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Isomorphisms and strictly singular operators in mixed Tsirelson spaces

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We study the family of isomorphisms and strictly singular operators in mixed Tsirelson spaces and their modified versions setting. We show sequential minimality of modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n,\theta_n)_n]$ satisfying some regularity conditions and present results on existence of strictly singular non-compact operators on subspaces of mixed Tsirelson spaces defined by the families $(\mathcal{A}_n)_n$ and $(\mathcal{S}_n)_n$.

0. Introduction

In the celebrated paper [20] W.T. Gowers started his classification program for Banach spaces. The goal is to identify classes of Banach spaces which are

1. hereditary, i.e. if a space belongs to a given class, then all of its closed infinite dimensional subspaces as well,
2. inevitable, i.e. any Banach space contains an infinite dimensional subspace in one of those classes,
3. defined in terms of richness of family of bounded operators in the space.

The famous Gowers' dichotomy brought first two classes: spaces with unconditional basis and hereditarily indecomposable spaces. The further classification, described in terms of isomorphisms, concerned minimality and strict quasiminimality. A Banach space X is *minimal* if every closed infinite dimensional subspace of X contains a further subspace isomorphic to X. A Banach space X is called *quasiminimal* if any two infinite dimensional subspaces Y, Z of X contain further isomorphic subspaces. The classical spaces ℓ_p, $1 \leqslant p < \infty$, c_0 are minimal and the Tsirelson space $T[\mathcal{S}_1, 1/2]$ is the first known strictly quasiminimal space (i.e. without minimal subspaces) [15]. The results of W.T. Gowers led to the question of the refinement of the classes and classification of already known Banach spaces. A further step in the first direction was made by the third named author [31], who proved that a strictly quasiminimal Banach space contains a subspace with no subsymmetric sequence. An extensive refinement of list of the classes and study of examples were made recently by V. Ferenczi and C. Rosendal [16,17].

The mixed Tsirelson spaces $T[(\mathcal{M}_n,\theta_n)_n]$, for $\mathcal{M}_n = \mathcal{A}_n$ or $\mathcal{S}_n$, as the basic examples of spaces not containing ℓ_p or c_0, form a natural class to be studied with respect to the classification program. The first step was made by T. Schlumprecht [5],

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who proved that his famous space $S = T[(\mathcal{A}_n, 1/\log_2(n+1))_n]$ is complementably minimal. The result of Schlumprecht holds for a certain class of mixed Tsirelson spaces $T[(\mathcal{A}_{k_n}, \theta_n)_n]$ by [27]. On the other hand, the Tzafriri space $T[(\mathcal{A}_n, c/\sqrt{n})_n]$ [36] is not minimal by [21]. However the original Tsirelson space $T[\mathcal{S}_1, 1/2]$ is not minimal [15], every normalized block sequence is equivalent to a subsequence of the basis (a property studied also in [33]). We show that mixed Tsirelson spaces $T[(\mathcal{A}_n, \theta_n)_n]$, for which Tzafriri space is a prototype, are saturated with subspaces with this "blocking principle".

V. Ferenczi and C. Rosendal [16] introduced and studied a stronger notion of quasiminimality. A Banach space X with a basis is *sequentially minimal* [16], if any block subspace of X contains a block sequence (x_n) such that every block subspace of X contains a copy of a subsequence of (x_n). The related notions in mixed Tsirelson spaces defined by families $(\mathcal{S}_n)$ and their relation to existence of ℓ_1^ω-spreading models were studied in [25,22]. In [29] it was shown that the spaces $T[(\mathcal{A}_n, \theta_n)_n]$, as well as $T[(\mathcal{S}_n, \theta_n)_n]$ satisfying the regularity condition $\theta_n/\theta^n \searrow$, where $\theta = \lim_n \theta_n^{1/n}$, are sequentially minimal. We show that the modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)_n]$ with the above property are also sequentially minimal.

The major tool in the study of mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ are the tree-analysis of norming functionals and the special averages introduced in [7], see also [11]. The basic idea to prove quasiminimality is to produce in every subspace a sequence of appropriate special averages of rapidly increasing lengths and show these sequences span isomorphic subspaces. The major obstacle in study of modified mixed Tsirelson spaces is estimating the norms of splitting of a vector into pairwise disjoint parts instead of consecutive parts as in non-modified setting. In order to overcome it, we introduced special types of averages, so-called Tsirelson averages, describing in fact local representation of the Tsirelson space $T[\mathcal{S}_1, \theta]$, with $\theta = \sup_n \theta_n^{1/n}$, in the considered space. Then we are able to estimate the action of a norming functional on a linear combination of Tsirelson averages by the action of a norming functional on suitable averages in the Tsirelson space $T[\mathcal{S}_1, \theta]$. Using those estimations we prove the sequential minimality of modified mixed Tsirelson space satisfying the regularity condition. Special averages, a weaker form of Tsirelson averages, are also the main tool for proving arbitrary distortability of $T_M[(\mathcal{S}_n, \theta_n)]$ in case $\theta_n/\theta^n \searrow 0$, the result known before in non-modified setting under the condition $\theta_n/\theta^n \to 0$ [3].

In the second part of the paper we deal with the existence of strictly singular non-compact operators in mixed Tsirelson spaces. The existence of non-trivial strictly singular operators, i.e. operators whose none restriction to an infinite dimensional subspace is an isomorphism, was also studied in context of classification program of Banach space, both in search for sufficient conditions and examples on known spaces. A space on which all the bounded operators are compact perturbations of multiple of the identity was constructed recently by S.A. Argyros and R. Haydon [10], who solved "scalar-plus-compact" problem. The existence of strictly singular non-compact operators was shown on Gowers–Maurey space and Schlumprecht space [6], as well as on a class of spaces defined by families $(\mathcal{S}_n)_n$ [19]. T. Schlumprecht [35], studying the richness of the family of operators on a Banach space in connection with the "scalar-plus-compact" problem, defined two classes of Banach spaces. Class 1 refers to a variation of a "blocking principle", while a space belongs to Class 2 if and only if it admits a strictly singular non-compact operator in any subspace (see Definition 3.3). T. Schlumprecht asked if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2. We show that a mixed Tsirelson space with the canonical form $T[(\mathcal{A}_n, \frac{c_n}{n^{1/q}})_n]$ belongs to Class 1 if $\inf_n c_n > 0$ and to Class 2 if $\lim_n c_n = 0$.

In [23] a block sequence $(x_n)_{n\in\mathbb{N}}$ generating ℓ_1-spreading model was constructed in Schlumprecht space S. This result combined with the result of I. Gasparis [19] led to the question if some biorthogonal sequence to $(x_n)_n$ generates a c_0-spreading model in S^*. We remark that this is not the case. In general, it is still unknown if any sequence in S^* generates a c_0-spreading model. Finally we show that in (modified) mixed Tsirelson spaces defined by $(\mathcal{S}_n)_n$ containing a block sequence generating ℓ_1^ω-spreading model there is a strictly singular non-compact operator on a subspace.

We describe now briefly the content of the paper. In the first section we recall the basic notions of the theory of mixed Tsirelson spaces and their modified versions, including the canonical representation of these spaces and the notion of a tree-analysis of a norming functional (Definition 1.8). The second section is devoted to the study of modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)_n]$ satisfying the regularity condition. We extend the notion of an averaging tree (Definition 2.5) and present the notions of averages of different types, providing also upper (Lemma 2.10) and lower (Lemma 2.14) "Tsirelson-type" estimates. We conclude the section with the result on arbitrary distortion for spaces with $\theta_n/\theta^n \searrow 0$ (Theorem 2.19) and sequential minimality (Theorem 2.20). In the last section we study the existence of non-compact strictly singular operators in mixed Tsirelson spaces $T[(\mathcal{A}_n, \theta_n)_n]$ (Theorem 3.4). We discuss the behavior of a biorthogonal sequence to the sequence generating ℓ_1-spreading model in Schlumprecht space (Proposition 3.6) and the case of (modified) mixed Tsirelson spaces defined by families $(\mathcal{S}_n)_n$ admitting ℓ_1^ω-spreading model (Theorem 3.8). We finish with the comments and questions concerning the Tzafriri space and richness of the set of subsymmetric sequences in a Banach space.

1. Preliminaries

We recall the basic definitions and standard notation.

By a *tree* we shall mean a non-empty partially ordered set $(\mathcal{T}, \leqslant)$ for which the set $\{y \in \mathcal{T}\colon\ y \leqslant x\}$ is linearly ordered and finite for each $x \in \mathcal{T}$. If $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $(\mathcal{T}', \leqslant)$ is a *subtree* of $(\mathcal{T}, \leqslant)$. The tree $\mathcal{T}$ is called *finite* if the set $\mathcal{T}$ is finite. The *root* is the smallest element of the tree (if it exists). The *terminal* nodes are the maximal elements. A *branch* in $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The *immediate successors* of $x \in \mathcal{T}$, denoted by $\operatorname{succ}(x)$, are all the nodes $y \in \mathcal{T}$ such that $x \leqslant y$ but there is no $z \in \mathcal{T}$ with $x \leqslant z \leqslant y$. A node x is a *sibling* of a node y, if $x, y \in \operatorname{succ}(z)$ for some $z \in \mathcal{T}$. If X is a linear space, then a *tree in X* is a tree whose nodes are vectors in X.

Let X be a Banach space with a basis (e_i). The *support* of a vector $x = \sum_i x_i e_i$ is the set $\operatorname{supp} x = \{i \in \mathbb{N}\colon\ x_i \neq 0\}$, the *range* of x, denoted by $\operatorname{range} x$ is the minimal interval containing $\operatorname{supp} x$. Given any $x = \sum_i a_i e_i$ and finite $E \subset \mathbb{N}$ put $Ex = x_E = \sum_{i \in E} a_i e_i$. We write $x < y$ for vectors $x, y \in X$, if $\max\operatorname{supp} x < \min\operatorname{supp} y$. A *block sequence* is any sequence $(x_i) \subset X$ satisfying $x_1 < x_2 < \cdots$. A closed subspace spanned by an infinite block sequence (x_n) is called a *block subspace* and denoted by $[(x_n)]$.

Notation 1.1. Given any two vectors $x, y \in X$ we write $x \preccurlyeq y$, if $\operatorname{supp} x \subset \operatorname{supp} y$, and we say that x and y are *incomparable*, if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$.

Given a block sequence $(x_n) \subset X$ and a functional $f \in X^*$ we say that f begins in x_n, if $\operatorname{minsupp} f \in (\operatorname{maxsupp} x_{n-1}, \operatorname{maxsupp} x_n]$.

A basic sequence (x_n) *C-dominates* a basic sequence (y_n), $C \geqslant 1$, if for any scalars (a_n) we have

$$\Big\| \sum_n a_n y_n \Big\| \leqslant C \Big\| \sum_n a_n x_n \Big\|.$$

Two basic sequences (x_n) and (y_n) are *C-equivalent*, $C \geqslant 1$, if (x_n) C-dominates (y_n) and (y_n) C-dominates (x_n).

We shall use the notions describing different ways of asymptotic representation of ℓ_p, $1 \leqslant p < \infty$, and c_0 in a Banach space.

Definition 1.2. Let E be a Banach space with a 1-subsymmetric basis (u_n), i.e. 1-equivalent to any of its infinite subsequences. Let (x_n) be a seminormalized basic sequence in a Banach space X. We say that $(x_n)_n$ generates (u_n) as a spreading model, if for any $k \in \mathbb{N}$ and any $(a_i)_{i=1}^k \subset \mathbb{R}$ we have

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \cdots \lim_{n_k \to \infty} \Big\| \sum_{i=1}^k a_i x_{n_i} \Big\|_X = \Big\| \sum_{i=1}^k a_i u_i \Big\|_E.$$

By [13] any seminormalized basic sequence admits a subsequence generating spreading model. We say that (x_n) generates ℓ_p- (resp. c_0-)spreading model, if (u_n) is equivalent to the u.v.b. of ℓ_p (resp. c_0).

We say that a Banach space X with a basis is ℓ_p-asymptotic, $1 \leqslant p \leqslant \infty$, if any block sequence $n \leqslant x_1 < \cdots < x_n$ is C-equivalent to the u.v.b. of ℓ_p^n, for any $n \in \mathbb{N}$ and some universal $C \geqslant 1$.

We work on two types of families of finite subsets of $\mathbb{N}$: $(\mathcal{A}_n)_{n \in \mathbb{N}}$ and $(\mathcal{S}_\alpha)_{\alpha < \omega_1}$. Let

$$\mathcal{A}_n = \{F \subset \mathbb{N}\colon\ \#F \leqslant n\}, \quad n \in \mathbb{N}.$$

Schreier families $(\mathcal{S}_\alpha)_{\alpha < \omega_1}$, introduced in [1], are defined by induction:

$$\mathcal{S}_0 = \big\{\{k\}\colon\ k \in \mathbb{N}\big\} \cup \{\emptyset\},$$
$$\mathcal{S}_{\alpha+1} = \{F_1 \cup \cdots \cup F_k\colon\ k \leqslant F_1 < \cdots < F_k,\ f_1, \ldots, F_k \in \mathcal{S}_\alpha\}, \quad \alpha < \omega_1.$$

If α is a limit ordinal, choose $\alpha_n \nearrow \alpha$ and set

$$\mathcal{S}_\alpha = \{F\colon\ F \in \mathcal{S}_{\alpha_n} \text{ and } n \leqslant F \text{ for some } n \in \mathbb{N}\}.$$

Given a family $\mathcal{M} = \mathcal{A}_n$ or $\mathcal{S}_n$ we say that a sequence $E_1, \ldots, E_k$ of subsets of $\mathbb{N}$ is

1. $\mathcal{M}$-*admissible*, if $E_1 < \cdots < E_k$ and $(\min E_i)_{i=1}^k \in \mathcal{M}$,
2. $\mathcal{M}$-*allowable*, if $(E_i)_{i=1}^k$ are pairwise disjoint and $(\min E_i)_{i=1}^k \in \mathcal{M}$.

Let X be a Banach space with a basis. We say that a sequence $x_1 < \cdots < x_n$ is $\mathcal{M}$-*admissible* (resp. -*allowable*), if $(\operatorname{supp} x_i)_{i=1}^n$ is $\mathcal{M}$-admissible (resp. -allowable).

Definition 1.3 *(Mixed and modified mixed Tsirelson space).* Fix a sequence of families $(\mathcal{M}_n) = (\mathcal{A}_{k_n})$ or $(\mathcal{S}_{k_n})$ and sequence $(\theta_n) \subset (0,1)$ with $\lim_{n \to \infty} \theta_n = 0$. Let $K \subset c_{00}$ be the smallest set satisfying the following:

1. $(\pm e_n^*)_n \subset K$,
2. for any $f_1 < \cdots < f_k$ in K, if $(f_i)_{i=1}^k$ is $\mathcal{M}_n$-admissible for some $n \in \mathbb{N}$, then $\theta_n(f_1 + \cdots + f_k) \in K$.

We define a norm on c_{00} by $\|x\| = \sup\{f(x)\colon\ f \in K\}$, $x \in c_{00}$. The *mixed Tsirelson space* $T[(\mathcal{M}_n, \theta_n)_n]$ is the completion of $(c_{00}, \|\cdot\|)$.

The *modified mixed Tsirelson space* $T_M[(\mathcal{M}_n, \theta_n)_n]$ is defined analogously, by replacing admissibility by allowability of the sequences.

It is standard to verify that the norm $\|\cdot\|$ defined above is the unique norm on c_{00} satisfying the equation

$$\|x\| = \max\left\{\|x\|_\infty, \sup\left\{\theta_n \sum_{i=1}^{k} \|E_i x\|:\ (E_i)_{i=1}^{k} - \mathcal{M}_n\text{-admissible (resp. -allowable)},\ n \in \mathbb{N}\right\}\right\}.$$

It follows immediately that the u.v.b. (e_n) is 1-unconditional in the space $T[(\mathcal{M}_n,\theta_n)_n]$ and its modified version. It was proved in [7] that any $T[(\mathcal{S}_{k_n},\theta_n)_n]$ is reflexive, also any $T[(\mathcal{A}_{k_n},\theta_n)_n]$ is reflexive, provided $\theta_n > \frac{1}{k_n}$ for at least one $n \in \mathbb{N}$ [11].

Taking $\mathcal{M}_n = \mathcal{M}$ and $\theta_n = \theta$ for any n we obtain the classical Tsirelson-type space $T[\mathcal{M},\theta]$. Recall that $T[\mathcal{A}_n,\theta] = c_0$ if $\theta \leqslant 1/n$ and $T[\mathcal{A}_n,\theta] = \ell_p$, if $\theta = 1/\sqrt[q]{n}$ for q satisfying $1/p + 1/q = 1$ [12,11]. The space $T[\mathcal{S}_1, 1/2]$ is the Tsirelson space.

Schlumprecht space S is the space $T[(\mathcal{A}_n, \frac{1}{\log_2(n+1)})_n]$, Tzafriri space is $T[(\mathcal{A}_n, \frac{c}{\sqrt{n}})_n]$ for $0 < c < 1$. Modified Tsirelson-type spaces are isomorphic to their non-modified version [12,15,28], whereas the situation is quite different in mixed setting [9].

We present now the canonical form of a (modified) mixed Tsirelson space in both cases $\mathcal{M}_n = \mathcal{A}_{k_n}$ or $\mathcal{S}_{k_n}$, $n \in \mathbb{N}$.

Definition 1.4. (See [27].) A mixed Tsirelson space $T[(\mathcal{A}_{k_n},\theta_n)_{n\in\mathbb{N}}]$ is called a *p-space*, for $p \in [1,\infty)$, if there is a sequence $(p_N)_N \subset (1,\infty)$ such that

1. $p_N \to p$ as $N \to \infty$, and $p_N \geqslant p_{N+1} > p$ for any $N \in \mathbb{N}$,
2. $T[(\mathcal{A}_{k_n},\theta_n)_{n=1}^N]$ is isomorphic to ℓ_{p_N} for any $N \in \mathbb{N}$.

A p-space $T[(\mathcal{A}_n,\theta_n)_{n\in\mathbb{N}}]$ is called *regular*, if $\theta_n \searrow 0$ and $\theta_{nm} \geqslant \theta_n\theta_m$ for any $n,m \in \mathbb{N}$. Recall that any p-space is isometric to a regular p-space [29].

Notation 1.5. Let $T[(\mathcal{A}_n,\theta_n)_{n\in\mathbb{N}}]$ be a regular p-space. If we set $\theta_n = 1/n^{1/q_n}$ with $q_n \in (1,\infty)$, $n \in \mathbb{N}$, then $q = \lim_n q_n = \sup_n q_n \in (0,\infty]$, where $1/p + 1/q = 1$, with usual convention $1/\infty = 0$.

In the situation as above let $c_n = \theta_n n^{1/q} \in (0,1)$, $n \in \mathbb{N}$, if $p > 1$. To unify the notation put $c_n = \theta_n$, $n \in \mathbb{N}$, in case $p = 1$.

A space $T_M[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$ with $\theta_n \searrow 0$ and $\theta_{n+m} \geqslant \theta_n\theta_m$ is called a *regular* space. Notice that any modified mixed Tsirelson space $T_M[(\mathcal{S}_{k_n},\theta_n)_{n\in\mathbb{N}}]$ is isometric to a regular modified mixed Tsirelson space (cf. [3]).

Notation 1.6. For a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n,\theta_n)_n]$ let $\theta = \lim_n \theta_n^{1/n} = \sup_n \theta_n^{1/n} \in (0,1]$. We shall use also the following condition:

$$(\clubsuit)\quad \left(\theta_n/\theta^n\right)_n \searrow \quad \text{i.e.}\quad \theta_{n+m} \leqslant \theta_n\theta^m \quad \text{for any } n,m \in \mathbb{N}.$$

Given two families $\mathcal{M}$, $\mathcal{N}$ of finite subsets of $\mathbb{N}$ define

$$\mathcal{M}[\mathcal{N}] = \left\{F_1 \cup \cdots \cup F_k:\ F_1,\ldots,F_k \in \mathcal{N},\ (F_1,\ldots,F_k)\ \mathcal{M}\text{-admissible},\ k \in \mathbb{N}\right\}.$$

It follows straightforwardly that $\mathcal{S}_n[\mathcal{S}_m] = \mathcal{S}_{n+m}$, for any $n,m \in \mathbb{N}$.

Lemma 1.7. *The space $T_M[(\mathcal{S}_n[A_2],\theta_n)_n]$ is 3-isomorphic to $T_M[(\mathcal{S}_n,\theta_n)_n]$.*

The proof of the above follows that of [29, Lemma 4.5] with “admissible” sequences replaced by “allowable” ones.

The following notion provides a useful tool for estimating norms in mixed Tsirelson spaces and their modified versions:

Definition 1.8 *(The tree-analysis of a norming functional).* Let $f \in K$, the norming set of $T[(\mathcal{M}_n,\theta_n)_n]$ (resp. $T_M[(\mathcal{M}_n,\theta_n)_n]$). By a *tree-analysis* of f we mean a finite family $(f_\alpha)_{\alpha\in\mathcal{T}}$ indexed by a tree $\mathcal{T}$ with a unique root $0 \in \mathcal{T}$ (the smallest element) such that the following hold

1. $f_0 = f$ and $f_\alpha \in K$ for all $\alpha \in \mathcal{T}$,
2. $\alpha \in \mathcal{T}$ is maximal if and only if $f_\alpha \in (\pm e_n^*)$,
3. for every not maximal $\alpha \in \mathcal{T}$ there is some $n \in \mathbb{N}$ such that $(f_\beta)_{\beta\in\operatorname{succ}(\alpha)}$ is an $\mathcal{M}_n$-admissible (resp. -allowable) sequence and $f_\alpha = \theta_n(\sum_{\beta\in\operatorname{succ}(\alpha)} f_\beta)$. We call θ_n the *weight* of f_α.

For any $\alpha \in \mathcal{T}$, $\alpha > 0$, we define the tag $t(\alpha) = t(f_\alpha)$ as $t(\alpha) = \prod_{\alpha>\beta\geqslant 0} weight(f_\beta)$.

For any $\alpha \in \mathcal{T}$ we define also inductively the order of α as follows: $\operatorname{ord}(0) = 0$ and for any $\beta \in \operatorname{succ}(\alpha)$ we put $\operatorname{ord}(\beta) = \operatorname{ord}(\alpha) + n$, where $weight(f_\alpha) = \theta_n$.

Notice that every functional $f \in K$ admits a tree-analysis, not necessarily unique.
We shall use repeatedly the following

Remark 1.9. Let $X = T_M[(\mathcal{S}_n, \theta_n)_n]$ with (♣). Let $(f_\alpha)_{\alpha\in\mathcal{T}}$ be a norming tree of a norming functional $f \in K$ and $\alpha \in \mathcal{T}$ not a terminal node. Let $f_\alpha = \theta_{r_\alpha} \sum_{\beta\in\mathrm{succ}(\alpha)} f_\beta$. Then by definition of Schreier families given any $k \in [\mathrm{ord}(\alpha), \mathrm{ord}(\alpha) + r_\alpha]$ we can write f_α as follows

$$f_\alpha = \theta_{r_\alpha} \sum_{t\in A_\alpha} \sum_{s\in F_t} f_s$$

where $(f_s)_{s\in F_t}$ is $\mathcal{S}_{r_\alpha-(k-\mathrm{ord}(\alpha))}$-allowable, for any $t \in A_\alpha$, and $(g_t)_{t\in A_\alpha}$ is $\mathcal{S}_{k-\mathrm{ord}(\alpha)}$-allowable, with $g_t = \theta_{r_\alpha-(k-\mathrm{ord}(\alpha))} \sum_{s\in F_t} f_t$, $t \in A_\alpha$. In particular for any f_α and $x \in X$ with non-negative coefficients we get by (♣)

$$f_\alpha(x) = \frac{\theta_{r_\alpha}}{\theta_{r_\alpha-(k-\mathrm{ord}(\alpha))}} \sum_{t\in A_\alpha} g_t(x) \leqslant \theta^{k-\mathrm{ord}(\alpha)} \sum_{t\in A_\alpha} g_t(x).$$

As $t(\alpha) \leqslant \theta_{\mathrm{ord}(\alpha)} \leqslant \theta^{\mathrm{ord}(\alpha)}$ it follows that $t(\alpha) f_\alpha(x) \leqslant \theta^k \sum_{t\in A_\alpha} g_t(x)$.

2. Modified mixed Tsirelson spaces defined on Schreier families

In this section we present the main results on sequential minimality and arbitrary distortability of regular modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)]$ with (♣). In the first subsection we discuss the notion of an average, in the next two subsections we present "Tsirelson-type" estimations needed for the proof of main theorems in the last subsection. Since the u.v.b. in any (modified) mixed Tsirelson space and its dual is unconditional, we work in the sequel on functionals and vectors with non-negative coefficients. From now on we fix a regular modified mixed Tsirelson space $X = T_M[(\mathcal{S}_n, \theta_n)_{n\in\mathbb{N}}]$.

2.1. Averages

In this part we recall the notion of an average [7] and present basic facts.

Definition 2.1. A vector x in a Banach space X with a basis is called an (M, ε)-average of a block sequence $(x_i)_i \subset X$, for $M \in \mathbb{N}$ and $\varepsilon > 0$, if $x = \sum_{i\in G} a_i x_i$ for some $G \in \mathcal{S}_M$ and $(a_i)_{i\in G} \subset (0, 1]$ with $\sum_{i\in G} a_i = 1$ and for any $F \in \mathcal{S}_{M-1}$ we have $\sum_{i\in F} a_i < \varepsilon$.

We shall use the following facts in the sequel.

Fact 2.2. (See [8, Lemma 4.9].) Let $x = \sum_{i\in F} a_i x_i$ be an (M, ε)-average of normalized vectors $(x_i)_{i\in F}$, $M \in \mathbb{N}$, $\varepsilon > 0$ and $\mathcal{E}$ an $\mathcal{S}_{M-1}$-allowable family of sets. Let $G = F \setminus K$, where $K = \{i \in F\colon\ \exists E \in \mathcal{E},\ E \text{ begins in } x_i\}$. Then for every $i \in G$ the set $\{Ex_i\colon\ E \in \mathcal{E},\ Ex_i \neq 0\}$ is $\mathcal{S}_1$-allowable and

$$\sum_{E\in\mathcal{E}} \|Ex\| \leqslant \sum_{E\in\mathcal{E}} \left\| E\left(\sum_{i\in G} a_i x_i\right)\right\| + 2\varepsilon/\theta_M.$$

Fact 2.3. Let $x = \sum_{i\in F} a_i x_i$ be an (M, ε)-average of normalized vectors $(x_i)_{i\in F}$, $M \in \mathbb{N}$, $\varepsilon > 0$ and f a norming functional with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$. Then there is subtree $\mathcal{T}'$ of $\mathcal{T}$ such that any terminal node of $\mathcal{T}'$ has order at least M and the functional f' defined by the tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}'}$ satisfies $f(x) \leqslant f'(x) + 2\varepsilon$.

Proof. Let $\mathcal{E}$ be the collection of all terminal nodes of $\mathcal{T}$ of order smaller than M. Let $G = \{i \in F\colon$ some f_α begins in x_i, $\alpha \in \mathcal{E}\}$. Since the set $(f_\alpha)_{\alpha\in\mathcal{E}}$ is $\mathcal{S}_{M-1}$-allowable, it follows $G \setminus \{\min G\} \in \mathcal{S}_{M-1}$ and $f(\sum_{i\in G} a_i x_i) \leqslant a_{\min G} + \sum_{i\in G\setminus\{\min G\}} a_i \leqslant 2\varepsilon$. Let $\mathcal{T}'$ be the tree $\mathcal{T}$ with removed nodes from the family $\mathcal{E}$. Then $f(x) \leqslant f'(x) + f(\sum_{i\in G} a_i x_i) \leqslant f'(x) + 2\varepsilon$. □

Lemma 2.4. *Let X satisfy (♣). Let $x = \sum_i a_i x_i$ be an (M, ε)-average of a normalized block sequence $(x_i)_i \subset X$, $M \in \mathbb{N}$. Then for any $j \in \mathbb{N}$, $j < M$ and $\mathcal{S}_j$-allowable $(E_l)_l$ we have*

$$\sum_l \|E_l x\| \leqslant \theta_1^{-1} \theta^{M-j-1} \sum_l \sum_i a_i \|E_l x_i\| + 4\varepsilon/\theta_M.$$

In particular $\|x\| \leqslant \theta_1^{-1} \theta^{M-1} + 4\varepsilon/\theta_M$.

Proof. Take an $\mathcal{S}_j$-allowable sequence $(E_l)_l$. For any l take a norming functional f_l with $\|E_l x\| = f_l(x)$ and its tree-analysis $(f^l_\alpha)_{\alpha\in\mathcal{T}_l}$. Let $\mathcal{E}$ be the collection of all terminal nodes $\alpha \in \mathcal{T}_l$ for all l, such that $\operatorname{ord}_{\mathcal{T}_l}(\alpha) \leqslant M-1-j$. Then the set $(f_\alpha)_{\alpha\in\mathcal{E}}$ is $\mathcal{S}_{M-1}$-allowable. By Fact 2.3 we can assume with error 2ε that all terminal nodes of all $\mathcal{T}_l$ have order at least $M-j$.

We will add in the tree-analysis $(f^l_\alpha)_{\alpha\in\mathcal{T}_l}$ additional nodes $(h_t)_t$ of order $M-j-1$, by grouping some of nodes of $\mathcal{T}_l$, and by (♣) obtain the desired estimation.

For any l let $\mathcal{E}_l$ be collection of all $\alpha \in \mathcal{T}_l$ which are maximal with respect to the property $\operatorname{ord}_{\mathcal{T}_l}(\alpha) \leqslant M-j-1$. Fix $\alpha \in \mathcal{E}_l$. Then by the above reduction α is not terminal, so $f^l_\alpha = \theta_{r_\alpha}\sum_{s\in\operatorname{succ}(\alpha)} f^l_s$ for some $\mathcal{S}_{r_\alpha}$-allowable (f^l_s). By Remark 1.9 for $k = M-j-1$ there are $\mathcal{S}_{M-j-1-\operatorname{ord}(\alpha)}$-allowable functionals $(h_t)_{t\in A_\alpha}$ with

$$t(\alpha) f^l_\alpha(x) \leqslant \theta^{M-j-1}\sum_{t\in A_\alpha} h_t(x).$$

It follows that $(h_t)_{t\in A_l}$ is $\mathcal{S}_{M-j-1}$-allowable, where $A_l = \bigcup_{\alpha\in E_l} A_\alpha$. Now we have

$$\begin{aligned}\|E_l x\| = f_l(x) &= \sum_{\alpha\in\mathcal{E}_l} t(\alpha) f^l_\alpha(E_l x)\\ &\leqslant \sum_{\alpha\in\mathcal{E}_l}\theta^{M-j-1}\sum_{t\in A_\alpha} h_t(E_l x) = \theta^{M-j-1}\sum_{t\in A_l} h_t(E_l x).\end{aligned}$$

Taking into account the error from erasing nodes with too small orders we obtain

$$\sum_l \|E_l x\| \leqslant \theta^{M-j-1}\sum_l\sum_{t\in A_l} h_t(E_l x) + 2\varepsilon \leqslant \cdots.$$

Notice that $(h_t)_{t\in A}$ is $\mathcal{S}_{M-1}$-allowable, where $A = \bigcup_l A_l$. By Fact 2.2 with error $2\varepsilon/\theta_M$ we assume that the family $(h_t(x_i))_{t:\, h_t(x_i)\neq 0}$ is $\mathcal{S}_1$-allowable for each i and thus we have:

$$\begin{aligned}\cdots &\leqslant \theta^{M-j-1}\sum_l\sum_i a_i \sum_{t:\, h_t(x_i)\neq 0} h_t(E_l x_i) + 4\varepsilon/\theta_M\\ &\leqslant \theta^{M-j-1}\theta_1^{-1}\sum_l\sum_i a_i\|E_l x_i\| + 4\varepsilon/\theta_M. \qquad \square\end{aligned}$$

2.2. Averaging trees

In order to control the norm of splitting of a vector of special type into allowable, not only admissible parts, we compare it to the norm of splitting of a corresponding vector in the original Tsirelson space $T[\mathcal{S}_1,\theta]$. In this section we present the upper "Tsirelson-type" estimate for (M,ε)-averages with more refined structure. We shall use the notion of an averaging admissible tree [3], with additional features:

Definition 2.5. We call a tree $(x^j_i)^{M,N^j}_{j=0,\,i=1}$ in X with weights $(N^j_i)^{M,N^j}_{j=1,\,i=1} \subset \mathbb{N}$ and errors $(\varepsilon^j_i)^{M,N^j}_{j=1,\,i=1} \subset (0,1)$, an averaging tree, if

1. $(x^j_i)_{i\in I_j}$ is a block sequence for any j, $1 = N^M \leqslant \cdots \leqslant N^0$.
   Moreover for any $j = 1,\dots,M$ and $i = 1,\dots,N^j$ we have the following:
2. there exists a non-empty interval $I^j_i \subset \{1,\dots,N^{j-1}\}$ with $\# I^j_i = N^j_i$ such that $\operatorname{succ}(x^j_i) = (x^{j-1}_s)_{s\in I^j_i}$,
3. $x^j_i = 1/N^j_i \sum_{s\in I^j_i} x^{j-1}_s$,
4. $2/\varepsilon^j_i < N^j_i \leqslant \operatorname{minsupp} x^j_i$,
5. $\varepsilon^j_{i+1} < 1/(2^i \operatorname{maxsupp} x^j_i)$, $\operatorname{maxsupp} x^j_i < N^j_{i+1}$.

Remark 2.6. In the situation as above we define coefficients $(a^j_i)^{M,N^j}_{j=0,\,i=1} \subset (0,1]$, as satisfying $x^M = \sum_{i=1}^{N^j} a^j_i x^j_i$. It follows straightforwardly that for any $j = 0,\dots,M$, $i = 1,\dots,N^j$ we have the following

6. $\sum_{i=1}^{N^j} a^j_i = 1$,
7. $a^j_i = \prod_{r=j+1}^M \frac{1}{N^r_{i_r}}$, where $x^r_{i_r} \succcurlyeq x^j_i$ for each $M \geqslant r > j$,
8. $a^j_i = \sum_{m:\, x^0_m \preccurlyeq x^j_i} a^0_m$.

Notice that any x^j_i is a (j,ε^j_i)-average of $(x^0_m)_{x^0_m \preccurlyeq x^j_i}$.

To show the last statement notice that by (4) for any $j, i \geqslant 1$ the block sequence $\operatorname{succ}(x_i^j)$ is $\mathcal{S}_1$-admissible, thus any block sequence $(x_m^0)_{x_m^0 \preccurlyeq x_i^j}$ is $\mathcal{S}_j$-admissible. To complete the proof notice that by the standard reasoning (cf. for example [30], last part of the proof of Proposition 3.6) we have the following fact:

Fact. Fix a block sequence $(x_m)_m$ and let $(x_i)_{i=1}^N$ be a block sequence of $(M-1, \varepsilon_i)$-averages of $(x_m)_{m \in A_i}$ such that $N > 2/\varepsilon$ and $\varepsilon_{i+1} < 1/2^i \operatorname{maxsupp} x_i$. Then $x = \frac{1}{N}(x_1 + \cdots + x_N)$ is an (M, ε)-average of $(x_m)_{m \in A_i, i=1,\dots,N}$.

The above remark together with the construction of an averaging tree presented in [3] yields the standard

Fact 2.7. For any block sequence $(x_m)_m$ of X, any $\varepsilon > 0$ and any $M \in \mathbb{N}$ there is an (M, ε)-average x of (x_m) with an averaging tree $(x_i^j)_{j=0, i=1}^{M, N^j}$ in X with suitable weights and errors $(\varepsilon_i^j)_{j=1, i=1}^{M, N^j}$ such that $x_1^M = x$, $\varepsilon_1^M = \varepsilon$ and $(x_m^0)_m \subset (x_m)_m$.

In order to deal with allowable splittings, we need the next result, stating – roughly speaking – that a restriction of an average x with an averaging tree high enough is still an average y, with a strict control on the error on the new average y – depending on the error in the averaging tree of x corresponding to $\operatorname{minsupp} y$.

Lemma 2.8. *Let (x_i^j), (N_i^j), (a_i^j), (ε_i^j) form an averaging tree for an $(M + \tilde{M}, \varepsilon)$-average x, $M, \tilde{M} \in \mathbb{N}$, $\varepsilon > 0$, of a normalized block sequence $(x_i^0)_i$, satisfying for any i, j the following*

1. *$N_i^j = 2^{k_i^j}$ for some k_i^j,*
2. *$\varepsilon_1^j \leqslant \theta_M \varepsilon/2$, $\varepsilon_{i+1}^j \leqslant \theta_M \varepsilon / 2^i \operatorname{maxsupp} x_i^j$.*

Then for any $I \subset \mathbb{N}$ with $N_{\min I}^M \sum_{i \in I} a_i^M \in \mathbb{N}$ the vector $y = \sum_{i \in I} a_i^M x_i^M$ is a restriction of an $(M, \varepsilon_{\min I}^M)$-average of some block sequence (y_k^0) with $\|y_k^0\| \leqslant 1$ and such that the following property holds:

(P) *for every k, i, l either $x_i^M \preccurlyeq y_k^l$ or $x_i^M \succcurlyeq y_k^l$ or x_i^M and y_k^l are incomparable, where $(y_k^l)_{k,l}$ is the family of nodes of averaging tree of y.*

Proof. Let $\varepsilon_I = \varepsilon_{\min I}^M$. We represent $y = \sum_{i \in I} a_i^M x_i^M$ as a restriction of an (M, ε_I)-average. We construct inductively on $l = M, M-1, \dots, 0$ an averaging tree $(y_k^l)_{l=0, k=1}^{M, K_l}$ with weights (W_k^l) and coefficients (c_k^l), where $y_k^l = 1/W_k^l \sum_{s \in J_k^l} y_s^{l-1}$ and $c_k^l = \prod_{r>l:\ y_k^l \preccurlyeq y_{k_r}^r} \frac{1}{W_{k_r}^r}$, such that $y_1^M = y$ and the following is satisfied

(P_0) $c_k^l y_k^l = \sum_{m \in A_k^l} a_m^0 x_m^0$, $c_k^l = \sum_{m \in A_k^l} a_m^0$ for every k and $l < M$,
(P_1) for every k, i, l either $x_i^l \preccurlyeq y_k^l$ or x_i^l is incomparable with y_k^l,
(P_2) for every i, j, k, l either $x_i^j \preccurlyeq y_k^l$ or $x_i^j \succcurlyeq y_k^l$ or x_i^j and y_k^l are incomparable,
(P_3) for every k, l we have $W_k^l = \min\{N_i^l:\ x_i^l \preccurlyeq y_k^l\}$.

We allow one difference from the original definition: $\# J_1^M = L = N_{\min I}^M \sum_{i \in I} a_i^M$, not W_1^M, otherwise $\# J_k^l = W_k^l$ for any $l < M$.

Let $y_1^M = \sum_{i \in I} a_i^M x_i^M = \sum_{m \in A} a_m^0 x_m^0$, $c_1^M = 1$, $A_1^M = A$ and $W_1^M = N_{\min I}^M \leqslant \operatorname{minsupp} y$. All properties ($P_0$)–($P_3$) are obviously satisfied.

Assume we have defined $(y_k^l)_k$, $(W_k^l)_k$ and $(c_k^l)_k$ for some $M \geqslant l > 2$ satisfying the above. Fix k and consider A_k^l. Pick any $m \in A_k^l$. By (P_1) in inductive assumption we have $x_{i_r}^r \preccurlyeq y_{k_r}^r$ for any $l \leqslant r \leqslant M$, i_r, k_r with $x_m^0 \preccurlyeq x_{i_r}^r$ and $x_m^0 \preccurlyeq y_{k_r}^r$. Thus $N_{i_r}^r \geqslant W_{k_r}^r$ for any $l \leqslant r \leqslant M$, i_r, k_r as above. By Remark 2.6 and (P_3) we have

$$a_m^0 = \prod_{r=1}^M \frac{1}{N_{i_r}^r} \leqslant \prod_{r=l}^M \frac{1}{N_{i_r}^r} \leqslant \prod_{r=l}^M \frac{1}{W_{k_r}^r} = \frac{c_k^l}{W_k^l}.$$

Recall that all coefficients $a_m^0, c_k^l, 1/W_k^l$ are some powers of $1/2$ and $(a_m^0)_m$ is non-increasing. Moreover for $l < M$ we have $\sum_{m \in A_k^l} a_m^0 = c_k^l$, hence we can split A_k^l into W_k^l-many successive sets $(A_s^{l-1})_{s=1}^{W_k^l}$ such that for each s we have

$$\sum_{m \in A_s^{l-1}} a_m^0 = \frac{c_k^l}{W_k^l}.$$

In case $l=M$ we have $\sum_{m\in A_1^M} a_m^0 = L/W_1^M$, hence we can split A_1^M into L-many sets $(A_s^{M-1})_{s=1}^L$ such that for each s we have

$$\sum_{m\in A_s^{M-1}} a_m^0 = \frac{c_1^M}{W_1^M} = \frac{1}{W_1^M}.$$

We define then $(y_s^{l-1})_s$ and $(c_s^{l-1})_s$ by

$$\frac{c_k^l}{W_k^l} y_s^{l-1} = \sum_{m\in A_s^{l-1}} a_m^0 x_m^0, \qquad c_s^{l-1} = \frac{c_k^l}{W_k^l}.$$

Hence obviously $y_k^l = 1/W_k^l \sum_s y_s^{l-1}$. Let also $W_s^{l-1} = \min\{N_i^{l-1}\colon\ x_i^{l-1} \preccurlyeq y_s^{l-1}\}$ and thus we finish construction of vectors on level $l-1$ satisfying (P_0) and (P_3).

Now we verify property (P_1). Notice that by property (P_1) on level l for each k we have $\operatorname{supp} y_k^l = \bigcup\{\operatorname{supp} x_i^l\colon\ x_i^l \preccurlyeq y_k^l\} = \bigcup\{\operatorname{supp} x_s^{l-1}\colon\ x_s^{l-1} \preccurlyeq y_k^l\}$. In case $l<M$ by Remark 2.6 and (P_0) for l we have

$$\sum_{r\colon\ y_r^{l-1}\preccurlyeq y_k^l} c_r^{l-1} = W_k^l \frac{c_k^l}{W_k^l} = c_k^l = \sum_{m\in A_k^l} a_m^0 = \sum_{s\colon\ x_s^{l-1}\preccurlyeq y_k^l} a_s^{l-1},$$

and as in the construction each $a_s^{l-1} \leqslant c_k^l/W_k^l = c_r^{l-1}$. In case of $l=M$ we have

$$\sum_{r\colon\ y_r^{M-1}\preccurlyeq y_k^M} c_r^{M-1} = L\frac{c_1^M}{W_1^M} = \frac{L}{W_k^l} = \sum_{m\in A_1^M} a_m^0 = \sum_{s\colon\ x_s^{M-1}\preccurlyeq y_k^M} a_s^{M-1},$$

and each $a_s^{M-1} \leqslant 1/W_1^M = c_r^{M-1}$. Since all coefficients are powers of $1/2$ and the sequence $(a_s^{l-1})_s$ is non-increasing we can partition the set $\{s\colon\ x_s^{l-1} \preccurlyeq y_k^l\}$ into $\bigcup\{B_r\colon\ y_r^{l-1} \preccurlyeq y_k^l\}$ such that for any r we have $c_r^{l-1} = \sum_{s\in B_r} a_s^{l-1}$. Consequently for any $y_r^{l-1} \preccurlyeq y_k^l$ and $x_s^{l-1} \preccurlyeq y_k^l$ we have either $y_r^{l-1} \succcurlyeq x_s^{l-1}$ or y_r^{l-1} and x_s^{l-1} are incomparable.

The property (P_2) is verified analogously by induction. If for some l,k,j we have $\operatorname{supp} y_k^l = \bigcup\{\operatorname{supp} x_i^j\colon\ x_i^j \preccurlyeq y_k^l\}$, then we show that for any $y_r^{l-1} \preccurlyeq y_k^l$ and $x_s^{j-1} \preccurlyeq y_k^l$ we have either $y_r^{l-1} \succcurlyeq x_s^{j-1}$ or y_r^{l-1} and x_s^{j-1} are incomparable. The same argument works if $\operatorname{supp} x_i^j = \bigcup\{\operatorname{supp} y_k^l\colon\ x_i^j \succcurlyeq y_k^l\}$ for some i,j,l.

Define the error δ_k^l for each $l=M,\dots,1$ and $k=1,\dots,K_l$. For $k=1$ and any $l=M,\dots,1$ let $\delta_1^l = \varepsilon_I$. By property (P_1) for any l,k there is some $i_k \geqslant k$ with

$$\operatorname{maxsupp} y_k^l \leqslant \operatorname{maxsupp} x_{i_k}^l < \operatorname{minsupp} x_{i_k+1}^l \leqslant \operatorname{minsupp} y_{k+1}^l.$$

Let $\delta_{k+1}^l = \varepsilon_{i_k+1}^l$ for any $k \geqslant 1$. We verify condition (5) of Definition 2.5. For $k=1$ and $l=M,\dots,1$ we have $W_1^l \geqslant N_{\min I}^M \geqslant 2/\varepsilon_{\min I}^M = 2/\delta_1^l$. On the other hand we have for any $l=M-1,\dots,1$ and $k=1,\dots,K_l-1$

$$\delta_{k+1}^l = \varepsilon_{i_k+1}^l < 1/2^{i_k} \operatorname{maxsupp} x_{i_k}^l \leqslant 1/2^k \operatorname{maxsupp} y_k^l,$$

and $W_{k+1}^l \geqslant N_{i_k+1}^l > 2/\varepsilon_{i_k+1}^l = 2/\delta_{k+1}^l$.

Hence $(y_k^l)_{k,l}$, $(W_k^l)_{k,l}$, $(c_k^l)_{k,l}$, $(\delta_k^l)_{k,l}$ form an averaging tree and thus y is (M,ε_I)-average of $(y_k^0)_k$. Notice that

$$\|c_k^0 y_k^0\| = \left\| \sum_{m\in A_k^0} a_m^0 x_m^0 \right\| \leqslant \sum_{m\in A_k^0} a_m^0 = c_k^0,$$

therefore $\|y_k^0\| \leqslant 1$. Moreover property (P_2) includes property (P). □

Remark 2.9. Note that by the construction each sequence $(y_s^{l-1})_{s\in J_k^l}$ is $\mathcal{S}_1$-admissible for any k,l. Hence it readily follows that for every set F of incomparable nodes (y_k^l) the functional $\sum_{y_k^l\in F} \theta^{M-l} e^*_{\operatorname{minsupp} y_k^l}$ is a norming functional on the space $T[\mathcal{S}_1,\theta]$.

Note also that for any k_0,l_0 the family $(x_i^M\colon\ x_i^M \preccurlyeq y_{k_0}^{l_0},\ l_0 = \min\{m \geqslant l \geqslant 0\colon\ x_i^M \prec y_k^l \text{ for some } k\})$ is $\mathcal{S}_1$-admissible.

The next lemma provides a "Tsirelson-type" upper estimate for the norms of averages.

Lemma 2.10. *Let (x_i^j), (N_i^j), (a_i^j), (ε_i^j) form an averaging tree for a $(2M-3,\varepsilon)$-average x, $M>1$, $\varepsilon>0$, of a normalized block sequence $(x_i^0)_i$, satisfying for any i, j the following*:

1. $N_i^j = 2^{k_i^j}$ *for some* k_i^j,
2. $\varepsilon_1^j \leqslant \theta_M \varepsilon/2$, $\varepsilon_{i+1}^j \leqslant \theta_M \varepsilon/2^i \operatorname{maxsupp} x_i^j$.

Fix an $\mathcal{S}_{M-4}$*-allowable family* $\mathcal{E}$ *of subsets of* $\mathbb{N}$*, such that the family* $\{E \in \mathcal{E}\colon\ Ex_i^M \neq 0\}$ *is* $\mathcal{S}_1$*-allowable for any i, and coefficients* $(t_E)_{E\in\mathcal{E}} \subset [0,1]$.

Then there is a partition $(V_E)_{E\in\mathcal{E}}$ *of nodes* $(x_i^0)_i$*, with* $\operatorname{minsupp} x^0_{\min V_E} \geqslant \min E$*, such that*

$$\sum_{E\in\mathcal{E}} t_E \|Ex\| \leqslant C \sum_{E\in\mathcal{E}} t_E \left\| \sum_{i\in V_E} a_i^0 e_{\operatorname{minsupp} x_i^0} \right\|_{T[\mathcal{S}_1,\theta]} + C\varepsilon$$

for some universal constant C depending only on θ_1 *and* θ.

Proof. STEP 1. Let us recall that x is an $(M-3,\varepsilon)$-average of $(x_i^M)_i$. First let $\mathcal{E}_i = \{E \in \mathcal{E}\colon\ E \text{ begins at } x_i^M\}$ and $J = \{i\colon\ \mathcal{E}_i \neq \emptyset\}$. As $(x_i^M)_{i\in J\setminus \min J}$ is $\mathcal{S}_{M-4}$-admissible, we have

$$\sum_{E\in\mathcal{E}} t_E \left\| E \sum_{i\in J} a_i^M x_i^M \right\| \leqslant \sum_{i\in J} a_i^M \sum_{E\in\mathcal{E}} \|Ex_i^M\| \leqslant \theta_1^{-1} \sum_{i\in J} a_i^M \leqslant \theta_1^{-1} 2\varepsilon.$$

For any $E\in\mathcal{E}$ let $I_E = \{i \notin J\colon\ Ex_i^M \neq 0\}$, $i_E = \min I_E$ and $\varepsilon_E = \varepsilon_{i_E}^M$. Compute

$$\begin{aligned}\sum_{E\in\mathcal{E}} \varepsilon_E &\leqslant \varepsilon\theta_M \sum_{i\in J}\sum_{E\in\mathcal{E}_i} 1/2^{i_E-1} \operatorname{maxsupp} x^M_{i_E-1} \\ &\leqslant \varepsilon\theta_M \sum_{i\in J} \operatorname{maxsupp} x_i^M / 2^i \operatorname{maxsupp} x_i^M \leqslant \varepsilon\theta_M.\end{aligned}$$

STEP 2. Fix $E\in\mathcal{E}$. Let $\sum_{i\in I_E} a_i^M x_i^M = \sum_{m\in K} a_m^0 x_m^0$. Notice that each $a_m^0 \leqslant 1/N_{i_E}^M$ and $(a_m^0)_m$ is non-increasing, therefore we can partition K into intervals $A<B$ with $\sum_{m\in A} a_m^0 = L/N_{i_E}^M$ and $\sum_{m\in B} a_m^0 = \delta/N_{i_E}^M$ for some $L\in\mathbb{N}$ and $0\leqslant\delta<1$. Hence we can erase $\sum_{m\in B} a_m^0 x_m^0$ with error $\delta/N_{i_E}^M \leqslant 1/N_{i_E}^M \leqslant \varepsilon_E$.

After this reduction by Lemma 2.8 the vector $y = \sum_{i\in I_E} a_i^M x_i^M$ is a restriction of an $(M-2,\varepsilon_E)$-average $\sum_k c_k^2 y_k^2$ with $\|y_k^2\| \leqslant 1$ and property (P) given by a suitable averaging tree $(y_k^l)_{k,l}$ with proper weights, coefficients and errors.

We take the family $K = \{k\colon\ \operatorname{minsupp} x_i^M \in \operatorname{range} y_k^2 \text{ for some } x_i^M\}$. Since $(x_i^M)_i$ is an $\mathcal{S}_{M-3}$-admissible family and y is an $(M-2,\varepsilon_E)$-average of (y_k^2), we can erase $\sum_{k\in K} c_k^2 y_k^2$ with error $2\varepsilon_E$. For any i let

$$l_{E,i} = \min\bigl\{M \geqslant l \geqslant 0\colon\ y_k^l \succcurlyeq x_i^M \text{ for some } k\bigr\}.$$

By the above reduction and (P) we can assume that $l_{E,i} \geqslant 2$ for all $i\in I_E$. Let

$$K_{E,i} = \bigl\{k\colon\ y_k^2 \preccurlyeq x_i^M\bigr\} \quad \text{for any } i \in I_E.$$

Compute by Lemma 2.4 for the $(M-2,\varepsilon_E)$-average $\sum_k c_k^2 y_k^2$ and $j=0$

$$\begin{aligned}\|Ex\| = \left\| E\sum_k c_k^2 y_k^2 \right\| &\leqslant \left\| \sum_{k\notin K} c_k^2 E y_k^2 \right\| + 2\varepsilon_E \\ &\leqslant \theta_1^{-1}\theta^{M-3} \sum_{i\in I_E}\sum_{k\in K_{E,i}} c_k^2 \|Ey_k^2\| + 6\varepsilon_E/\theta_M \\ &= \theta_1^{-1}\theta^{M-3} \sum_{i\in I_E} a_i^M \sum_{k\in K_{E,i}} \left\| \frac{c_k^2}{a_i^M} E y_k^2 \right\| + 6\varepsilon_E/\theta_M.\end{aligned}$$

STEP 3. Fix $i\notin J$. Put $\mathcal{F}_i = \{E\in\mathcal{E}\colon\ i\in I_E\} = \{E\in\mathcal{E}\colon\ Ex_i^M \neq 0\}$. For any $E\in\mathcal{F}_i$ and $k\in K_{E,i}$ let $w_k = \frac{c_k^2}{a_i^M} E y_k^2$. Notice that $(w_k)_k$ is a partition of x_i^M. For each $k\in K_{E,i}$ take the norming functional f_k with $f_k(w_k) = \|w_k\|$ and $\operatorname{supp} f_k \subset \operatorname{supp} w_k$.

We gather all the terminal nodes in the tree-analysis of f_k for all $k\in K_{E,i}$, $E\in\mathcal{F}_i$, of order smaller than $M - l_{E,i}$. By the assumption on $\mathcal{E}$ and the fact that $l_{E,i} \geqslant 2$ they form an $\mathcal{S}_{M-1}$-allowable family, hence as x_i^M is an (M,ε_i^M)-average, we can erase these nodes with total error $2\varepsilon_i^M$.

By Remark 1.9, adding nodes in the tree-analysis of each f_k, $k \in K_{E,i}$, on the level $M - l_{E,i}$, we get $\|w_k\| \leqslant \theta^{M-l_{E,i}} \sum_l f_k^l(x_i^M)$ for some $\mathcal{S}_{M-l_{E,i}}$-allowable functionals $(f_k^l)_l$. Pick E_i with $t_{E_i}\theta^{-l_{E_i,i}} = \max\{t_E\theta^{-l_{E,i}}\colon\ E \in \mathcal{F}_i\}$. Let $l_i = l_{E_i,i}$ and compute

$$\sum_{E\in\mathcal{F}_i} t_E \sum_{k\in K_{E,i}} \left\| \frac{c_k^2}{a_i^M} E y_k^2 \right\| \leqslant \sum_{E\in\mathcal{F}_i} t_E \theta^{M-l_{E,i}} \sum_{k\in K_{E,i}} \sum_l f_k^l\big(x_i^M\big) + 2\varepsilon_i^M \leqslant \cdots.$$

Notice again that $(f_k^l)_{l,k\in K_{E,i},\,E\in\mathcal{F}_i}$ is an $\mathcal{S}_{M-1}$-allowable family (as before by $l_{E,i} \geqslant 2$ and assumption on $\mathcal{E}$). As x_i^M is an (M,ε_i^M)-average of suitable $(x_m^0)_m$, by Fact 2.2 with error $2\varepsilon_i^M/\theta_M$, we may assume that for any m the family $(\operatorname{supp} f_k^l \cap \operatorname{supp} x_m^0)_{l,k\in K_{E,i},\,E\in\mathcal{F}_i}$ is $\mathcal{S}_1$-allowable. Therefore we continue the estimation

$$\cdots \leqslant t_{E_i}\theta^{M-l_i} \sum_{E\in\mathcal{F}_i} \sum_{k\in K_{E,i}} \sum_l f_k^l\big(x_i^M\big) + 4\varepsilon_i^M/\theta_M \leqslant \theta_1^{-1} t_{E_i}\theta^{M-l_i} + 4\varepsilon_i^M/\theta_M.$$

STEP 4. We define $J_E = \{i\colon\ E = E_i\} \subset I_E$ for any $E \in \mathcal{E}$. Notice that $(J_E)_{E\in\mathcal{E}}$ are pairwise disjoint and compute, using the previous steps

$$\begin{aligned}
\sum_{E\in\mathcal{E}} t_E\|Ex\| &\leqslant \sum_{E\in\mathcal{E}} t_E \left\| E\sum_{i\in J} a_i^M x_i^M \right\| + \sum_{E\in\mathcal{E}} t_E \left\| E\sum_{i\in I_E} a_i^M x_i^M \right\| \quad (\text{STEP 1})\\
&\leqslant 2\theta_1^{-1}\varepsilon + \theta_1^{-1}\theta^{M-3} \sum_{E\in\mathcal{E}} \sum_{i\in I_E} a_i^M \sum_{k\in K_{E,i}} t_E \left\| \frac{c_k^2}{a_i^M} E y_k^2 \right\| + 6\sum_{E\in\mathcal{E}} \varepsilon_E/\theta_M \quad (\text{STEP 2})\\
&\leqslant \theta_1^{-1}\theta^{M-3} \sum_{i\notin J} a_i^M \sum_{E\in\mathcal{F}_i} \sum_{k\in K_{E,i}} t_E \left\| \frac{c_k^2}{a_i^M} E y_k^2 \right\| + \big(6 + 2\theta_1^{-1}\big)\varepsilon\\
&\leqslant \theta_1^{-2}\theta^{M-3} \sum_{i\notin J} a_i^M t_{E_i}\theta^{M-l_i} + 4\sum_i \varepsilon_i^M/\theta_M + \big(6 + 2\theta_1^{-1}\big)\varepsilon \quad (\text{STEP 3})\\
&\leqslant \theta_1^{-2}\theta^{M-3} \sum_{E\in\mathcal{E}} t_E \sum_{i\in J_E} a_i^M\theta^{M-l_i} + \big(10 + 2\theta_1^{-1}\big)\varepsilon \leqslant \cdots.
\end{aligned}$$

By Remark 2.9 for any $E \in \mathcal{E}$ the formula $\sum_{i\in I_E} \theta^{M-l_i+1} e^*_{\operatorname{minsupp} x_i^M}$ defines a norming functional in $T[\mathcal{S}_1,\theta]$. Therefore for any $E \in \mathcal{E}$ we have

$$\sum_{i\in J_E} a_i^M\theta^{M-l_i} \leqslant \theta^{-1} \left\| \sum_{i\in J_E} a_i^M e_{\operatorname{minsupp} x_i^M} \right\|_{T[\mathcal{S}_1,\theta]},$$

and we continue the above estimation

$$\cdots \leqslant \theta_1^{-2}\theta^{M-4} \sum_{E\in\mathcal{E}} t_E \left\| \sum_{i\in J_E} a_i^M e_{\operatorname{minsupp} x_i^M} \right\|_{T[\mathcal{S}_1,\theta]} + \big(10 + 2\theta_1^{-1}\big)\varepsilon \leqslant \cdots.$$

Consider $z_i^M = 1/a_i^M \sum_{x_m^0 \preccurlyeq x_i^M} a_m^0 e_{\operatorname{minsupp} x_m^0}$, for $i = 1,\dots,N^M$, which are (M,ε_i^M)-averages in $T[\mathcal{S}_1,\theta]$ by Remark 2.6. As $\|z_i^M\|_{T[\mathcal{S}_1,\theta]} \geqslant \theta^M$ for each i, we continue

$$\begin{aligned}
\cdots &\leqslant \theta_1^{-2}\theta^{-4} \sum_{E\in\mathcal{E}} t_E \left\| \sum_{i\in J_E} a_i^M z_i^M \right\|_{T[\mathcal{S}_1,\theta]} + \big(10 + 2\theta_1^{-1}\big)\varepsilon\\
&\leqslant \theta_1^{-2}\theta^{-4} \sum_{E\in\mathcal{E}} t_E \left\| \sum_{i\in J_E} \sum_{x_m^0 \preccurlyeq x_i^M} a_m^0 e_{\operatorname{minsupp} x_m^0} \right\|_{T[\mathcal{S}_1,\theta]} + C\varepsilon,
\end{aligned}$$

which ends the proof with $C = 10 + 2\theta_1^{-2}\theta^{-4}$ and $V_E = \{m\colon\ x_m^0 \preccurlyeq x_i^M,\, i \in J_E\}$ for each $E \in \mathcal{E}$. $\square$

2.3. Special types of averages

For the rest of this subsection we assume that the considered regular modified mixed Tsirelson space $X = T_M[(\mathcal{S}_n,\theta_n)_n]$ satisfies (♣). In this setting we present the lower "Tsirelson-type" estimate, using special types of averages. We start with [29, Corollary 4.10] recalled below.

Proposition 2.11. *For any block subspace Y of X, any $M\in\mathbb{N}$ and $0<\varepsilon<\theta_M\theta^M/4$, there is an (M,ε)-average $x\in Y$ of some normalized block sequence in Y such that*

$$\theta^{M-j}D\geqslant\sup\Big\{\sum_i\|E_ix\|:\ \mathcal{S}_j\text{-allowable }(E_i)\Big\}\geqslant\theta^{M-j}/D$$

for any $0\leqslant j\leqslant M$ and some universal constant D depending only on θ_1 and θ.

Proof. We recall [29, Lemma 4.9], whose proof is valid, line by line, also in the modified case. [29, Lemma 4.9] and Lemma 2.4 yield the proposition. □

Definition 2.12. A special (M,ε)-average x, $M\in\mathbb{N}$, $\varepsilon>0$, is any (M,ε)-average satisfying assertion of Proposition 2.11.

For the next lemma we shall need the following observation.

Fact 2.13. Fix $M\in\mathbb{N}$. Then for any $G\in\mathcal{S}_M$ and any $z=\sum_{i\in G}a_ie_i\in T[\mathcal{S}_1,\theta]$, $(a_i)_{i\in G}\subset[0,1]$, there is a norming functional f with a tree-analysis with height at most M, such that $\|z\|_{T[\mathcal{S}_1,\theta]}\leqslant 2f(z)$.

Proof. Take a norming functional g with a tree-analysis $(g_t)_{t\in T}$ satisfying $g(z)=\|z\|_{T[\mathcal{S}_1,\theta]}$. Let I be the set of all terminal nodes of $\mathcal{T}$ with order at most M and let g_1 be the restriction of g to I and $g_2=g-g_1$. If $g_1(z)\geqslant g_2(z)$ then let $f=g_1$. Assume that $g_1(z)\leqslant g_2(z)$ and compute

$$g(z)\leqslant 2g_2(z)\leqslant 2\theta^{M+1}\sum_{i\in G\setminus I}a_i\leqslant 2\theta^M\sum_{i\in G}a_i=2f(z),$$

where $f=\theta^M\sum_{i\in G}e_i^*$, which ends the proof. □

The major obstacle in obtaining the lower "Tsirelson-type" estimate for norm is the fact that given an (M,ε)-average $x=\sum_{i\in F}a_ix_i$ we do not control the norm of $\sum_{i\in G}a_ix_i$, $G\subset F$, in general case. The next result provides a block sequence(x_i) for which any $\mathcal{S}_M$-admissible subsequence dominates suitable subsequence of the basis in the original Tsirelson space. This result is a generalization in the setting of mixed Tsirelson spaces of [8, Prop. 3.3].

Lemma 2.14. *For every block subspace Y of X and every $M\in\mathbb{N}$, $\delta>0$, there exists a block sequence (x_i) of Y satisfying for any $G\in\mathcal{S}_M$ and scalars $(a_i)_{i\in G}$*

$$\Big\|\sum_{i\in G}a_ix_i\Big\|\geqslant\frac{1}{2}(1-\delta)\Big\|\sum_{i\in G}a_i\|x_i\|e_{\operatorname{minsupp}x_i}\Big\|_{T[\mathcal{S}_1,\theta]}. \tag{2.1}$$

Proof. Assume the contrary. Notice first that for any $M\in\mathbb{N}$ we have

$$\big(\sqrt[m]{\theta_m}\big)^M\leqslant\sqrt[m]{\theta_{Mm}}\leqslant\sqrt[m]{\theta^{mM}},$$

thus $\lim_{m\to\infty}\sqrt[m]{\theta_{Mm}}=\theta^M$. Pick $m\in\mathbb{N}$ such that $\sqrt[m]{\theta_{Mm}}>\sqrt[m]{D^2}(1-\delta)\theta^M$ with D as in Proposition 2.11. Take a block sequence $(x_i^0)_i$ of special (Mm,ε)-averages, for some $\varepsilon>0$.

Since (2.1) fails there is an infinite sequence G_k^1 of successive elements of $\mathcal{S}_M$ and coefficients $(a_i^1)_{i\in G_k^1}$ such that

$$\Big\|\sum_{i\in G_k^1}a_i^1x_i^0\Big\|<\frac{1}{2}(1-\delta)\Big\|\sum_{i\in G_k^1}a_i^1\|x_i^0\|e_{m_i^0}\Big\|_{T[\mathcal{S}_1,\theta]},$$

where $m_i^0=\operatorname{minsupp}x_i^0$ for each i. Set $x_k^1=\sum_{i\in G_k^1}a_i^1x_i^0$, $k\in\mathbb{N}$, and by Fact 2.13 take norming functionals f_k^1 of the space $T[\mathcal{S}_1,\theta]$ with a tree-analysis of height at most M with

$$\Big\|\sum_{i\in G_k^1}a_i^1\|x_i^0\|e_{m_i^0}\Big\|_{T[\mathcal{S}_1,\theta]}\leqslant 2f_k^1\Big(\sum_{i\in G_k^1}a_i^1\|x_i^0\|e_{m_i^0}\Big).$$

Assume that we have defined $(x_k^{j-1})_k$ and $(f_k^{j-1})_k$ for some $j<m$. Then the failure of (2.1) implies the existence of a sequence $(G_k^j)_k$ of successive elements of $\mathcal{S}_M$ and a sequence $(a_i^j)_{i\in G_k^j}$ such that

$$\Big\|\sum_{i\in G_k^j}a_i^jx_i^{j-1}\Big\|<\frac{1}{2}(1-\delta)\Big\|\sum_{i\in G_k^j}a_i^j\|x_i^{j-1}\|e_{m_i^{j-1}}\Big\|_{T[\mathcal{S}_1,\theta]},$$

where $m_i^{j-1} = \operatorname{minsupp} x_i^{j-1}$. Set $x_k^j = \sum_{i\in G_k^j} a_i^j x_i^{j-1}$, for $k\in\mathbb{N}$, and take norming trees f_k^j of the space $T[\mathcal{S}_1,\theta]$ with a tree-analysis of height at most M such that

$$\Big\| \sum_{i\in G_k^j} a_i^j \|x_i^{j-1}\| e_{m_i^{j-1}} \Big\|_{T[\mathcal{S}_1,\theta]} \leqslant 2 f_k^j\Big(\sum_{i\in G_k^j} a_i^j \|x_i^{j-1}\| e_{m_i^{j-1}} \Big).$$

The inductive construction ends once we get the vector x_1^m and the functional f_1^m.

Each functional f_k^j is of the form $\sum_{i\in G_k^j} \theta^{l_i^j} e^*_{m_i^{j-1}}$, by construction satisfying

$$\|x_k^j\| < (1-\delta) \sum_{i\in G_k^j} \theta^{l_i^j} a_i^j \|x_i^{j-1}\|. \tag{2.2}$$

Inductively, beginning from f_1^m we produce a tree-analysis of some norming functional f on $T[\mathcal{S}_1,\theta]$ by substituting each terminal node $e^*_{m_k^j}$, $j=1,\dots,m$, by the tree-analysis of the functional f_k^j.

Put $G = \bigcup_{k_{m-1}\in G_1^m} \bigcup_{k_{m-2}\in G_{k_{m-1}}^{m-1}} \cdots \bigcup_{k_1\in G_{k_2}^2} G_{k_1}^1$. Let $(l_i)_{i\in G}$ be such that $f = \sum_{i\in G} \theta^{l_i} e^*_{m_i^0}$. Notice that $l_i \leqslant mM$ for any $i\in G$, as the height of each f_i^j does not exceed M. We compute the norm of x_1^m, which is of the form

$$x_1^m = \sum_{k_{m-1}\in G_1^m} \sum_{k_{m-2}\in G_{k_{m-1}}^{m-1}} \cdots \sum_{k_1\in G_{k_2}^2} \sum_{i\in G_{k_1}^1} a_{k_{m-1}}^m \dots a_i^1 x_i^0 = \sum_{i\in G} b_i x_i^0.$$

Since each x_i^0 is a special (mM,ε)-average, for some $\mathcal{S}_{mM-l_i}$-allowable sequence $(E_l)_{l\in L_i}$ we have $\|x_i^0\| \leqslant D^2\theta^{mM-l_i} \sum_{l\in L_i} \|E_l x_i^0\|$.

We have on one hand by repeated use of (2.2)

$$\begin{aligned} \|x_1^m\| &\leqslant (1-\delta)^m \sum_{i\in G} \theta^{l_i} b_i \|x_i^0\| \\ &\leqslant (1-\delta)^m D^2 \sum_{i\in G} \theta^{l_i} b_i \theta^{mM-l_i} \sum_{l\in L_i} \|E_l x_i^0\| \\ &= (1-\delta)^m D^2 \theta^{mM} \sum_{i\in G} b_i \sum_{l\in L_i} \|E_l x_i^0\|. \end{aligned}$$

On the other hand notice that $(E_l)_{l\in\bigcup_{i\in G} L_i}$ is $\mathcal{S}_{mM}$-allowable by the definition of f and $(l_i)_{i\in G}$, thus

$$\|x_1^m\| \geqslant \theta_{mM} \sum_{i\in G} b_i \sum_{l\in L_i} \|E_l x_i^0\|,$$

which brings $\theta_{mM} \leqslant (1-\delta)^m D^2 \theta^{mM}$, a contradiction with the choice of m. □

Definition 2.15. A Tsirelson (M,ε)-average $x\in X$, $M\in\mathbb{N}$, $\varepsilon>0$, is an (M,ε)-average $x=\sum_{i\in F} a_i x_i$ of a normalized block sequence (x_i) satisfying the assertion of Lemma 2.14 with $\delta = 1/2$.

Notice that by Lemma 2.4 every Tsirelson average is also a special average (with a possibly different constant).

Definition 2.16. A RIS of (special, Tsirelson) averages is any block sequence of (special, Tsirelson) $(n_k,\varepsilon/2^k)$-averages $(x_k)\subset X$ for $\varepsilon>0$ and $(n_k)_k\subset\mathbb{N}$ satisfying

$$\theta_{l_{k+1}} \|x_k\|_{\ell_1} \leqslant \frac{\varepsilon}{2^{k+1}}, \quad k\in\mathbb{N},$$

where $l_k = \max\{l\in\mathbb{N}\colon 4l\leqslant n_k\}$, $k\in\mathbb{N}$.

We need the following technical lemma, mostly reformulating [22, Lemma 7]:

Fact 2.17. Take RIS of normalized averages (x_k), for some $(n_k)\subset\mathbb{N}$ and $\varepsilon>0$, and some $x=\sum_k b_k x_k$ with $(b_k)\subset[0,1]$. Then for any norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$ there is a subtree $\mathcal{T}'$ such that the corresponding functional f' defined by the tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}'}$ satisfies $f(x)\leqslant f'(x)+3\varepsilon$ and the following hold for any k

(a) any node α of $\mathcal{T}'$ with $f_\alpha(x_k)\neq 0$ satisfies $\operatorname{ord}(\alpha) < n_{k+1}/4$,
(b) any terminal node α of $\mathcal{T}'$ with $f_\alpha(x_k)\neq 0$ satisfies $\operatorname{ord}(\alpha)\geqslant n_k$.

Proof. In order to prove (a) we repeat the reasoning from the proof of [22, Lemma 7]. For any k let $\mathcal{F}_k$ be the collection of all nodes in $\mathcal{T}$ which are minimal with respect to the property $\mathrm{ord}(\alpha) \geqslant n_{k+1}/4$ and $f_\alpha(x_k) \neq 0$. Then

$$\sum_{\alpha\in\mathcal{F}_k} t(\alpha) f_\alpha(x_k) \leqslant \theta_{l_{k+1}} \|x_k\|_{\ell_1} \leqslant \frac{\varepsilon}{2^{k+1}}.$$

Thus we can erase all nodes from $\mathcal{F}_k$ restricted to supports of x_k, for all k, with error $\sum_k b_k \frac{\varepsilon}{2^{k+1}} \leqslant \varepsilon$.

For (b) we use Fact 2.3 for erasing all terminal nodes α of $\mathcal{T}$ with $f_\alpha(x_k) \neq 0$ with error $2\varepsilon_k$, for any k. □

Lemma 2.18. *Let $x = \sum_k a_k x_k$ be an (M,ε)-average of RIS of normalized special averages (x_k), for $(n_k) \subset \{M+3, M+4, \dots\}$ and $0 < \varepsilon < \theta_M$.*

Then $\|x\| \leqslant D'\theta_M$, for some universal constant D' depending only on θ and θ_1.

Proof. Take a norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$ such that $\|x\| = f(x)$. Using Fact 2.17 pick the subtree $\mathcal{T}'$ satisfying (a) and (b) and the corresponding functional f'.

Let $\mathcal{E}$ be collection of all $\alpha \in \mathcal{T}'$ maximal with respect to the property $\mathrm{ord}(\alpha) \leqslant M-1$. Notice that $\mathcal{E}$ is $\mathcal{S}_{M-1}$-allowable.

Fix $\alpha \in \mathcal{E}$. Then α is not terminal, so $f_\alpha = \theta_{r_\alpha} \sum_{s\in\mathrm{succ}(\alpha)} f_s$. As in Remark 1.9 we partition $\mathrm{succ}(\alpha) = \bigcup_{t\in A_\alpha} F_t$ in such a way that $(f_s)_{s\in F_t}$ is $\mathcal{S}_{\mathrm{ord}(s)-(M-1)}$-allowable for every $t \in A_\alpha$ and $(g_t)_{t\in A_\alpha}$ is $\mathcal{S}_{M-1-\mathrm{ord}(\alpha)}$-allowable, where $g_t = \sum_{s\in F_t} f_s$. Let $A = \bigcup_{\alpha\in\mathcal{E}} A_\alpha$ and notice that $(g_t)_{t\in A}$ is $\mathcal{S}_{M-1}$-allowable. Let H denote the set of all k such that some g_t, $t \in A$, begins in x_k. Since x is an (M,ε)-average we have $\|\sum_{k\in H} a_k x_k\| \leqslant \sum_{k\in H} a_k \leqslant 2\varepsilon$.

By definition of H for any $\alpha \in \mathcal{E}$ and $k \notin H$ with $f_\alpha(x_k) \neq 0$ there is an immediate successor of α beginning before x_k. Thus by (a) we have for any $k \notin H$

(c) for any $\alpha \in \mathcal{E}$ with $f_\alpha(x_k) \neq 0$ the order of immediate successors of α is at most $n_k/4$,
(d) $\{g_t\colon t \in A,\ g_t(x_k) \neq 0\}$ restricted to $\operatorname{supp} x_k$ is $\mathcal{S}_1$-allowable.

Fix $k \notin H$ and $t \in A$ with $g_t(x_k) \neq 0$ and let $B_t^k = \{s \in F_t\colon f_s(x_k) \neq 0\}$.

Fix $s \in B_t^k$ and take the subtree $\mathcal{T}_s$ of $\mathcal{T}'$ consisting of s (as a root) and of all successors of s in $\mathcal{T}'$. By Remark 1.9, using (b) and (c) we can add nodes in $\mathcal{T}_s$ on level $n_k - \mathrm{ord}(s)$ obtaining $(h_{s,r})_{r\in C_s}$ which is $\mathcal{S}_{n_k-\mathrm{ord}(s)}$-allowable satisfying

$$f_s(x_k) \leqslant \sum_{r\in C_s} \theta^{n_k-\mathrm{ord}(s)} h_{s,r}(x_k).$$

Compute for $k \notin H$ using the above and (♣)

$$\begin{aligned} f'(x_k) &= \sum_{t\in A}\sum_{s\in B_t^k} t(s) f_s(x_k) \\ &\leqslant \theta_M \sum_{t\in A}\sum_{s\in B_t^k} \theta^{\mathrm{ord}(s)-M} \sum_{r\in C_s} \theta^{n_k-\mathrm{ord}(s)} h_{s,r}(x_k) \\ &\leqslant \theta_M \sum_{t\in A}\sum_{s\in B_t^k}\sum_{r\in C_s} \theta^{n_k-M} h_{s,r}(x_k). \end{aligned}$$

Notice that the family $\{h_{s,r}\colon r \in C_s,\ s \in B_t^k\}$ for any fixed $t \in A, k \notin H$ is $\mathcal{S}_{n_k-M+1}$-allowable. Therefore by (d) the family $\{h_{s,r}\colon r \in C_s,\ s \in B_t^k,\ t \in A\}$ for any fixed $k \notin H$ is $\mathcal{S}_{n_k-M+2}$-allowable and hence since x_k is a normalization of an (n_k,ε_k)-special average, we continue the estimation

$$\cdots \leqslant \theta_M \theta^{n_k-M} D^2 \theta^{-n_k+M-2} = D^2\theta^{-2}\theta_M.$$

We compute

$$f(x) \leqslant f'(x) + 3\varepsilon \leqslant \sum_{k\notin H} a_k f'(x_k) + 5\varepsilon \leqslant D^2\theta^{-2}\theta_M + 5\varepsilon \leqslant \big(D^2\theta^{-2} + 5\big)\theta_M,$$

which ends the proof of the lemma. □

2.4. Main results

Recall that a Banach space $(X, \|\cdot\|)$ is λ-distortable, $\lambda > 1$, if X admits an equivalent norm $|\cdot|$, such that for any infinite dimensional subspace Y of X we have $\sup\{|x|/|y|\colon x, y \in Y,\ \|x\| = \|y\| = 1\} \geqslant \lambda$. A Banach space X is arbitrarily distortable, if it is λ-distortable for any $\lambda > 1$. It is an open question if there exists a Banach space λ-distortable for some $\lambda > 1$, but not arbitrarily distortable. A natural candidate for such an example is the Tsirelson space $T[\mathcal{S}_1, 1/2]$.

Theorem 2.19. *Let X be a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n,\theta_n)_n]$. If $\theta_n/\theta^n \searrow 0$, then X is arbitrarily distortable.*

Proof. We show that the norm defined as $\|x\|_n=\sup\{\sum_i\|E_ix\|,\ (E_i)\ \mathcal{S}_n\text{-admissible}\}$, $x\in X$, $c\theta^n/\theta_n$-distorts X, for any $n\in\mathbb{N}$ and universal $c>0$. Clearly $\|\cdot\|\leqslant\|\cdot\|_n\leqslant 1/\theta_n\|\cdot\|$, $n\in\mathbb{N}$. Fix an infinite dimensional subspace Y of X. By Proposition 2.11 there is $y\in Y$ with $\|y\|\geqslant\theta_n/D$ and $\|y\|_n\leqslant D$. On the other hand, again by Proposition 2.11 and Lemma 2.18 there is $x\in Y$ with $\|x\|\leqslant D'\theta_n$ and $\|x\|_n\geqslant 1$. Considering $x/\|x\|$ and $y/\|y\|$ we obtain $c\theta^n/\theta_n$-distortion of X with $c=1/D^2D'$. □

Recall that a Banach space X with a basis is called sequentially minimal [16], if any block subspace of X contains a block sequence (x_n) such that every block subspace of X contains a copy of a subsequence of (x_n). Notice that this property implies quasiminimality of X.

Theorem 2.20. *Let X be a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n,\theta_n)_n]$. If $\theta_n/\theta^n \searrow$, then X is sequentially minimal.*

The theorem follows immediately from the following result:

Lemma 2.21. *Let $(x_k)_k$, $(y_k)_k$ be RIS of Tsirelson $(2M_k-3,\varepsilon_k)$-averages, $M_k>4$, $\varepsilon<(6C)^{-1}$, with C as in Lemma 2.10, such that*

1. *x_k has an averaging tree $(x^j_{k,i})_{i,j}$, $(N^j_{k,i})_{i,j}$, $(\varepsilon^j_{k,i})_{i,j}$, $(a^j_{k,i})_{i,j}$, y_k has an averaging tree $(y^j_{k,i})_{i,j}$, $(N^j_{k,i})_{i,j}$, $(\varepsilon^j_{k,i})_{i,j}$, $(a^j_{k,i})_{i,j}$, both satisfying conditions (1) and (2) of Lemma 2.10 for any k,*
2. *$\operatorname{minsupp} x^0_{k,i}=\operatorname{minsupp} y^0_{k,i}$ and $\|x^0_{k,i}\|=\|y^0_{k,i}\|=1$ for any k,i,*
3. *$\varepsilon_k\leqslant\theta_{2M_k-3}\theta^{2M_k-3}\varepsilon/2^{k+2}$ for any k.*

Then $(x_k/\|x_k\|)_k$ and $(y_k/\|y_k\|)_k$ are equivalent.

Notice first that the lemma above yields Theorem 2.20, as given a block sequence (w_n) in X, a block subspace Y of X and $k\in\mathbb{N}$, we can choose block sequences $(u_i)\subset[(w_n)]$ and $(v_i)\subset Y$ satisfying the assertion of Lemma 2.14 for $2M_k-3$. Passing to subsequences if necessary and using small perturbations we obtain block sequences (u'_i) and (v'_i) of the form $u'_i=u_i+\delta_ie_{m_i}$, $v'_i=v_i+\delta_ie_{m_i}$, for some $(m_i)\subset\mathbb{N}$ with $m_i=\operatorname{minsupp}u'_i=\operatorname{minsupp}v'_i$ for each i and small $(\delta_i)\subset(0,1)$, which are equivalent to (u_i) and (v_i) respectively and still satisfy the assertion of Lemma 2.14 for $2M_k-3$. Then by Fact 2.7 construct on these sequences two Tsirelson $(2M_k-3,\varepsilon_k)$-averages with averaging trees as in Lemma 2.21 with equal systems of weights, errors and coefficients, obtaining x_k and y_k for each $k\in\mathbb{N}$.

Now we proceed to the proof of Lemma 2.21.

Proof of Lemma 2.21. Notice first that by Lemma 2.4 and definition of a Tsirelson average we have estimation

$$\theta^{2M_k-3}/4\leqslant\|x_k\|\leqslant 5\theta_1^{-2}\theta^{2M_k-3},\quad k\in\mathbb{N},$$

and the same estimation for $\|y_k\|$, $k\in\mathbb{N}$.

We show first that $(y_k/\|y_k\|)_k$ dominates $(x_k/\|x_k\|)_k$. Let $x=\sum_k d_kx_k/\|x_k\|$ be of norm 1, with $(d_k)\subset[0,1]$, and take its norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$. Let $y=\sum_k d_ky_k/\|y_k\|$. By Fact 2.17 we can assume with error ε that $\operatorname{ord}(\alpha)<M_{k+1}/4\leqslant M_{k+1}-4$ for any $\alpha\in\mathcal{T}$ with $f_\alpha(x_k)\neq 0$. For any $k>1$ let

$$\mathcal{E}_k=\{\alpha\in\mathcal{T}\colon\ f_\alpha\text{ begins at }x_k\text{ and has a sibling beginning before }x_k\}.$$

By our reduction $\operatorname{ord}(\alpha)<M_k-4$ for any $\alpha\in\mathcal{E}_k$, $k\geqslant 2$. We replace in the tree-analysis of f each functional f_α, $\alpha\in\mathcal{E}_k$, by two functionals $g_\alpha=f_\alpha|_{\operatorname{supp}x_k}$ and $k_\alpha=f_\alpha-g_\alpha$, obtaining a tree-analysis of a functional g on the space $X_2=T[(\mathcal{S}_n[\mathcal{A}_2],\theta_n)_n]$, which by Lemma 1.7 is 3-isomorphic to X.

Notice that $(g_\alpha)_{\alpha\in\mathcal{E}_k,k\geqslant 2}$ have pairwise disjoint supports and $(\bigcup_{\alpha\in\mathcal{E}_k}\operatorname{supp}g_\alpha)\cap\operatorname{supp}x_k=\operatorname{supp}f\cap\operatorname{supp}x_k$, hence $f|_{\operatorname{supp}x_k}=\sum_{\alpha\in\mathcal{E}_k}t(\alpha)g_\alpha$. For each $k\geqslant 2$ consider the set $J_k=\{i\colon$ some g_α begins at $x^{M_k}_{k,i}\}$. Notice that by our reduction $(g_\alpha)_{\alpha\in\mathcal{E}_k}$ is $\mathcal{S}_{M_k-4}$-allowable, thus $(x^{M_k}_{k,i})_{i\in J_k\setminus\min J_k}$ is $\mathcal{S}_{M_k-4}$-admissible and recall that x_k is an (M_k-3,ε_k)-average of $(x^{M_k}_{k,i})$. Let g'_α, $\alpha\in\mathcal{E}_k$, be the restriction of g_α to $\bigcup_{i\notin J_k}\operatorname{supp}x^{M_k}_{k,i}$. Then we have the following estimation

$$\begin{aligned}
f(x)&=\frac{d_1}{\|x_1\|}f(x_1)+\sum_{k\geqslant 2}\frac{d_k}{\|x_k\|}f(x_k)\\
&\leqslant\frac{d_1}{\|x_1\|}f(x_1)+\sum_{k\geqslant 2}\frac{d_k}{\|x_k\|}\sum_{\alpha\in\mathcal{E}_k}t(\alpha)g'_\alpha(x_k)+\sum_k\frac{d_k}{\|x_k\|}\sum_{i\in J_k}a^{M_k}_{k,i}\|x^{M_k}_{k,i}\|\\
&\leqslant\frac{d_1}{\|x_1\|}f(x_1)+\sum_{k\geqslant 2}\frac{d_k}{\|x_k\|}\sum_{\alpha\in\mathcal{E}_k}t(\alpha)g'_\alpha(x_k)+8\sum_k\varepsilon_k\theta^{-2M_k+3}
\end{aligned}$$

$$\leqslant \frac{d_1}{\|x_1\|} f(x_1) + \sum_{k\geqslant 2} \frac{d_k}{\|x_k\|} \sum_{\alpha\in\mathcal{E}_k} t(\alpha) g'_\alpha(x_k) + \varepsilon.$$

Fix $k \geqslant 2$. Notice that by definition the set $\{g'_\alpha\colon\ g'_\alpha(x^{M_k}_{k,i}) \neq 0\}$ restricted to the support of $x^M_{k,i}$ is $\mathcal{S}_1$-allowable for any i. Therefore by Lemma 2.10 we pick suitable partition $(V_\alpha)_{\alpha\in\mathcal{E}_k}$ of nodes $(x^0_{k,i})_i$ with $\operatorname{minsupp} x^0_{k,\min V_\alpha} \geqslant \operatorname{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$ and by definition of a Tsirelson average obtain

$$\begin{aligned}
\sum_{\alpha\in\mathcal{E}_k} t(\alpha) g'_\alpha(x_k) &\leqslant C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \Big\| \sum_{i\in V_\alpha} a^0_{k,i} e_{\operatorname{minsupp} x^0_{k,i}} \Big\|_{T[\mathcal{S}_1,\theta]} + C\varepsilon_k \\
&\leqslant C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \Big\| \sum_{i\in V_\alpha} a^0_{k,i} e_{\operatorname{minsupp} y^0_{k,i}} \Big\|_{T[\mathcal{S}_1,\theta]} + C\varepsilon_k \\
&\leqslant 4C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \Big\| \sum_{i\in V_\alpha} a^0_{k,i} y^0_{k,i} \Big\| + C\varepsilon_k \\
&\leqslant 4C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) h_\alpha(y_k) + C\varepsilon_k,
\end{aligned}$$

where h_α is a norming functional on X with $h_\alpha(y_k) = \|\sum_{i\in V_\alpha} a^0_{k,i} y^0_{k,i}\|$ and $\operatorname{supp} h_\alpha \subset \bigcup_{i\in V_\alpha} \operatorname{supp} y^0_{k,i}$, thus $\operatorname{minsupp} h_\alpha \geqslant \operatorname{minsupp} x^0_{k,\min V_\alpha} \geqslant \operatorname{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$.

We modify the tree-analysis of g, replacing each node g_α, $\alpha \in \mathcal{E}_k$, $k \geqslant 2$, by the functional h_α. As $\operatorname{minsupp} h_\alpha \geqslant \operatorname{minsupp} g_\alpha$ for each α, we obtain a tree-analysis of some norming functional h on X_2. We compute, by Lemma 1.7 and the above estimations including the estimation on the norms of $(x_k)_k$ and $(y_k)_k$,

$$\begin{aligned}
1 = f(x) &\leqslant d_1 + \sum_{k\geqslant 2} \frac{d_k}{\|x_k\|} \sum_{\alpha\in\mathcal{E}_k} t(\alpha) g_\alpha(x_k) + \varepsilon \\
&\leqslant d_1 + 80C\theta^{-2} \sum_{k\geqslant 2} \frac{d_k}{\|y_k\|} \sum_{\alpha\in\mathcal{E}_k} t(\alpha) h_\alpha(y_k) + 4C \sum_{k\geqslant 2} \frac{\varepsilon_k}{\theta_{2M_k-3}} + \varepsilon \\
&\leqslant d_1 + 80C\theta^{-2} h\Big(\sum_{k\geqslant 2} \frac{d_k}{\|y_k\|} y_k \Big) + 3C\varepsilon \\
&\leqslant 241C\theta^{-2}\|y\| + 1/2,
\end{aligned}$$

which means that $(y_k/\|y_k\|)_k$ dominates $(x_k/\|x_k\|)_k$. Since the conditions are symmetric, the opposite domination follows analogously. □

3. Strictly singular non-compact operators

3.1. Spaces defined by families $(\mathcal{A}_n)_n$

In spaces defined by families $(\mathcal{A}_n)_n$ the crucial tool is formed by ℓ_p-averages.

Definition 3.1. A vector $x \in X$ is called a C–ℓ_p*-average* of length m, for $p \in [1,\infty]$, $m \in \mathbb{N}$ and $C \geqslant 1$ if $x = \sum_{i=1}^m x_i / \|\sum_{i=1}^m x_i\|$ for some normalized block sequence $(x_n)_{n=1}^m$ which is C-equivalent to the unit vector basis of ℓ_p^m.

Recall that an operator on a Banach space X is called *strictly singular* if its restriction to any infinite dimensional subspace of X is not an isomorphism.

Definition 3.2. (See [35].) Let X be a Banach space with a basis (e_n). Then X is in

1. Class 1, if any normalized block sequence in X has a subsequence equivalent to a subsequence of (e_n).
2. Class 2, if each block sequence has further normalized block sequences (x_n) and (y_n) such that the map $x_n \mapsto y_n$ extends to a bounded strictly singular operator between $[(x_n)]$ and $[(y_n)]$.

T. Schlumprecht in [35] asked if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2 and gave the following sufficient condition for the existence of strictly singular non-compact operator in the space.

Theorem 3.3. *(See [35, Thm. 1.1].) Let (x_n) and (y_n) be two normalized basic sequences generating spreading models (u_n) and (v_n) respectively. Assume that (u_n) is not equivalent to the u.v.b. of c_0 and (u_n) strongly dominates (v_n), i.e.*

$$\left\| \sum_{i=1}^{\infty} a_i v_i \right\| \leqslant \max_{n\in\mathbb{N}} \delta_n \max_{\#F\leqslant n} \left\| \sum_{i\in F} a_i u_i \right\|$$

for some sequence (δ_n) with $\delta_n \searrow 0$, $n \to \infty$. Then the map $x_n \mapsto y_n$ extends to a bounded strictly singular operator between $[(x_n)]$ and $[(y_n)]$.

Theorem 3.4. *Let $X = T[(\mathcal{A}_n, \frac{c_n}{n^{1/q}})_n]$ be a regular p-space, with $p \in [1, \infty)$. Then*

1. *if $\inf_n c_n > 0$, then X is saturated with subspaces of Class 1,*
2. *if $c_n \to 0$, $n \to \infty$, then X is in Class 2.*

Proof. (1). We show that any block subspace of X contains a normalized block sequence $(u_s)_s$ with the following "blocking principle" ("shift property" in [33]): any normalized block sequence $(y_j)_j$ is equivalent to any $(u_{k_j})_j$, with $y_j < u_{k_{j+1}}$ and $u_{k_j} < y_{j+1}$. It follows that the subspace $[(u_s)]$ is sequentially minimal.

By [29, Prop. 2.10] any block subspace of X contains an ℓ_p-asymptotic subspace of X. Let W be such ℓ_p-asymptotic subspace, spanned by a normalized block sequence $(w_k)_k$. Let C be the asymptotic constant of W, i.e. any normalized block sequence $(z_i)_{i=1}^n$ with $z_1 > n$ in W is C-equivalent to the u.v.b. of ℓ_p^n.

For any subspace Y of X spanned by normalized block sequence (y_n) let $\|\sum_n a_n y_n\|_{Y,\infty} = \sup_{n\in\mathbb{N}} |a_n|$.

Fix two strictly increasing sequences of integers $(m_n)_n \subset \mathbb{N}$ and $(N_j)_j \subset \mathbb{N}$ and take normalized block sequences $(v_n)_n$ of $(w_k)_k$ and $(u_j)_j$ of $(v_n)_n$ such that

1. $v_n > m_n$ in W for any n,
2. for any $y \in [(v_i)_{i>n}]$ we have $\|y\|_{W,\infty} < 1/(8m_n^5)$, for any n,
3. $u_j > N_j$ in $V = [(v_n)_n]$ for any j,
4. for any $y \in [(u_i)_{i>j}]$ we have $\|y\|_{V,\infty} < 1/(8N_j^5)$, for any j,
5. $\sqrt[p]{N_j} \geqslant C2^{j+7}$ for any j,
6. $N_j\theta_{m_n} < 1/2^{n+5}$ for any $n \geqslant j$ (in particular $m_n \geqslant N_j$ for any $n \geqslant j$),
7. $\theta_{m_n} \sum_{i<n} \#\operatorname{supp} v_i < 1/2^{n+5}$ for any n.

Notice that every vector $y \in [(v_i)_{i>n}]$ is a $2C$–ℓ_p-average of length m_n of some normalized block sequence $(y_i)_{i=1}^{m_n}$ of $(w_k)_k$. Indeed, by [29, Claim 3.8] and condition (2) split y into $(Fy_i)_{i=1}^{m_n}$ with almost equal norm, obtaining by condition (1) and ℓ_p-asymptoticity of W that y is a suitable average. The same holds in V: every vector $y \in [(u_i)_{i>j}]$ is a $2C$–ℓ_p-average of length N_j of some normalized block sequence $(y_i)_{i=1}^{N_j}$ (block with respect to $(v_n)_n$).

We show that in such setting we can prove the above theorem repeating the proof of [29, Theorem 3.1]. We consider any normalized block sequence (y_j) of (u_j) and as (z_j) we take (u_{k_j}) with $y_j < u_{k_{j+1}}$ and $u_{k_j} < y_{j+1}$. By the above observation $y_j = (y_1^j + \cdots + y_{N_j}^j)/\|y_1^j + \cdots + y_{N_j}^j\|$ and $u_{k_j} = (u_1^j + \cdots + u_{N_j}^j)/\|u_1^j + \cdots + u_{N_j}^j\|$, where $(y_j^i)_{i=1}^{N_j}$ and $(u_j^i)_{i=1}^{N_j}$ are normalized block sequences with respect to $(v_j)_j$. Notice that (N_j) are big enough by condition (5). We again use the above observation obtaining that each y_j^i and v_j^i is an ℓ_p-average of a block sequence of $(w_k)_k$, of suitable length with parameters satisfying the assertion of a version of [29, Lemma 3.2] for C-averages instead of 2-averages (by conditions (6) and (7)). Therefore repeating the proof of [29, Theorem 3.1] we obtain uniform equivalence of (y_j) and (u_{k_j}) and hence "blocking principle" stated above.

(2). Fix a block subspace Y of X. By [29, Theorem 2.9] p is in Krivine set of Y. Take finite normalized block sequences $(y_i)_i$ such that for some $(m_i)_i \subset \mathbb{N}$

1. each y_i is $2 - \ell_p$-averages of length $N_i \geqslant (2m_i)^p$,
2. $\theta_{m_i} \sum_{j<n} \#\operatorname{supp} y_j \leqslant 1/2^{i+5}$ for any i,
3. $2^{i+5}\theta_{m_i} \to 0$, $i \to \infty$.

Passing to a subsequence we can assume that (y_i) generates a spreading model (v_i).

Lemma 3.5. *The spreading model (v_i) is strongly dominated by the u.v.b. of ℓ_p.*

Proof. Take $k \in \mathbb{N}$ and $(a_i)_{i=1}^N \in c_{00}$ with $\|(a_i)\|_\infty \leqslant 1/k^2$ and $\|(a_i)\|_{\ell_p} = 1$. Choose M by (3) in definition of (y_i) with $N\theta_{m_i} \leqslant 1/2^{i+5}$ for any $i > M$ and $1/2^M \leqslant 1/k$. We have $\|\sum_{i=1}^N a_i v_i\| \leqslant 2\|\sum_{i=1+M}^{N+M} \tilde{a}_i y_i\|$, where $\tilde{a}_{i+M} = a_i$, $i = 1, \dots, N$.

Take a norming functional f with a tree-analysis $(f_t)_{t\in\mathcal{T}}$ and $\operatorname{supp} f \subset \operatorname{supp} y$, where $y=\sum_{i=1+M}^{N+M}\tilde{a}_i y_i$. By [29, Lemma 2.5] up to multiplying by 36 we can assume that for any f_t and y_i we have either $\operatorname{supp} f_t \subset y_i$, $\operatorname{supp} f_t \supset \operatorname{supp} y_i \cap \operatorname{supp} f$ or $\operatorname{supp} f_t \cap \operatorname{supp} y_i = \emptyset$. We say that f_t covers y_i, if t is maximal in $\mathcal{T}$ with $\operatorname{supp} f_t \supset \operatorname{supp} y_i \cap \operatorname{supp} f$.

Let $A=\{t\in T\colon\ f_t \text{ covers some } y_i\}$. Given any $t\in A$ let $I_t=\{i=1+M,\dots,N+M\colon\ f_t \text{ covers } y_i\}$. Let θ_{m_t} be the weight of f_t. If $m_t>m_i$ for some $i\in I_t$ let i_t be the maximal element of I_t with this property. Otherwise let $i_t=0$.

For any $i\in I_t$ let $J_i=\{s\in \operatorname{succ}(t)\colon\ \operatorname{supp} f_s\subset \operatorname{supp} y_i\}$. By [29, Lemma 2.8] we have $\sum_{s\in J_i} f_s(y_i)\leqslant 8(\#J_i)^{1/q}$ for each $i\in I_t$, $i>i_t$.

First let $L_t=\{i\notin I_t\colon\ \operatorname{supp} y_i\cap \operatorname{supp} f\subset \operatorname{supp} f_t\}$. Notice that for any $i\in L_t$ there is some f_{t_i} – successor of f_t so that $\operatorname{supp} y_i\cap\operatorname{supp} f\subset \operatorname{supp} f_{t_i}$. Hence

$$f_t\biggl(\sum_{i\in L_t}\tilde{a}_i y_i\biggr)\leqslant \theta_{m_{i_t}}\biggl(\sum_{i\in L_t} f_{t_i}(\tilde{a}_i y_i)\biggr)\leqslant N\theta_{m_{i_t}}\leqslant 1/2^{i_t+2}.$$

Thus $f(\sum_{t\in A, i\in L_t} y_i)\leqslant 1/2^M$ and we erase this part for all t with error $\leqslant 1/k$. Notice that by condition (2) in choice of (y_i) we have

$$f_t\biggl(\sum_{i\in I_t,\, i<i_t} y_i\biggr)\leqslant \theta_{m_{i_t}}\sum_{i<i_t}\#\operatorname{supp} y_i\leqslant 1/2^{i_t+2},$$

so we can again erase this part for all t with error $1/k$.

Let g be the restriction of f to $\bigcup_{t\in A}\operatorname{supp} y_{i_t}$ and $h=f-g$. First we consider $g(y)=\sum_{t\in A} t(f_t)\tilde{a}_{i_t} f_t(y_{i_t})$. Let $B=\{t\in A\colon \operatorname{ord}(f_t)\leqslant k\}$, hence $\#B\leqslant k$. Then $\sum_{t\in B}\tilde{a}_{i_t} f_t(y_{i_t})\leqslant \#B/k^2\leqslant 1/k$, hence we can erase this part with error $1/k$. Notice that $\sum_{t\in A\setminus B}\frac{1}{\operatorname{ord}(f_t)^{1/q}}e^*_{i_t}$ is a norming functional on ℓ_p, hence

$$\sum_{t\in A\setminus B}\tilde{a}_{i_t} t(f_t) f_t(y_{i_t})\leqslant \sum_{t\in A\setminus B}\tilde{a}_{i_t}\frac{c_{\operatorname{ord}(f_t)}}{(\operatorname{ord}(f_t))^{1/q}}\leqslant \max_{n\geqslant k} c_n\bigl\|(\tilde{a}_{i_t})_{t\in A\setminus B}\bigr\|_{\ell_p}\leqslant \max_{n\geqslant k} c_n.$$

We consider $h(y)=\sum_{t\in A}\sum_{i\in I_t,\, i>i_t}\tilde{a}_i\sum_{s\in J_i} t(f_s) f_s(y_i)$. Let $D=\{s\in J_i,\ i\in I_t,\ i>i_t,\ t\in A\colon\ \operatorname{ord}(f_s)\leqslant k\}$. Then

$$\sum_{t\in A}\sum_{i\in I_t,\, i>i_t}\sum_{s\in J_i\cap D}\tilde{a}_i f_s(y_i)\leqslant \#D/k^2\leqslant 1/k,$$

and we again erase this part with error $1/k$. For any $i\in I_t$, $i>i_t$ for some $t\in A$ let $r_i=\operatorname{ord}(f_t)m_t$ and compute, using Hölder inequality,

$$\begin{aligned}\sum_{t\in A}\sum_{i\in I_t,\, i>i_t}\sum_{s\in J_i\setminus D}\tilde{a}_i t(f_s) f_s(y_i) &\leqslant \sum_{t\in A}\sum_{i\in I_t,\, i>i_t}\tilde{a}_i 8(\#J_i)^{1/q}\theta_{r_i}\\ &\leqslant 8\max_{n\geqslant k} c_n\sum_{t\in A}\sum_{i\in I_t,\, i>i_t}\tilde{a}_i\frac{(\#J_i)^{1/q}}{r_i^{1/q}}\\ &\leqslant 8\max_{n\geqslant k} c_n\bigl\|(\tilde{a}_i)_{i\in I_t,\, i>i_t,\, t\in A}\bigr\|_{\ell_p}\leqslant 8\max_{n\geqslant k} c_n.\end{aligned}$$

We put all the estimates together obtaining

$$f(y)\leqslant 36\Bigl(9\max_{n\geqslant k} c_n+4/k\Bigr).$$

Therefore we proved that $\Delta_\varepsilon=\sup\{\|\sum_{i\in\mathbb{N}} a_i v_i\|\colon\ \sup_{i\in\mathbb{N}}|a_i|\leqslant\varepsilon,\ \ \|(a_i)_{i\in\mathbb{N}}\|_{\ell_p}=1\}$ converges to zero, as $\varepsilon\to 0$. By [35, Lemma 2.4] there are some $(\delta_n)_n\subset(0,\infty)$ with $\delta_n\searrow 0$ such that for any $(a_i)_i\in c_{00}$

$$\biggl\|\sum_i a_i v_i\biggr\|\leqslant \max_{n\in\mathbb{N}}\delta_n\max_{\#F\leqslant n}\bigl\|(a_i)_{i\in F}\bigr\|_{\ell_p},$$

which ends the proof of the lemma. □

We continue the proof of Theorem 3.4. By the proof of [29, Thm 2.9], p is in the Krivine set of Y in Lemberg sense [24], i.e. for any n there is a normalized block sequence $(x_i^{(n)})_i\subset Y$ generating spreading model $(u_i^{(n)})_i$ such that $(u_i^{(n)})_{i=1}^n$ is 1-equivalent to the u.v.b. of ℓ_p^n.

Pick $(m_n)_n$ such that $\delta_{m_n}\leqslant 1/4^n$. Apply [4, Prop. 3.2] to constants $C_n=2^n$, $n\in\mathbb{N}$ and normalized block sequences $(x_i^{(m_n)})_i$ generating spreading models $(u_i^{(m_n)})_i$. We obtain thus a seminormalized block sequence $(x_i)_i$ generating spreading model $(u_i)_i$ which C_n dominates $(u_i^{(m_n)})_i$ for any $n\in\mathbb{N}$. By Lemma 3.5 we have

$$\begin{aligned}
\Big\|\sum_i a_i v_i\Big\| &\leqslant \max_{n\in\mathbb{N}} \delta_n \max_{\#F\leqslant n} \big\|(a_i)_{i\in F}\big\|_{\ell_p} \\
&\leqslant \max_{n\in\mathbb{N}} \delta_{m_n} \max_{\#F\leqslant m_{n+1}} \big\|(a_i)_{i\in F}\big\|_{\ell_p} \\
&\leqslant \max_{n\in\mathbb{N}} 1/4^n \max_{\#F\leqslant m_{n+1}} \Big\|\sum_{i\in F} a_i u_i^{(m_{n+1})}\Big\| \\
&\leqslant \max_{n\in\mathbb{N}} C_{n+1}/4^n \max_{\#F\leqslant m_{n+1}} \Big\|\sum_{i\in F} a_i u_i\Big\| \\
&\leqslant \max_{n\in\mathbb{N}} 2/2^n \max_{\#F\leqslant m_{n+1}} \Big\|\sum_{i\in F} a_i u_i\Big\|.
\end{aligned}$$

Notice that $(u_i)_i$ is not equivalent to the u.v.b. of c_0, thus by Theorem 3.3 we finish the proof. □

In [19] the construction of non-compact strictly singular operators was based on c_0-spreading model of higher order in the dual space. However this method does not follow straightforwardly in case of p-spaces, as the observation below shows. In [23] it was shown that Schlumprecht space $S = T[(\mathcal{A}_n, \frac{1}{\log_2(n+1)})_n]$ introduced in [34] contains a block sequence (y_k) generating an ℓ_1-spreading model. We show that no biorthogonal sequence to (y_k) generates a c_0-spreading model. An analogous example is constructed in [8]: a mixed Tsirelson space defined by $(\mathcal{S}_n)_n$ which admits a block sequence (z_k) generating an ℓ_1^ω-spreading model (cf. Definition 3.7), such that no biorthogonal sequence to (z_k) generates c_0^ω-spreading model.

Proposition 3.6. *Consider the sequence (y_k) generating an ℓ_1-spreading model constructed in* [23]*, $y_k = \sum_{m=1}^k v_{k,m}$, $k \in \mathbb{N}$. Take any block sequence $(y_k^*) \subset S^*$ so that $y_k^*(y_l) = \delta_{l,k}$. Then the sequence (y_k^*) does not generate a c_0-spreading model.*

Proof. We can assume that $\operatorname{supp} y_k^* = \operatorname{supp} y_k$, $k \in \mathbb{N}$. Consider two cases:

CASE 1. There is $m_0 \in \mathbb{N}$, $\delta > 0$ and an infinite $K \subset \mathbb{N}$ with $|y_k^*(\sum_{m=1}^{m_0} v_{k,m})| \geqslant \delta$ for any $k \in K$.

Let z_k^* be the restriction of y_k^* to the support of $\sum_{m=1}^{m_0} v_{k,m}$, $k \in K$. Then $(z_k^*)_{k\in K}$ is a seminormalized block sequence in S^*, majorized by $(y_k^*)_{k\in K}$. Since by the form of $(v_{m,k})$ the length of $\operatorname{supp}(\sum_{m=1}^{m_0} v_{k,m})$ is constant, we can pick some subsequence $(z_k^*)_{k\in L}$ of $(z_k^*)_{k\in K}$ consisting of, up to controllable error, equally distributed vectors, i.e. for some finite sequence $(a_i)_{i\in I} \subset \mathbb{R}$ and $(n_{k,i})_{i\in I,\,k\in L}$ with $n_{k,i} < n_{k,i+1}$ and $n_{k,\max I} < n_{k+1,1}$ we have $\|z_k - \sum_{i\in I} a_i e_{n_{k,i}}\| \leqslant 1/2^k$, $k \in L$. As the u.v.b. in S is subsymmetric, the same holds for $(z_k^*)_{k\in L}$, thus $(z_k^*)_{k\in L}$ is equivalent to spreading model generated by itself. It follows that (y_k^*) cannot generate c_0-spreading model.

CASE 2. If the first case does not hold, pick increasing $(N_j)_j \subset \mathbb{N}$ so that

$$\left| y_{N_j}^* \left(\sum_{m=1}^{N_{j-1}} v_{N_j,m} \right) \right| \leqslant 1/2^j, \quad j \in \mathbb{N}.$$

Consider the norm of vectors $z_j^* = y_{N_1}^* + \cdots + y_{N_j}^*$, $j \in \mathbb{N}$. Put

$$x_{N_1} = y_{N_1}, \qquad x_{N_j} = \sum_{m=N_{j-1}+1}^{N_j} v_{N_j,m}, \quad j > 1.$$

By the choice of $(N_j)_j$ we have $y_{N_j}^*(x_{N_j}) \geqslant 1 - 1/2^j$ for any $j \in \mathbb{N}$.

We estimate the norm of $x_j = x_{N_1} + \cdots + x_{N_j}$, $j \in \mathbb{N}$. We can assume that $(N_j)_j$ was chosen to increase fast enough so that (x_{N_j}) is D-equivalent to the unit basis of S (see Remark 5, Lemma 2 [23]). Thus $\|x_j\| \leqslant Dj/f(j)$ for every $j \in \mathbb{N}$. By the choice of $(N_j)_j$ and definition of x_{N_j} we have $z_j^*(x_j) \geqslant j - 1$. Hence

$$\|z_j^*\| \geqslant z_j^*(x_j)/\|x_j\| \geqslant f(j)(j-1)/Dj \geqslant f(j)/2D, \quad j \in \mathbb{N}.$$

Notice that the same scheme works if we replace $N_1, \dots, N_j$ by any $N_{n_1}, \dots, N_{n_j}$ in definition of z_j, hence no subsequence of (y_k^*) can produce a c_0-spreading model. □

3.2. Spaces defined by families $(\mathcal{S}_n)_n$

Regarding the existence of strictly singular operators from subspaces of mixed Tsirelson spaces we prove the following result, which is a “localization” of Schlumprecht result in mixed Tsirelson spaces. First recall the definition of higher order ℓ_1-spreading models.

Definition 3.7. We say that a normalized basic sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space generates a $C-\ell_1^\alpha$-spreading model, $\alpha<\omega_1$, $C\geqslant 1$, if for any $F\in\mathcal{S}_\alpha$ the sequence $(x_n)_{n\in F}$ is C-equivalent to the u.v.b. of $\ell_1^{\#F}$.

Notice that in case of $\alpha=1$ we obtain the classical ℓ_1-spreading model. We recall that $[M]$, $M\subset\mathbb{N}$, denotes the family of all infinite subsequences of M, $[M]^<$ – the family of all finite subsequences of M.

Theorem 3.8. *Let $X=T[(\mathcal{S}_n,\theta_n)_n]$ or $T_M[(\mathcal{S}_n,\theta_n)_n]$ be a regular (modified) mixed Tsirelson space. If X contains a block sequence (y_n) generating an ℓ_1^ω-spreading model then there are a subspace $Y\subset[(y_n)]$ and a strictly singular non-compact operator $T:Y\to X$.*

We recall that in [25] it was proved that if a sequence (θ_n) satisfies $\lim_m\limsup_n\frac{\theta_{m+n}}{\theta_n}>0$ then the regular mixed Tsirelson space $X=T[(\mathcal{S}_n,\theta_n)_n]$ is subsequentially minimal if and only if any block subspace of X admits an ℓ_1^ω-spreading model, if and only if any block subspace of X has Bourgain ℓ_1-index greater than ω^ω. These conditions hold in particular if $\sup\theta_n^{1/n}=1$ [27]. In [8,22] analogs of these results were studied in the partly modified setting.

To prove the theorem we first define an index measuring the best constant of ℓ_1^α-spreading models generated by subsequences of a given sequence. Let $\vec{x}:=(x_n)_{n\in\mathbb{N}}$ be a normalized block sequence in X. Set

$$\delta_\alpha(\vec{x})=\sup\bigl\{\delta>0\colon\ \exists M\in[\mathbb{N}]\text{ such that }(x_n)_{n\in M}\text{ generates }\delta-\ell_1^\alpha\text{-spreading model}\bigr\}.$$

The following properties of $\delta_\alpha(\vec{x})$ follow readily from the definition.

a) $\delta_\alpha((x_n)_{n\in\mathbb{N}})=\delta_\alpha((x_n)_{n\geqslant n_0})$ for all $n_0\in\mathbb{N}$.
b) $\delta((x_n)_{n\in M})\leqslant\delta_\alpha((x_n)_{n\in\mathbb{N}})$ for all $M\in[\mathbb{N}]$.
c) $(\delta_\alpha(\vec{x}))_{\alpha<\omega_1}$ is a non-increasing family.

By standard arguments we may stabilize $\delta_\alpha(\vec{x})$. Namely passing to a subsequence we may assume that $\delta_\alpha((x_n)_{n\in\mathbb{N}})=\delta_\alpha((x_n)_{n\in M})$ for every $M\in[\mathbb{N}]$.

By the reflexivity of the space X [9], Bourgain theorem yields that $\delta_\alpha(\vec{x})>0$ for countably many α's, enumerate them as $(\alpha_n)_n$. As X is an ℓ_1-asymptotic space it follows that $\delta_n(\vec{x})>0$ for all $n\in\mathbb{N}$.

Inductively we choose $M_1\supset M_2\supset\cdots$ infinite subsets of $\mathbb{N}$ such that

$$\delta_{\alpha_n}\bigl((x_j)_{j\in M_n}\bigr)=\delta_{\alpha_n}(x_j)_{j\in L}\quad\forall L\in[M_n].$$

We define the family

$$\mathcal{F}_n=\bigl\{A\in[\mathbb{N}]^<\colon\ \exists x^*\in B_{X^*}\text{ with }x^*(x_i)>2\delta_{\alpha_n}\bigl((x_j)_{j\in M_n}\bigr)\text{ for all }i\in A\bigr\}.$$

By [18, Theorem 1.1] there exists $N\in[M_n]$ such that

$$\text{either}\quad\mathcal{S}_{\alpha_n}\cap[N]\subset\mathcal{F}_n\quad\text{or}\quad\mathcal{F}_n\cap[N]\subset\mathcal{S}_{\alpha_n}.$$

In the first case by 1-unconditionality of the basis it follows that $(x_n)_{n\in N}$ and hence $(x_k)_{k\in M_n}$ contains a subsequence which generates $2\delta_{\alpha_n}-\ell_1^{\alpha_n}$-spreading model, a contradiction. Hence additionally by the above and [30] we may assume that $(M_n)_{n\in\mathbb{N}}$ satisfies also the following

$$\mathcal{F}_n(M_n)\subset\mathcal{S}_{\alpha_n},\tag{3.1}$$

$$\mathcal{S}_{\alpha_{n-1}}\cap\{F\subset\mathbb{N}\colon\ \min F\geqslant\min M_n\}\subset\mathcal{S}_{\alpha_n}.\tag{3.2}$$

Let $M=(m_i)_i$, where $m_i=\min M_i$, be a diagonal set and let $\delta_{\alpha_n}=\delta_{\alpha_n}((x_{m_i})_i)$. Passing to a subsequence we may assume that $\sum_n n\delta_{\alpha_n}<0.25$. Let $\|\sum_i a_ix_{m_i}\|=1$ and let $x^*\in B_{X^*}$ such that $\sum_i a_ix^*(x_{m_i})=1$. By the unconditionality we may assume that $x^*(x_{m_i})\geqslant 0$ for every i. Let $2\delta_{\alpha_0}=1$ and

$$F_k=\bigl\{i\colon\ x^*(x_{m_i})\in(2\delta_{\alpha_k},2\delta_{\alpha_{k-1}}]\bigr\}$$

and $F_k^1=F_k\cap\{1,\dots,k-1\}$, $F_k^2=F_k\cap\{k,k+1,\dots\}$.

From (3.1), (3.2) we get $F_k^2\in\mathcal{S}_{\alpha_k}\cap\{F\subset\mathbb{N}\colon\ \min F\geqslant k\}=\mathcal{G}_k$. It follows

$$\begin{aligned}\Bigl\|\sum_i a_ix_{m_i}\Bigr\|&=\sum_i a_ix^*(x_{m_i})=\sum_{k=1}^\infty\sum_{i\in F_k}a_ix^*(x_{m_i})\\&=\sum_{k=1}^\infty\Bigl(\sum_{i\in F_k^1}a_ix^*(x_{m_i})+\sum_{i\in F_k^2}a_ix^*(x_{m_i})\Bigr)\end{aligned}$$

$$\leqslant \sum_{k=2}^{\infty} 2\delta_{\alpha_{k-1}}(k-1)\max_i |a_i| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sum_{i\in F_k^2} |a_i|$$

$$\leqslant 0.5\Big\| \sum_i a_i x_{m_i} \Big\| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i|.$$

Therefore by the above inequalities we have

$$\Big\| \sum_i a_i x_{m_i} \Big\| \leqslant 4\sum_{k=1}^{\infty} \delta_{\alpha_{k-1}}\big((x_{m_i})_i\big) \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i| \quad \text{for all } (a_i)_i, \tag{3.3}$$

where $\mathcal{G}_k = \mathcal{S}_{\alpha_k} \cap \{F \subset \mathbb{N}\colon\ \min F \geqslant k\}$.

Proof of Theorem 3.8. Let $\vec{e} = (e_n)_{n\in\mathbb{N}}$ be the basis of X. Recall [7,9] that for every $j \in \mathbb{N}$ and every (n, θ_n)-average $\sum_{i\in F} a_i e_i$ (special convex combination) of the basis we have

$$\theta_n \leqslant \Big\| \sum_{i\in F} a_i e_i \Big\| \leqslant 2\theta_n.$$

It follows readily that $\delta_n(\vec{e}) \in [\theta_n, 2\theta_n]$ and $\delta_\omega(\vec{e}) = 0$.

Let $(y_n)_{n\in\mathbb{N}}$ be a normalized block sequence $(y_n)_{n\in\mathbb{N}}$ generating ℓ_1^ω-spreading model, i.e. for some $c \geqslant 1$

$$\Big\| \sum_i a_i y_i \Big\| \geqslant c\sum_{i\in F} |a_i| \quad \forall n \in \mathbb{N},\ F \in \mathcal{S}_n,\ \min F \geqslant n.$$

By the previous reasoning for $(x_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ we pick an $M = (m_i) \in [\mathbb{N}]$ and a sequence $\alpha_k \nearrow \omega$ such that $\sum_k k\delta_{\alpha_k}((e_{m_i})_i) < \infty$ and (3.3) holds. Setting $D = \sum_k \theta_{\alpha_{k-1}} \leqslant \sum_k \delta_{\alpha_{k-1}}((e_{m_i})_i)$ we have

$$\Big\| \sum_i a_i e_{m_i} \Big\| \leqslant 8\sum_k \theta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i|$$

$$\leqslant \frac{8D}{c} \sup_k c \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i|$$

$$\leqslant \frac{8D}{c} \Big\| \sum_i a_i y_i \Big\|.$$

It follows that the operator extending the mapping $y_n \to e_{m_n}$ factors through a c_0-saturated space and hence is strictly singular and non-compact (as $(y_n)_{n\in\mathbb{N}}$ is normalized). □

3.3. Remarks and questions

As a corollary to Theorem 3.4, part (1), we obtain that the (non-modified) Tzafriri space Y has an ℓ_2-asymptotic subspace Z which satisfies a blocking principle in the sense of [14], called a "shift property" in [33]. The only known spaces with a blocking principle so far were similar to T, T^* and their variations. The two major ingredients used in [14] for proving the minimality of T^* are the blocking principle and the saturation with ℓ_∞^n's. It is shown in [21] that Tzafriri space Y contains uniformly ℓ_∞^n's. It is not known whether Y is uniformly saturated with ℓ_∞^n's. In the opposite direction, we do not know if Z contains a convexified Tsirelson space $T^{(2)}$ (which is equivalent to its modified version).

Aside from the main topic of our paper, we want to finish with some observations about subsymmetric sequences (i.e. basic sequences equivalent to all its subsequences) in two concrete spaces and pose corresponding related questions about the richness of the set of subsymmetric sequences in a Banach space. In 1977 Altshuler [2] (cf. e.g. [26]) constructed a Banach space with a symmetric basis which contains no ℓ_p or c_0, and all its symmetric basic sequences are equivalent. In 1981 C. Read [32] constructed a space with, up to equivalence, precisely two symmetric bases. More precisely, Read proved that any symmetric basic sequence in his space CR is equivalent either to the u.v.b. of ℓ_1 or to one of the two symmetric bases of CR. A careful look at the papers of Altshuler and Read shows that their proofs work similarly for the more general case of all subsymmetric basic sequences. This observation leads to the following questions:

Question 1. Does there exist a space in which all subsymmetric basic sequences are equivalent to one basis, and that basis is not symmetric?

We remark that Altshuler's space has a natural subsymmetric version but we do not know if it satisfies the above property.

Question 2. Does there exist a space with exactly two subsymmetric bases, which are not symmetric?

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