$ or not.

Some classes of matrices are well known to belong to class $\mathscr{D}$: quasi-diagonal-dominant matrices [17], M-matrices with inverted sign, and sign-stable matrices with nonzero diagonal entries [8, 15, 18].

In this paper, the class $\mathscr{D}$ is characterized in a simple (algebraic) way, within the class of indecomposable quasi-Jacobi matrices [13]. Another subclass of $\mathscr{D}$, namely the class $\mathscr{D}_2$ defined as $\mathscr{D}_2 = \{A = (a_{ij}): PA + A'P = -Q,\ P$ and Q both positive definite and diagonal$\}$, is also characterized. In both cases, a pair (P, Q) corresponding to the definitions is explicitly given.

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The paper is organized as follows. In Section II, basic results and definitions are recalled. In Section III quasi-Jacobi matrices are considered; its main result is Theorem III.1, which gives (algebraic) necessary and sufficient conditions that characterize indecomposable quasi-Jacobi matrices of class $\mathscr{D}$. The conditions are similar to the determinantal conditions of Kotelyanski and Sevastianov [19], related to M-matrices. The characterization of the class $\mathscr{D}_2$ is given in Section IV.

In what follows (unless otherwise stated) all matrices are $n \times n$, real, and denoted by Latin capital letters $A, B,$ etc. The elements of $A, B, \dots$ are denoted by $a_{ij}, b_{ij}, \dots$ $(i, j = 1, 2, \dots, n)$. The notation $A > 0$ means that the matrix A is positive definite. The ith leading principal minor of A is denoted by $\det_i(A)$. Diagonal matrices are denoted by $D = \operatorname{diag}(d_i)$.

II. PRELIMINARIES

The aim of this section is to provide definitions and basic results which are used in the subsequent sections. Some definitions given below, though familiar, are recalled for the sake of clearness, since they are not uniformly adopted in the literature.

Given a matrix A, we associate with it a digraph $D(A)$ containing n nodes $(\nu_i,\ i = 1, 2, \dots, n)$ and a directed edge (ν_i, ν_j) from node ν_i to node ν_j iff $a_{ij} \neq 0$, $i \neq j$. Assigning to each edge of $D(A)$ the weight a_{ij}, we associate with A the weighted digraph $D_w(A)$ and vice versa. The matrix associated with a digraph $D_w(A)$ is called its *adjacency matrix*. We refer to the properties of a digraph associated with a given matrix as properties of the matrix, and vice versa.

A matrix A is combinatorially symmetric iff $a_{ij} \neq 0$ implies $a_{ji} \neq 0$ for $i \neq j$; in this case $D(A)$ is a symmetric graph. A chain of length (or order) r of A is the product $a_{i_1 i_2} \cdot a_{i_2 i_3} \cdots a_{i_{r-1} i_r} \cdot a_{i_r i_{r+1}}$, where $i_1, i_2, \dots, i_r$ are distinct and all the elements of this product are nonzero. A chain is a cycle of length (or order) r iff $i_{r+1} = i_1$. If A has no cycles of order $r \geqslant k$ $(k \geqslant 1)$, then A is said to be *acyclic-k*. Considering the case of combinatorially symmetric A, each edge of $D(A)$ is in a 2-cycle and $D(A)$ can be transformed in a (undirected) graph $G(A)$. Indecomposable Jacobi matrices (tridiagonal) and more generally quasi-Jacobi matrices [13] are acyclic-3, and the corresponding graphs $G(A)$ are trees. The edges of $G(A)$ are denoted by $|\nu_i, \nu_j|$.

A matrix A is said to be: (1) *stable* iff all its eigenvalues have strictly negative real parts; (2) *D-stable* iff DA is stable for any positive definite diagonal matrix D, and (3) *totally D-stable* [2] iff every principal submatrix of A is D-stable.

LEMMA II.1 [2]. *If $A \in \mathcal{D}$ then A is D-stable.*

LEMMA II.2 [16]. *A necessary condition for A to be totally D-stable is that every principal minor of even order is positive and every principal minor of odd order is negative.*

III. ACYCLIC-3 INDECOMPOSABLE (OR QUASI-JACOBI) MATRICES

The class of acyclic-3 indecomposable matrices was initially studied by Maybee [13], who called them "quasi-Jacobi" (the term "matrix whose digraph is a tree" has also been used [22]). Some of his results are used to derive our main theorem (Theorem III.1), which characterizes in a simple way acyclic-3 matrices of class $\mathcal{D}$.

The following well-known lemma characterizes the class of acyclic-3 matrices in "structural" terms. A simple proof is given here. A more complicated proof (for the "only if" part) was given by Quirk and Ruppert [18] (see also Maybee [13, 14]).

LEMMA III.1. *An indecomposable matrix A is acyclic-3 iff:* (i) *A is combinatorially symmetric and* (ii) *it has exactly $n-1$ nonzero elements above (and below) the main diagonal.*

Proof. Sufficiency: Condition (i) implies that $D(A)$ is symmetric, and thus one can define the undirected graph $G(A)$. Since A is indecomposable, $G(A)$ is connected. Moreover, from condition (ii) one concludes that $G(A)$ has $n-1$ edges. Thus, A is acyclic-3.

Necessity: If A is acyclic-3 and indecomposable, then $D(A)$ is symmetric and $G(A)$ has $n-1$ edges, or equivalently, A has $n-1$ nonzero elements above (and below) the main diagonal. ■

We introduce the term "symmetric in modulus" for a matrix A satisfying $|a_{ij}| = |a_{ji}|$ for all $i \neq j$. Then we have the following Lemma:

LEMMA III.2. *If A is indecomposable and acyclic-3, then there exists a similarity transformation given by a diagonal matrix $D = \text{diag}(d_i)$, such that $DAD^{-1} = \bar{A}$ is symmetric in modulus, $a_{ij} \cdot a_{ji} = \bar{a}_{ij} \cdot \bar{a}_{ji}$, and $a_{ii} = \bar{a}_{ii}$; furthermore, D can be chosen positive definite, i.e., $D > 0$.*

Proof. Let $\bar{N} = \{(p,q) : p \neq q \text{ and } a_{pq}a_{qp} \neq 0\}$. By Lemma III.1, $\bar{N}$ has $n-1$ elements. The matrix $\bar{A} = DAD^{-1}$ is symmetric in modulus if

$$\left|\frac{a_{ij}}{a_{ji}}\right| = \frac{d_j^2}{d_i^2}, \qquad (i,j) \in \bar{N}. \tag{III.1}$$

Considering $Z_k \triangleq d_k^2$ $(k = 1,\ldots,n)$ as unknowns of the algebraic system (III.1), we can show that consistent positive $(Z_k > 0)$ solutions exist. Indeed, we can construct an undirected graph as follows: (1) to the n values Z_k $(k = 1,2,\ldots,n)$ we associate the n nodes, and (2) to the ratios $|a_{ij}/a_{ji}|$ we associate the $n-1$ edges $|Z_i, Z_j|$ connecting nodes Z_i and Z_j.

Then, to an arbitrary node Z_r assign an arbitrary positive value. Since the graph is connected and acyclic, any other node can be reached through a unique path. Thus, using the relations (III.1), the (positive) values of all nodes can be found. The corresponding (2^n) solutions of (III.1) are given by $d_i = \pm\sqrt{Z_i}$ $(i = 1,\ldots,n)$. Any of these solutions, in particular the totally positive one, gives a matrix $D = \text{diag}(d_i)$ such that $\bar{A} = DAD^{-1}$ is symmetric in modulus. ■

REMARK. Symmetrization ideas and graph techniques similar to those in the above proof were also used in [5], [14], and [20].

Let A and B be two acyclic-3 matrices. Then A and B are said to be principally equal (denoted $A \underset{p}{=} B$) iff (1) $a_{ii} = b_{ii}$ and (2) $a_{jk}a_{kj} = b_{jk}b_{kj}$.

LEMMA III.3 [13]. *If $A \underset{p}{=} B$ then A and B have the same spectrum. This is equally true for all their corresponding principal submatrices. Also, if $A \underset{p}{=} B$ then their corresponding principal minors are equal.*

From a general indecomposable acyclic-3 matrix A, we can define a matrix $A^+ = (a_{ij}^+)$ with $a_{ij}^+ = a_{ij}$ if $a_{ij}\cdot a_{ji} > 0$, $a_{ij}^+ = a_{ji}^+ = 0$ if $a_{ij}\cdot a_{ji} \leqslant 0$, and $a_{ii}^+ = a_{ii}$ (A^+ was used in [5] for tridiagonal matrices).

THEOREM III.1. *Given an indecomposable acyclic-3 matrix A, then $A \in \mathcal{D}$ iff*

$$(-1)^i \det_i(A^+) > 0, \qquad i = 1,2,\ldots,n. \tag{III.2}$$

The corresponding diagonal matrix satisfying $DA + A'D = -Q < 0$ is given by a positive solution of (III.1).

Proof. Sufficiency: By Lemma III.2 there exists $D > 0$, diagonal, such that $DAD^{-1} = A_s$, where A_s is symmetric in modulus. Obviously A_s^+ is the symmetric part of A_s. Though A_s^+ can be decomposable (with strictly negative diagonal), A_s^+ is combinatorially symmetric and acyclic-3. Furthermore, $A_s^+ \underset{p}{=} A^+$, and by Lemma III.3, $(-1)^i \det_i(A_s^+) = (-1)^i \det_i(A^+) > 0$, i.e., $A_s^+ < 0$ and thus

$$\tfrac{1}{2}\left(IA_s + A_s' I\right) = A_s^+ < 0. \tag{III.3}$$

This relation can be written as

$$\tfrac{1}{2}\left(D^2 A + A' D^2\right) = DA_s^+ D < 0. \tag{III.4}$$

Hence, $A \in \mathcal{D}$ and a corresponding positive diagonal matrix which satisfies the Lyapunov equation is given by D^2.

Necessity: Supposing $A \in \mathcal{D}$, then there exists $T = \operatorname{diag}(t_i) > 0$ such that $TA + A'T = -R < 0$. This also means that all principal submatrices of A belong to $\mathcal{D}$. Hence, by Lemma II.1, all principal submatrices of A are D-stable, i.e., A is totally D-stable. Since A^+ and all its principal submatrices are principal submatrices of A, Lemma II.2 implies the determinantal conditions (III.2). ■

From the practical viewpoint, the conditions (III.2) can be simplified by applying them to the strongly connected (or indecomposable) principal submatrices A_k^+ $(k = 1, \dots, p)$ of A^+ (A_k^+ are the irreducible direct summands of A).

COROLLARY III.1. *Given an indecomposable acyclic*-3 *matrix* A, *the following conditions are equivalent*: (i) $A \in \mathcal{D}$, (ii) A *is totally D-stable, and* (iii) A *satisfies* (III.2).

This corollary represents a generalization (in the indecomposable case) for the characterization of totally D-stable tridiagonal matrices given in [5].

It is worth noting that the conditions (III.2), which are similar to the determinantal conditions of Kotelyanski and Sevastianov for M-matrices, are not sufficient to guarantee that a general matrix A have all principal minors satisfying the necessary condition of Lemma II.2. However, for indecomposable acyclic-3 matrices (as a consequence of Corollary III.1 and Lemma II.2) the latter condition is equivalent to the simplified conditions (III.2). In this case, instead of requiring sign conditions over all principal minors of A (as in

[4] and [5]), only the leading principal minors of A^+ (or A_k^+) have to be considered.

IV. CHARACTERIZATION OF THE CLASS $\mathscr{D}_2$

The problem of characterizing the class $\mathscr{D}_2$ is much simpler than the foregoing problem. The theorems given below are stated without the proofs (which are straightforward). Let us introduce the notation $\tilde{A} = A - \operatorname{diag}(a_{ii})$.

We say that a matrix A is D-skew-symmetric iff there exists a diagonal matrix $D > 0$ such that $D\tilde{A}$ is skew-symmetric.

The main theorem for the class $\mathscr{D}_2$ is:

THEOREM IV.1. *Necessary and sufficient conditions for a given matrix to be of class $\mathscr{D}_2$ are:* (i) *A has strictly negative diagonal and* (ii) *A is D-skew-symmetric. The diagonal matrix $D > 0$ that skew-symmetrizes $\tilde{A}$ satisfies the Lyapunov equation, i.e., $DA + A'D = -Q$, Q being also positive definite and diagonal.*

COROLLARY IV.1. *Necessary and sufficient conditions for A to be of class $\mathscr{D}_2$ are:* (i) $a_{ii} < 0$; (ii) *$\tilde{A}$ is sign skew-symmetric* ($\operatorname{sgn}\tilde{a}_{ij} = -\operatorname{sgn}\tilde{a}_{ji}$, $i \neq j$), *and* (iii) *all cycles of order $k \geqslant 3$ of A verify* $a_{i_1i_2}\cdot a_{i_2i_3}\cdots a_{i_{k-1}i_k}\cdot a_{i_ki_1} = (-1)^k a_{i_1i_k}\cdot a_{i_ki_{k-1}}\cdots a_{i_3i_2}\cdot a_{i_2i_1}$ (*distinct* $i_1, i_2, \dots, i_k$).

NOTE. The equality in condition (iii) has also appeared in Reference [14] as a condition for "pseudo-skew-symmetry," which is defined by the existence of a real diagonal matrix T such that $T^{-1}\tilde{A}T$ is skew-symmetric. It is easy to verify that the two definitions (D-skew-symmetry and pseudo-skew-symmetry) are equivalent (take $T^2 = D^{-1}$).

COROLLARY IV.2. *Let A be indecomposable and acyclic-3. Then $A \in \mathscr{D}_2$ iff A has strictly negative diagonal and $\tilde{A}$ is sign-skew-symmetric.*

Corollary IV.1 gives a test for the class $\mathscr{D}_2$. First check conditions (i) and (ii). Then, either check (iii) or proceed as follows: extract from A an acyclic-3 indecomposable submatrix $\hat{A}$, by eliminating nonzero pairs (a_{ij}, a_{ji}) from A, i.e., determine a spanning tree for $G(A)$; then solve for q_k ($k = 1, \dots, n-1$) the system $|\hat{a}_{ij}/\hat{a}_{ji}| = q_j^2/q_i^2$, $(i, j) \in \bar{N}$, as for the system (III.1); then check if $D\tilde{A}$ [$D = \operatorname{diag}(q_k^2)$] is skew-symmetric.

EXAMPLE.

$$A = \begin{bmatrix} -1 & +2 & +1 & -2 \\ -6 & -3 & 0 & 0 \\ -10 & 0 & -2 & +5 \\ +4 & 0 & -1 & -1 \end{bmatrix}$$

Conditions (i) and (ii) of Corollary IV.1 are verified. Take

$$\hat{A} = \begin{bmatrix} -1 & +2 & +1 & -2 \\ -6 & -3 & 0 & 0 \\ -10 & 0 & -2 & 0 \\ +4 & 0 & 0 & -1 \end{bmatrix}$$

(indecomposable, acyclic-3), obtained from A by eliminating (a_{43}, a_{34}). Solve

$$\left|\frac{\hat{a}_{12}}{\hat{a}_{21}}\right| = \frac{1}{3} = \frac{q_2^2}{q_1^2}, \qquad \left|\frac{\hat{a}_{13}}{\hat{a}_{31}}\right| = \frac{1}{10} = \frac{q_3^2}{q_1^2}, \qquad \left|\frac{\hat{a}_{14}}{\hat{a}_{41}}\right| = \frac{1}{2} = \frac{q_4^2}{q_1^2}.$$

Setting $q_1 = 1$, we obtain $q_2^2 = \frac{1}{3}$, $q_3^2 = \frac{1}{10}$, $q_4^2 = \frac{1}{2}$; $D = \operatorname{diag}(1, \frac{1}{3}, \frac{1}{10}, \frac{1}{2})$; and

$$DA = \begin{bmatrix} -1 & +2 & +1 & -2 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & -\frac{1}{5} & +\frac{1}{2} \\ +2 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus A is of class $\mathscr{D}_2$, since $D\tilde{A} = \widetilde{DA}$ is skew-symmetric.

In this low order example condition (iii) of Corollary IV.1 is not difficult to verify:

$$a_{13}a_{34}a_{41} = -a_{14}a_{43}a_{31} = 20.$$

The pair (P, Q) is given by $P = D$ and $Q = DA + A'D = \operatorname{diag}(2, 2, \frac{2}{5}, 1)$.

NOTE. After submission of this paper, the authors were informed about the work by Berman and Hershkowitz [4], presented in April 1982, containing a result close to our main result (Theorem III.1). However, the two results should be considered as independent, since Theorem III.1 first appeared in the first author's D.Sc. Dissertation, presented in February 1981 and published as a COPPE Technical Report PTS-11/81 in November 1981 [10].

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