Geometric representation of cubic graphs with four directions

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1. Introduction

A drawing of a graph is said to be a straight-line drawing if the vertices of $G$ are represented by distinct points in the plane and every edge is represented by a straight-line segment connecting the corresponding pair of vertices and not passing through any other vertex of $G$. Wade and Chu [8] defined the slope number $sl(G)$ of a graph $G$ to be the smallest number of distinct slopes used in a straight-line drawing of the graph. We may note that a vertex of degree $d$ requires at least $\lceil\frac{d}{2}\rceil$ slopes and hence, the bound on the slope number would depend on the maximum degree of the graph. Dujmović et al. [7] asked if the slope number of a graph with bounded maximum degree could be arbitrarily large. Pach and Pálvölgyi [1] and Barát, Matoušek, Wood [2] (independently) showed with a counting argument that the number of degree-5 graphs (graphs with maximum degree 5) exceeds the number of graphs drawn with a fixed number of slopes, thereby proving that a finite number of slopes are insufficient to draw all degree-5 graphs.

In [3] it was shown that cubic graphs could be drawn with five slopes. This involved showing (inductively) that any subcubic graph (a degree-3 graph with at least one vertex of degree two or less) can be drawn with four slopes. Therefore, a cubic graph requires one additional slope. We show in this paper that, for connected cubic graphs, four slopes suffice (Fig. 1).

Theorem 1.1. Every connected cubic graph has a straight-line drawing with only four slopes.

A majority of the paper is devoted to the same result established for triangle-free graphs:

Theorem 1.2. Every triangle-free connected cubic graph has a straight-line drawing with only four slopes.
We may note that this is best possible because $K_4$ requires a minimum of four slopes. An easy way to see this is to observe that a triangle in the $K_4$ uses up three slopes and the third edge from each vertex would have to use a slope not already used at the vertex, and then it is impossible to place the fourth vertex.

To prove the above theorem, we prove the following, which is technical and not hard.

**Theorem 1.3.** Let $L_i(x) = a_{i,0} + \sum_{j=1}^{n} a_{i,j} x_j$ for $1 \leq i \leq n$ be linear forms, such that all coefficients are non-negative. Define a directed graph, $\mathcal{G} = \mathcal{G}(\mathbb{Z})$ with vertex set $V(\mathcal{G}) = \{0, 1, \ldots, n\}$ and edge set $E(\mathcal{G}) = \{(j,i) \mid a_{i,j} \neq 0\}$. Let $g_i = \frac{L_i(x)}{x_i}$ for $1 \leq i \leq n$. Assume that in $\mathcal{G}(\mathbb{Z})$ every node can be reached from 0. Then

$$g_1(x) = g_2(x) = \cdots = g_n(x)$$

has an all-positive solution.

It was shown by Max Engelstein [5] that cubic graphs with a Hamiltonian cycle can be drawn with four slopes. Slope numbers of other subclasses of graphs like outerplanar graphs, 2-trees, 3-trees, 3-connected planar graphs have been discussed by Dujmović et al. in [6]. In particular, they show that every cubic 3-connected planar graph has a plane drawing with three slopes. In their paper they also give interesting bounds for minimizing the number of segments in the drawings of some graphs. Here segment is defined as the maximal set of edges in the drawing that form a line segment. It is easy to see that the minimum number of segments forms a natural upper bound for the slope number. For certain class of graphs like 3-connected plane graphs, they show that $\frac{5}{2}n$ segments and $2n$ slopes suffice.

A variation of the problem arises if (a) two vertices in a drawing have an edge between them if and only if the slope between them belongs to a certain set $S$ and, (b) collinearity of points is allowed. This violates the condition stated before that an edge cannot pass through vertices other than its end points. For instance, $K_n$ can be drawn with one slope. Under this set of conditions, [11] proves that degree-3 outerplanar graphs can be drawn with 3 slopes. If the set of slopes $S$ is $\{\pm 1, 0, \infty\}$, then the graphs are called queens graphs and [10] characterizes certain graphs as queens graphs. Graph theoretic properties of some specific queens graphs can be found in [12,13].

The geometric thickness of a graph is the smallest number of planar graphs a straight-line drawing of a graph can be decomposed into. For a straight-line drawing of a graph, it is easy to see that the subgraph obtained by picking all the edges with the same slope is planar. Hence, the slope number provides a natural upper bound for the geometric thickness of a graph. Barát, Matoušek, Wood in [2] also show that the geometric thickness of degree-9 graphs is unbounded. Duncan et al. [9] show that the geometric thickness of degree-4 graphs is 2.

Section 2 of this paper deals with an outline of the proof of Theorem 1.1. Section 3 is dedicated to the proof of Theorem 1.3. Open problems are discussed in the final section.

### 2. Outline of the proof

#### 2.1. Assumptions

We will assume in the rest of the paper that the graph is bridgeless and triangle-free. We will use the following theorem to see why these assumptions do not restrict generality.

**Theorem 2.1.** (See [3].) Let $G$ be a connected graph that is not a cycle and whose every vertex has degree at most three. Suppose that $G$ has at least one vertex of degree less than three and denote by $v_1, \ldots, v_m$ the vertices of degree at most two ($m \geq 1$).

Then, for any sequence $x_1, \ldots, x_m$ of real numbers, linearly independent over the rationals, $G$ has a straight-line drawing with the following properties:
(1) Vertex \( v_i \) is mapped into a point with x-coordinate \( x(v_i) = x_i \) (1 \( \leq i \leq m \)).
(2) The slope of every edge is 0, \( \pi/2 \), \( \pi/4 \), or \( -\pi/4 \).
(3) No vertex is to the North of any vertex of degree two.
(4) No vertex is to the North or to the Northwest of any vertex of degree one.

We would use this theorem to patch together different components of a cubic graph obtained after removal of some edges. For this we would want to note that we could rotate the components by any multiple of \( \pi/4 \) and still have a graph with the same slopes as before.

**Claim 2.2.** A cubic graph with a bridge or a minimal two-edge disconnecting set can be drawn with four slopes.

**Proof.** We note that the above method cannot be extended to a minimal disconnecting set with more edges, as then, one of the components might be a cycle and then the above theorem cannot be invoked.

Both components obtained by removing the bridge can be drawn with four slopes using Theorem 2.1. Both have the north direction free for the vertex of degree two. To put these together, rotate the second one by \( \pi \) and place the degree two vertices above each other. Move the components far enough so that none of the other vertices or edges overlap.

For a two-edge disconnecting set, we may note that these edges must be vertex-disjoint or the graph would contain a bridge. Then, the same procedure as above can be used, now keeping the distance between the two vertices of degree two the same in both components. \( \square \)

**Claim 2.3.** A cubic graph with a cut-vertex or a two-vertex disconnecting set can be drawn with 4 slopes.

**Proof.** If the graph has a cut-vertex, then it has a bridge. If it has a two-vertex disconnecting set, then it has a two-edge disconnecting set. In both cases we can then invoke Claim 2.2 to draw the graph with four slopes. \( \square \)

**Remark 2.4.** A consequence of the above discussion is that any cubic graph that cannot be drawn with the **standard** four slopes (N, E, NE, NW) must be three vertex and edge connected.

**Claim 2.5.** Any cubic graph with a triangle can be drawn with 4 slopes.

**Proof.** First we note that by using the above claims, we may assume that we only consider cubic graphs in which all triangles are connected to the rest of the graph by vertex disjoint edges. If not, then the graph is either \( K_4 \) or has a two-vertex disconnecting set. A \( K_4 \) can be drawn using the vertices of a square. In the later case, we can draw the graph with four slopes using Claim 2.3.

We now prove the claim by contradiction. Suppose there exist cubic graphs with triangles that cannot be drawn with four slopes. By the preceding discussion all triangles in these graphs are necessarily connected to the graphs with vertex-disjoint edges. Of all such graphs consider the one with minimum number of vertices, say \( G \). The graph \( G' \) obtained by contracting the edges of the triangle \( \{v_1, v_2, v_3\} \) is also cubic and has fewer vertices. Either all triangles in \( G' \) are connected to the rest of \( G' \) with vertex-disjoint edges, in which case we invoke the minimality of \( G \) to conclude that \( G' \) can be drawn with 4 slopes (note: here the method of drawing the graph is unknown. We just know there exists a drawing of \( G' \) with four slopes). Or, some triangles in \( G' \) could be connected to the rest of \( G' \) with edges that are not vertex-disjoint. Here we can use Theorem 2.1 and the argument of the preceding paragraph to draw \( G' \) using four slopes. And lastly, \( G' \) could be a triangle-free graph. In this case we use Theorem 1.2 to draw \( G' \). Hence, \( G' \) can always be drawn with four slopes. In \( G' \), we call the vertex formed by contracting the edges of the triangle as \( v \). Since there is one slope that is not used by the edges incident on \( v \), we draw a segment with this slope in a very small neighborhood of \( v \) as shown in the figure (Fig. 2), to

![Fig. 2. Adding the triangle to the drawing of \( G' \) with four slopes.](image)
obtain a drawing of $G$ with four slopes. This contradicts the existence of a minimal counterexample and hence all graphs with triangles can be drawn with four slopes. □

**Remark 2.6.** It must be noted that this also gives an algorithm for drawing cubic graphs with triangles, namely, we contract triangles until we get a graph that can be drawn with either the Claims 2.2, 2.3, 2.5 or Theorem 2.1 or with our drawing strategy for triangle-free bridgeless graphs. Then we can backtrack with placing a series of edges which give us back all the contracted triangles.

### 2.2. Drawing strategy

Because of the above claims, we would now only focus on graphs that are bridgeless and triangle-free. Since the graph is bridgeless, Petersen’s theorem implies that it has a matching. We fix the slope of all the edges in the matching to be $\pi/2$ so that they all lie on (distinct) vertical lines (Fig. 3). If this matching is removed, then the graph consists of disjoint cycles. Next we isolate one special edge from each cycle. Our method of drawing the graph with four slopes then is as follows: For each cycle, remove the selected edge and draw the remaining path by going between corresponding vertical lines of the cycle alternating with slopes $\pi/4, 3\pi/4$ depending on whether we draw the edges with increasing/decreasing $x$-coordinate. This ensures that the cycles all grow upwards. Since we have the freedom to place the cycles where we want, we place...
them vertically on the matching so that they are very far apart (non-intersecting). Also, if the special edge of each cycle was between adjacent vertical lines then this edge would not pass through any other vertex of the graph either. Then, the only thing we would need is that the final edge in each cycle is drawn with the same slope. Fig. 3 illustrates this and the next remark is followed by a formal description of the problem.

**Remark 2.7.** In [5] a similar strategy of drawing the matching on vertical lines was employed. However, the cycles were drawn with alternating $\pi/4, 3\pi/4$ slopes for adjacent edges, so that the cycles were not “growing upwards” as in our construction. It leads to a different algebraic formulation of the problem giving tight bounds for the case when the cubic graph contains a Hamiltonian cycle.

Let $M$ be a matching in $G$. Each cycle $C$ in $E(G) \setminus M$ can be represented as a cyclic sequence $C = (v_1, \ldots, v_k)$, where each $v_i$ is an element of $M$. The sequence represents the elements of $M$ as we go around the cycle. We can assume (by Claim 2.5) that $k \geq 4$. An edge of $C$ by definition is $(v_i, v_{i+1})$ (all indices are understood mod $k$), which is although formally a pair formed by two distinct elements of $M$, also corresponds to an actual edge of the cycle. Notice that each element of $M$ is either shared by two cycles or occurs twice in a single cycle.

We now want to pick a distinguished edge (as in Fig. 4) $(v_i, v_{i+1})$ in $C$ (and in other cycles) such that the set of distinguished cycle-edges will satisfy certain properties.

**Notation:** Each distinguished cycle-edge is adjacent with two edges from the matching. These would be called the distinguished matching-edges of the cycle. In particular, the collection of distinguished edges from all cycles form the set of distinguished matching-edges. We would hope that distinguished matching-edges corresponding to a distinguished cycle-edge can be drawn as adjacent vertical lines for all cycles so that this would naturally enforce that the distinguished cycle-edge would not go through any other vertex of the graph.

**Definition 2.8.** Two cycles are connected if they share a distinguished matching-edge, and two cycles belong to the same component if they can be reached one from another by going through connected cycles. (An alternate way of looking at this

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Fig. 4. Distinguished “matching-edges” of Fig. 1 are represented by dashed lines while distinguished cycle-edges are represented by dotted lines.

Fig. 5. Graph and its connectivity graph.
would be that two cycles are adjacent iff the sets of distinguished matching-edges corresponding to the two cycles have a non-empty intersection). In other words, we define a graph on the cycles that we call the **cycle-connectivity graph**. Notice that in this graph each cycle can have at most two neighbors, thus the graph is a union of paths and cycles. The set of distinguished matching-edges associated with the component where cycle \( C \) belongs is denoted by \( D(C) \). (Clearly, if \( C_1 \) and \( C_2 \) belong to the same component, then \( D(C_1) = D(C_2) \) (Fig. 5)).

**Remark 2.9.** We note that in the cycle-connectivity graph two cycles are not necessarily connected if they share a matching-edge but only if they share a distinguished matching-edge. We can define another graph, where two cycles are connected if they share any matching-edge. It is easy to see that \( G \) is connected iff the latter graph is connected.

**Remark 2.10.** We also note that we may get a multigraph for the cycle-connectivity graph in the event that two cycles pick distinguished cycle-edges between the same set of matching-edges. Condition I below avoids that scenario also.

**Condition I.** The cycle-connectivity graph does not contain cycles (only paths). Equivalently, we can enumerate the distinguished matching-edges associated with the cycles of a component in some linear order \( y_1,\ldots,y_l \) in such a way that the pairs of consecutive matching-edges of this order are exactly the distinguished cycle-edges associated with the cycles in the component.

**Condition II.** In each component there is at most one cycle \( C \) such that \( C \subseteq D(C) \).

Assume that the lines of the matching are ordered \( v_1,\ldots,v_n \). From Condition I, we can ensure that every distinguished cycle-edge takes up two adjacent lines in this ordering. A drawing of these lines would be completely determined by the distance between consecutive lines. If \( v_i, v_{i+1} \) form a distinguished cycle-edge of the \( k \)th cycle, then call the distance between these lines \( x_k \). Otherwise fix this distance to be some arbitrary positive constant \( c_i \). This is illustrated in Figs. 6 and 7.

Now draw a cycle by starting at one of the distinguished matching-edges and first drawing the path obtained by removing the distinguished cycle-edge. If an edge of the cycle is \( v_k, v_l \) where \( k < l \) then use a slope of \( \pi/4 \) and \( 3\pi/4 \) otherwise.

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**Fig. 6.** Definition of variables \( x_i \) and \( c_i \).

**Fig. 7.** Paths of cycles will have adjacent distinguished cycle-edges in the drawing (because of the distinguished matching-edge they share). Hence it is necessary to not have cycles in the connectivity graph.
Definition 2.11. We define \( r(i) = \text{dist}(0, i) \) in the above graph \( \mathcal{G}(\mathcal{C}) \) and for a cycle \( C \) if the variable was \( x_i \) for its distinguished cycle-edge, we would denote \( r(C) \) to mean \( r(i) \).

Theorem 2.12. If Conditions I and II hold then we can use Theorem 1.3 to prove that every connected graph \( G \) is implementable with four directions.

Proof. Condition I ensures that the slope associated with the distinguished cycle-edge of each cycle \( i \) can be expressed as \( g_i(x) \) (as we have seen). Condition II is sufficient for the reachability condition (for \( \mathcal{G} \)) of Theorem 1.3. We will in fact show that \( r(C) \leq 2 \) for every cycle \( C \). The linear expression for cycle \( C \) has a non-zero constant term iff \( C \subseteq D(C) \neq \emptyset \). Consider a fixed component. By Condition II all cycles, except perhaps one, have associated linear expressions with non-zero constant terms, therefore they have \( r = 1 \) (Fig. 8).

It is sufficient to show that the single cycle \( C \) for which \( C \subseteq D(C) \), if exists, has \( r(C) = 2 \). Indeed, let \( y_1, \ldots, y_l \) be the distinguished matching-edges belonging to this component in this linear order, and let \( y_p \) and \( y_{p+1} \) be the distinguished matching-edges that belong to cycle \( C \). Since \( C \) is at least a four cycle, it either contains some other \( y_{p'} \notin \{y_p, y_{p+1}\} \), in which case indeed, it is geometrically easy to see that none of the other variables from the component has to occur in \( L_C \). If \( C \) is a four cycle and both \( y_p \) and \( y_{p+1} \) occur with multiplicity two in it. In the latter case \( C \) would form a separate \( K_4 \) component, thus \( G = K_4 \). In the former case the variable has \( r = 1 \), so \( r(C) = 2 \). \( \square \)

We are left with proving that we can pick distinguished cycle-edges from the cycles such that Conditions I and II are satisfied. Indeed, start from any cycle, and pick an edge for a distinguished cycle-edge, which has at least one adjacent matching-edge \( y \) that is common with a different cycle. If there is none, the cycle is the single (Hamiltonian) cycle, and if we distinguish any edge, Conditions I and II are clearly satisfied. Otherwise, in the cycle that contains \( y \), pick one of the two edges adjacent to \( y \), look at the other adjacent matching-edge, \( y' \), of this edge, look for another cycle that is adjacent with \( y' \), etc. The process ends when we get back to any cycle (including the current one) that has already been visited. There is one reason for back-track and this is when we return to the other adjacent matching-edge, \( z \), of the starting edge. In this case we choose the other edge (recall we always have two choices). It would be fatal to get back to \( z \), since then Condition I would not hold.

Assume that the above procedure has gone through. Then we have distinguished at most three matching-edges adjacent to any cycle. But this is not all. We have to do the same procedure from \( z \) as well. The procedure terminates when we encounter a cycle that has already been encountered. Thus in the final step we might create a fourth distinguished matching-edge adjacent to one of the cycles, but only in one of them. This can be the single cycle \( C \) in the component for which \( C \subseteq D(C) \). And because the graph is triangle-free, all the other components would have \( C \not\subseteq D(C) \).

Once we are done with creating the first component, we select a cycle not involved in it, and start the same procedure as before with the only difference that in subsequent rounds we also stop if we encounter a cycle visited in one of the previous rounds. It is easy to see, that now for the distinguished cycle-edges that we have selected Conditions I and II hold.

3. Solvability

Before we prove Theorem 1.3, we will look at the following special case when all the constant terms in \( L_C \) are positive.
Theorem 3.1. Let $B_1, \ldots, B_n > 0$ be positive constants, $L_i(x) = \sum_{j=1}^n a_{ij}x_j$ for $1 \leq i \leq n$ be linear forms. Let $g_i = \frac{B_i + L_i(x)}{x_i}$ for $1 \leq i \leq n$. Then
g_1(x) = g_2(x) = \cdots = g_n(x) \tag{2}
has an all-positive solution.

Proof. The intuition behind the proof is this: Let $\epsilon$ be very small and $\alpha_1, \ldots, \alpha_n > 0$ be fixed. If we set $x_i = \epsilon B_i \alpha_i^{-1}$ then $g_i(x) \approx \epsilon^{-1} \alpha_i$. In particular, let $\alpha$ range in the $[1, 2]^n$ solid cube. Then, if $\epsilon$ is small enough, the vector $(g_1(x), \ldots, g_n(x))$ will range roughly in the $[\epsilon^{-1}, 2\epsilon^{-1}]^n$ cube, thus $\epsilon^{-1}(1.5, \ldots, 1.5)$, which is the center of this cube, has to be in the image.

To make this proof idea precise we will use the following version of Brouwer's well-known fix point theorem:

**Theorem 3.2 (Brouwer).** Let $f : [1, 2]^n \to [1, 2]^n$ be a continuous function. Then $f$ has a fix point, i.e. an $x_0 \in [1, 2]^n$ for which $f(x_0) = x_0$.

We will use the fix point theorem as below. We first define

$h(\alpha_1, \ldots, \alpha_n) = (\epsilon g_1(x), \ldots, \epsilon g_n(x))$.

where $x = (\alpha_1^{-1} B_1, \ldots, \alpha_n^{-1} B_n) = \epsilon x'$, and we think of $\epsilon$ as some fixed positive number. Notice that $x'$ is just a function of $\alpha$, independent of $\epsilon$. It is sufficient to show that if $\epsilon$ is small enough, there are $\alpha_1, \ldots, \alpha_n$ such that $h(\alpha) = (1.5, \ldots, 1.5)$, since then $x$ satisfies (2) with common value $1.5\epsilon^{-1}$. We have:

$\epsilon g_i(x) = \frac{B_i + L_i(x)}{\epsilon \alpha_i^{-1} B_i} = \alpha_i \left( 1 + \epsilon B_i^{-1} L_i(x') \right)$. 

Here we used that $L_i(\epsilon x') = \epsilon L_i(x')$. We would like to have

$\alpha_i \left( 1 + \epsilon B_i^{-1} L_i(x') \right) = 1.5$ \quad for $1 \leq i \leq n$. \tag{3}

Define

$K = \max_{\alpha \in [1,2]^n} \sup_{i} \alpha_i^{-1} L_i(x');$

$\epsilon = 1/(10K)$.

To use the fix point theorem we consider the map

$f : (\alpha_1, \ldots, \alpha_n) \mapsto \left( \frac{1.5}{1 + \epsilon B_1^{-1} L_1(x')}, \ldots, \frac{1.5}{1 + \epsilon B_n^{-1} L_n(x')} \right)$

on the cube $[1, 2]^n$. The image is contained in $[1, 2]^n$, since if $\alpha \in [1, 2]^n$ then for $1 \leq i \leq n$ we have

$1 < \frac{1.5}{1 + 0.1} = \frac{1.5}{1 + \epsilon K} \leq \frac{1.5}{1 + \epsilon B_i^{-1} L_i(x')} \leq \frac{1.5}{1 - \epsilon K} = \frac{1.5}{1 - 0.1} < 2$.

Therefore, by Theorem 3.2 there is an $\alpha \in [1, 2]^n$ such that $\alpha_i = \frac{1.5}{1 + \epsilon B_i^{-1} L_i(x')}$ for $1 \leq i \leq n$, which is equivalent to (3). \Box

In Theorem 3.1 all linear forms have non-zero constant terms. We can, however generalize this to Theorem 1.3. We discuss its proof below.

**Remark 3.3.** The non-negativity of the coefficients can be relaxed such that the theorem becomes a true generalization of Theorem 3.1. Since the more general condition is slightly technical, we will stay with the simpler non-negativity condition, which is sufficient for us.

Proof. For $1 \leq i \leq n$ let $r(i) = \text{dist}(0,i)$ in $G(L)$. (In Theorem 3.1 each $r(i)$ was 1.) Define

$x_i = \epsilon r(i) x'_i,$

where $\epsilon > 0$ will be a small enough number that we will appropriately fix later, but as of now we think about it as a quantity tending to zero. We can rewrite (2) as:

$\epsilon g_1(x) = \epsilon g_2(x) = \cdots = \epsilon g_n(x)$. 

If we fix $x'$ and take epsilon tending to zero, then

$\epsilon g_i(x) \to \frac{\beta_i(x')}{x'_i}$.
where \( \beta_i(x') = a_{i,0}/x'_i \) if \( r(i) = 1 \), otherwise
\[
\beta_i(x') = \sum_{j: r(j) = r(i) - 1} a_{i,j}x'_j.
\]
We can now solve the system
\[
\frac{\beta_i(x')}{x'_i} = 1.5
\]
and even the system
\[
\frac{\beta_i(x')}{x'_i} = a_i, \tag{4}
\]
where \( 1 \leq a_i \leq 2 \) for \( 1 \leq i \leq n \). Indeed, the solution can be obtained iteratively, by first computing the values of the variables \( x_i \) with \( r(i) = 0 \), then with \( r(i) = 1 \), etc. We can again use the fix point theorem of Brouwer to show that if \( \epsilon \) is sufficiently small, the system
\[
\epsilon g_i(x) = 1.5 \quad \text{for} \quad 1 \leq i \leq n
\]
has a solution. For this we again parameterize \( x' \) with \( \alpha \). When \( \alpha \) ranges in the solid cube \([1, 2]^n\) then \( x' \) will range in some domain \( D \), where we obtain \( D \) by solving the system \((4)\) for all \( a_i \in [1, 2]^n \). Now we have to set \( \epsilon \) small enough such that everywhere in \( D \) it should hold that
\[
0.9 \leq \frac{\beta_i(x')}{x'_i} = \frac{\alpha_i}{\epsilon g_i(x)} \leq 1.1 \quad \text{for} \quad 1 \leq i \leq n. \tag{5}
\]
This is easily seen to be possible, since \( D \) is contained in a closed cube in the strictly positive orthant. We then apply the fix point theorem to
\[
f : \mathbf{\alpha} \rightarrow \mathbf{y},
\]
where
\[
y_i = \frac{1.5a_i}{\epsilon g_i(x)}.
\]
The fix point theorem applies, since the range of \( f \) remains in the \([0.9 \cdot 1.5, 1.1 \cdot 1.5]^n\) cube by Eq. \((5)\). For the fixed point \( \alpha_i = \frac{1.5a_i}{\epsilon g_i(x)} \) for \( 1 \leq i \leq n \), which implies \( \epsilon g_i(x) = 1.5 \) for \( 1 \leq i \leq n \). \( \Box \)

### 4. Open problems

For disconnected cubic graphs in which all components are bridgeless and triangle-free, we can simultaneously solve the slope equations in all components and hence draw the graph with four slopes. If some of the components have a bridge then we use Theorem 2.1 to draw these components. Hence, in general, disconnected cubic graphs would require five slopes. This is the same as the bound achieved in [3].

Another natural question that arises by looking at the drawings of \( K_4 \) and Petersen graph is what cubic graphs can be drawn with only the standard four directions \( (N, E, \overline{SE}, NW) \)? From the results in Section 2.1, it is evident that cubic graphs with a cut-vertex or a two-vertex disconnecting set or a bridge or two-edge disconnecting set can be drawn with the standard directions. Hence, any graph that requires a non-standard slope must be three-vertex and edge connected. We note here that queens graphs are also precisely the graphs such that two vertices in a drawing are joined by an edge if and only if the slope between them is standard. We refer the reader to the introduction of this paper for the differences between queens graphs and the drawings we consider and also to [10] for the characterization of some graphs as queens graphs and some related open problems.

Another interesting question arises if we constrain that, given \( S \) as the set of slopes, any two vertices are connected by an edge if and only if the slope between them is in \( S \). This is a strengthening of requiring specific slopes for edges of the graph. It was shown in [4] that seven slopes suffice for cubic graphs. Here they use five slopes for subcubic graphs and two additional slopes are required for cubic graphs. Though, the authors believe that six slopes should suffice. It must be noted here that the slope parameter thus defined, like in the case of queens graphs, does not require that the edges do not pass through any other vertex. Although, in [4], the drawings obtained for cubic graphs are with no three collinear points. Some more results and problems on the slope parameter of graphs may be found in [11].

Another related open problem is to find bounds for the slope number of degree-4 graphs. It is believed that this is unbounded.

### Acknowledgements

We would like to thank our Referees for the valuable suggestions in making Definition 2.8 unambiguous and in making proof of Claim 2.4 complete and also for citing the literature on queens graphs.
References


