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A Galerkin boundary node method and its convergence analysis

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ABSTRACT

The boundary node method (BNM) exploits the dimensionality of the boundary integral equation (BIE) and the meshless attribute of the moving least-square (MLS) approximations. However, since MLS shape functions lack the property of a delta function, it is difficult to exactly satisfy boundary conditions in BNM. Besides, the system matrices of BNM are non-symmetric.

A Galerkin boundary node method (GBNM) is proposed in this paper for solving boundary value problems. In this approach, an equivalent variational form of a BIE is used for representing the governing equation, and the trial and test functions of the variational formulation are generated by the MLS approximation. As a result, boundary conditions can be implemented accurately and the system matrices are symmetric. Total details of numerical implementation and error analysis are given for a general BIE. Taking the Dirichlet problem of Laplace equation as an example, we set up a framework for error estimates of GBNM. Some numerical examples are also given to demonstrate the efficacity of the method.

1. Introduction

In recent years, the meshless (or meshfree) methods have attached much attention for solving boundary value problems [1,2]. The main feature of this type of method is the absence of an explicit mesh, and the approximate solutions are constructed entirely based on a cluster of scattered nodes. Although many types of meshless methods have been already proposed, these methods can be divided into two categories: the boundary type and the domain type. Several domain type meshless methods, such as the element free Galerkin method (EFGM) [3], the reproducing kernel particle method [4], the moving least-square reproducing kernel method [5,6], the finite point method [7] and the h–p meshless method [8] have achieved remarkable progress in solving a wide range of boundary value problems, and their mathematical backgrounds were investigated.

Boundary integral equations (BIEs) have been widely used for the solution of boundary value problems in potential theory and engineering. Based on coupling BIEs and the moving least-squares (MLS) approach [9,10], Mukherjee and Mukherjee [11] proposed a boundary type meshless method which they call the boundary node method (BNM). BNM requires only a nodal structure on the bounding surface of a body for approximation of boundary unknowns. Hence it is an attractive computational technique for linear problems compared with the domain type meshless methods. However, since the MLS approximation lacks the delta function property, BNM cannot exactly satisfy boundary conditions. And the strategy used in BNM to impose boundary conditions doubles the number of system equations. Xie et al. [12] proposed a radial boundary node method (RBNM) to overcome this difficulty by using radial basis functions instead of the MLS to construct the interpolation functions. Although RBNM has been applied to the linear elasticity problems, the accuracy of numerical results is affected by the shape parameters of radial basis functions (e.g. parameters in MQ and Gaussians basis functions [13]), and the optimal

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values of these parameters are still not determined theoretically. Moreover, as BNM, the system matrices of RBNM are non-symmetric, and the theoretical basis is just being studied and far from completion.

In this paper we present a Galerkin boundary node method (GBNM), which based on an equivalent variational form of a boundary integral formulation for the governing partial differential equation. The key ideas in GBNM are:

1. The MLS approximation is implemented to construct the trial and test functions of the variational form by a cluster of nodes instead of elements. Thus, the elements division in the boundary element method (BEM) can be avoided.
2. The 'stiffness' matrices are symmetric, which provides an added advantage in coupling GBNM with finite element method (FEM) [14] or other established meshless methods such as EFGM. This coupled technique is especially suited for the problems with an unbounded domain.
3. Although the shape functions of MLS approximation lack the delta function property, boundary conditions can be enforced by the variational formulation. Thus the implementation of boundary conditions in this method is much easier than that in other meshless methods such as in BNM or EFGM, in which the MLS is also introduced.

The rest of this paper is outlined as follows. In Section 2, we introduce some preliminaries to be used later. Section 3 gives a brief description of the MLS approximation and deduces its error estimates. Then, a detailed numerical implementation of GBNM is described and the theoretical analysis of this method in Sobolev spaces is provided in the next section. Section 5 provides some numerical tests on theoretical results of the proposed meshless method. Finally, the conclusion is presented in Section 6.

2. Preliminary

Let Ω be an open bounded domain in $\mathbb{R}^2$ with boundary Γ, the complement of $\bar{\Omega} = \Omega + \Gamma$ is denoted by Ω'. A generic point in $\mathbb{R}^2$ is denoted by $\mathbf{x} = (x_1, x_2)$ or $\mathbf{y} = (y_1, y_2)$.

For any $\mathbf{x} \in \Gamma$, assume that the influence domain of $\mathbf{x}$ is $\Re(\mathbf{x})$ with radius $r(\mathbf{x})$, then $\Re(\mathbf{x})$ is a piece of the boundary and can be represented by a curvilinear co-ordinate (here the arc length) s, i.e.,

$$\Re(\mathbf{x}(s)) := \left\{\mathbf{y}\left(\tilde{s}\right) \in \Gamma : \left|\tilde{s} - s\right| \le r(\mathbf{x})\right\}, \tag{1}$$

where $\tilde{s}$ is the curvilinear coordinate of the boundary point $\mathbf{y}$.

Obviously, if Γ is a C^{ℓ_Γ} curve, it is true that $\Re(\mathbf{x})$ is a C^{ℓ_Γ} curve, thus $\partial^m \mathbf{x}(s)/\partial s^m$ is bounded provided that $m \le \ell_\Gamma$.

Let $\mathbf{x}_i \in \Gamma$ $(1 \le i \le N)$ be a set of points which are called boundary nodes. On $\Re(\mathbf{x})$, the curvilinear co-ordinate of $\mathbf{x}_i \in \Re(\mathbf{x})$ is denoted by s_i. Besides, assume that there have $\kappa(\mathbf{x})$ boundary nodes that lie on $\Re(\mathbf{x})$. Then, we use the notation $I_1, I_2, \ldots, I_\kappa$ to express the global sequence number of these nodes, and define $\wedge(\mathbf{x}) := \{I_1, I_2, \ldots, I_\kappa\}$.

From (1) the influence domain of $\mathbf{x}_i$ is

$$\Re_i := \Re(\mathbf{x}_i(s)) = \left\{\mathbf{y}\left(\tilde{s}\right) \in \Gamma : \left|\tilde{s} - s\right| \le r(\mathbf{x}_i)\right\}, \quad 1 \le i \le N. \tag{2}$$

It is worth noting that the union of $\{\Re_i\}_{i=1}^N$ should be a finite open covering of Γ, i.e., $\Gamma \subset \bigcup_{i=1}^N \Re_i$.

Besides, we use

$$\Re^i := \{\mathbf{x} \in \Gamma : \mathbf{x}_i \in \Re(\mathbf{x})\}, \quad 1 \le i \le N, \tag{3}$$

to denote the set of boundary points whose influence domain including the boundary node $\mathbf{x}_i$. For a different boundary point $\mathbf{x}$, the influence domain $\Re(\mathbf{x})$ varies from point to point, hence $\Re^i \equiv \Re_i$ if and only if $r(\mathbf{x})$ is a constant for any $\mathbf{x} \in \Gamma$.

For convenience, we suppose that τ is real and we denote by $H^\tau(\Gamma)$ the Sobolev spaces as well as their interpolation spaces on Γ for noninteger τ [15]. Moreover, let m be a nonnegative integer, we define the following weighted Sobolev spaces [16]

$$W^m_{m-1}\left(\Omega'\right) := \left\{u \in \mathscr{D}'\left(\Omega'\right) : \frac{u}{\sqrt{1+r^2}\ln\left(2+r^2\right)} \in L^2\left(\Omega'\right), \left(1+r^2\right)^{(|\lambda|-1)/2} D^\lambda u \in L^2\left(\Omega'\right), 1 \le |\lambda| \le m\right\},$$

where $\lambda = (\lambda_1, \lambda_2)$, $|\lambda| = \lambda_1 + \lambda_2$, and $r = |\mathbf{x}|$ represents the distance from the origin to the point $\mathbf{x} \in \mathbb{R}^2$.

The norm in $W^m_{m-1}\left(\Omega'\right)$ is defined by

$$\|u\|_{W^m_{m-1}(\Omega')} := \left(\left\|\frac{u}{\sqrt{1+r^2}\ln\left(2+r^2\right)}\right\|^2_{L^2(\Omega')} + \sum_{|\lambda|=1}^{m} \left\|\left(1+r^2\right)^{(|\lambda|-1)/2} D^\lambda u\right\|^2_{L^2(\Omega')}\right)^{\frac{1}{2}}.$$

Observe that all the local properties of the space $W^m_{m-1}(\Omega')$ coincide with those of the Sobolev space $H^m(\Omega')$. As a consequence, the traces of these functions on Γ satisfy the usual trace theorems.

3. The moving least squares (MLS) method

The MLS as an approximation method has been introduced in [9,10]. Since the numerical approximations of MLS starting from a cluster of scattered nodes instead of interpolation on elements, there have many meshless methods based on the MLS method for the numerical solution of differential equations in recent years.

3.1. The MLS procedure

Assume that $\mathbf{x}(s) \in \Gamma$, the MLS approximation for a given function v on $\mathfrak{R}(\mathbf{x})$ is defined by

$$v(\mathbf{x}) \approx \mathcal{M}v\left(\mathbf{x}(s)\right) = \sum_{j=0}^{\beta} P_j(s)a_j(s) = \mathbf{P}^{\mathrm{T}}(s)\mathbf{a}(s), \tag{4}$$

where $\mathcal{M}$ is an approximation operator, a_j are coefficients to be determined, $\mathbf{P}(s) = \left[P_0(s), P_1(s), P_2(s), \ldots, P_\beta(s)\right]^{\mathrm{T}}$ is a vector of the polynomial basis, $\beta + 1$ is the number of monomials in the polynomial basis.

In our work, for a given evaluation point $\mathbf{x}(s^e)$ on $\mathfrak{R}(\mathbf{x})$,

$$\mathbf{P}(s) := \left[1, s - s^e, \left(s - s^e\right)^2, \ldots, \left(s - s^e\right)^\beta\right]^{\mathrm{T}}. \tag{5}$$

As a matter of fact, $s - s^e$ is the local relative coordinate of the boundary point s with respect to the evaluation point s^e. Hence, when the boundary point is an evaluation point, $s - s^e \equiv 0$, and $\mathbf{P}(s)|_{s=s^e} = [1, 0, 0, \ldots, 0]^{\mathrm{T}}$.

The coefficient vector $\mathbf{a}(s)$ is determined by minimizing a weighted discrete L^2 norm, defined as

$$J(s) = \sum_{i \in \wedge(s)} w_i(s)\left[\mathbf{P}^{\mathrm{T}}(s_i)\,\mathbf{a}(s) - v_i\right]^2,$$

where $w_i(s) := w(s - s_i)$, $i \in \wedge(s)$, are weight functions which belong to $C_0^{\ell_w}(\mathfrak{R}_i)$, $\ell_w \geq 0$, and satisfy $w_i(s) \geq 0$ and $\sum_{i=1}^{N} w_i(s) = 1$.

The stationary of $J(s)$ with respect to s leads to:

$$\mathbf{a}(s) = \mathbf{A}^{-1}(s)\mathbf{B}(s)\mathbf{q}, \tag{6}$$

where

$$[\mathbf{A}(s)]_{jk} = \sum_{i \in \wedge(s)} w_i(s)P_j(s_i)P_k(s_i), \quad 0 \leq j, k \leq \beta, \tag{7}$$

$$[\mathbf{B}(s)]_{jk} = w_{I_k}(s)P_j\left(s_{I_k}\right), \quad 0 \leq j \leq \beta, \quad 1 \leq k \leq \kappa(s), \quad I_k \in \wedge(s), \tag{8}$$

$$q_k = v_{I_k}, \quad 1 \leq k \leq \kappa(s), \quad I_k \in \wedge(s). \tag{9}$$

Substituting (6) into (4) yields

$$v(\mathbf{x}) \approx \mathcal{M}v\left(\mathbf{x}(s)\right) = \sum_{k=1}^{\kappa(s)} \psi_k(s)q_k, \tag{10}$$

where

$$\psi_k(s) := \sum_{j=0}^{\beta} P_j(s)\left[\mathbf{A}^{-1}(s)\,\mathbf{B}(s)\right]_{jk}. \tag{11}$$

Denoted by

$$\Phi_i(s) := \begin{cases} \psi_k(s), & i = I_k \in \wedge(s), \\ 0, & i \notin \wedge(s), \end{cases} \quad 1 \leq i \leq N, \tag{12}$$

then (10) can be rewritten as

$$v(\mathbf{x}) \approx \mathcal{M}v(\mathbf{x}) = \sum_{i \in \wedge(\mathbf{x})} \Phi_i(\mathbf{x})v_i = \sum_{i=1}^{N} \Phi_i(\mathbf{x})v_i, \tag{13}$$

which is the MLS approximation for $v(\mathbf{x})$.

Note that in general the MLS approximation do not satisfy the usual interpolation condition, that is $\mathcal{M}v(s_i) \neq v(s_i)$. In fact, Lancaster and Salkauskas [9] called this as the non-interpolation interpolant.

In order to make (6) meaningful for any $\mathbf{x}(s) \in \Gamma$, the matrix $\mathbf{A}(s)$ must be invertible. In fact, we have [17]:

Proposition 3.1. *For any $\mathbf{x}(s) \in \Gamma$, a necessary condition for $\mathbf{A}(s)$ to be invertible is that there are at least β boundary nodes that lie on $\mathfrak{R}(\mathbf{x})$.*

Besides, the MLS shape functions have the following propositions.

Proposition 3.2 *([8]). If $P_j \in C^{\ell_p}$, $0 \leq j \leq \beta$, $\ell_p \geq 0$ and $w_i \in C^{\ell_w}(\mathfrak{R}_i)$, $1 \leq i \leq N$, $\ell_w \geq 0$, then $\Phi_i \in C^{\min(\ell_p, \ell_w)}(\Gamma)$.*

Notation 3.1. In all what follow we will denoted by $\gamma := \min\left(\ell_p, \ell_w, \ell_\Gamma\right)$, where ℓ_Γ is the continuous order of boundary curve Γ.

Proposition 3.3 ([6]). $\sum_{i\in\wedge(s)} D^j\Phi_i\,(s)\,(s_i - s)^k = k!\delta_{jk}$, $0 \le j \le \gamma$, $0 \le k \le \beta$.

Moreover, from (3) and (12) it follows that:

Proposition 3.4. *The MLS shape generating functions have compact supports, namely,* $\Phi_i\,(\mathbf{x}) \in C_0^\gamma\left(\mathfrak{R}^i\right)$, $i \in \wedge(\mathbf{x})$.

3.2. Error estimates for the MLS approximation

When the function to be approximated is continuous, Armentano and Duran [18] and Zuppa [19] have obtained error estimates of MLS approximation in the one dimensional case and higher dimensions, respectively. Since for many cases the function to be approximated is less regular, it is of importance to establish the rate of convergence for the MLS approximation in Sobolev spaces under weaker regularity suppositions.

In order to prove some theorems, we impose the following conditions which will be assumed from now on.

Assumption 1. There is a constant h such that $h = \sup_{\mathbf{x}\in\Gamma}\{r(\mathbf{x})\}$, which implies that the radii of any boundary point's influence domain is less than h.

Assumption 2. For any $\mathbf{x} \in \Gamma$, there exist nonnegative integers $K_1\,(\mathbf{x}) \ge \beta$ and $K_2\,(\mathbf{x})$ such that there are at least $K_1(\mathbf{x})$ boundary nodes, and at most $K_2\,(\mathbf{x})$ boundary nodes lie on the influence domain of $\mathbf{x}$.

Assumption 3. For any $\mathbf{x}(s) \in \Gamma$, there are numbers $C_{w_i}\,(\mathbf{x})$ such that the weight functions satisfying $D^j w_i\,(\mathbf{x}) = C_{w_i}\,(\mathbf{x})\,h^{-j}$, $0 \le j \le \gamma$, $i \in \wedge(\mathbf{x})$. Besides, there exist constants C_{w1} and C_{w2} independent with the parameter h such that $C_{w1} \le \left\|C_{w_i}\,(\mathbf{x})\right\|_{L^\infty(\Gamma)} \le C_{w2}$, $i \in \wedge(\mathbf{x})$.

Lemma 3.1. *There exist constants* $C_{\Phi1}$ *and* $C_{\Phi2}$ *independent with* h *such that*

$$C_{\Phi1}h^{-j} \le \left\|D^j\Phi_i\,(\mathbf{x})\right\|_{L^\infty(\Gamma)} \le C_{\Phi2}h^{-j}, \quad i \in \wedge(\mathbf{x})\,, \quad 0 \le j \le \gamma. \tag{14}$$

Proof. Since $s_I \in \mathfrak{R}(s)$ when $I \in \wedge(s)$, we have bounded constants ρ_I such that $s_I = s^e + \rho_I h$, in which s^e is a given evaluation point on $\mathfrak{R}(s)$.

It follows from (5) and (7) that

$$[\mathbf{A}(s)]_{jk} = \sum_{I\in\wedge(s)} w_I\,(s)\,P_j\,(s_I)\,P_k\,(s_I) = \sum_{I\in\wedge(s)} w_I\,(s)\,(\rho_I h)^{j+k} = h^{j+k}a_{jk}\,(s)\,, \tag{15}$$

where $0 \le j, k \le \beta$ and $a_{jk}(s) := \sum_{I\in\wedge(s)} w_I(s)\rho_I^{j+k}$.

From Assumptions 2 and 3 it is evident that $a_{jk}\,(s)$ are bounded, thus there exist bounded numbers $J\,(s)$ and computable numbers $\bar{a}_{jk}(s)$ independent with h such that

$$\det\left(\mathbf{A}(s)\right) = J\,(s)\,h^{\beta(\beta+1)}, \tag{16}$$

and

$$\left[\mathbf{A}^{-1}(s)\right]_{jk} = h^{-j-k}\bar{a}_{jk}(s), \quad 0 \le j, k \le \beta. \tag{17}$$

Therefore, according to (15) and Assumption 3 we have

$$D^m\,[\mathbf{A}(s)]_{jk} = \sum_{I\in\wedge(s)} (\rho_I h)^{j+k}\,D^m w_I(s) = h^{j+k-m}\sum_{I\in\wedge(s)} \rho_I^{j+k}C_{w_I}\,(s)\,, \quad 0 \le m \le \gamma.$$

Besides, from (8) one gets

$$[\mathbf{B}(s)]_{ki} = w_{I_i}(s)P_k\left(s_{I_i}\right) = w_{I_i}(s)\left(\rho_{I_i}h\right)^k, \quad 0 \le k \le \beta, \quad 1 \le i \le \kappa(s).$$

Hence using Assumption 3 yields

$$D^m[\mathbf{B}(s)]_{ki} = \left(\rho_{I_i}h\right)^k D^m w_{I_i}(s) = h^{k-m}\rho_{I_i}^k C_{w_{I_i}}(s), \quad 0 \le m \le \gamma.$$

Let

$$\mathbf{E}(s) := \mathbf{A^{-1}}(s)\mathbf{B}(s), \tag{18}$$

then

$$\begin{aligned} D^m[\mathbf{E}(s)]_{ji} &= \sum_{k=0}^{\beta}\left[\mathbf{A}^{-1}(s)\right]_{jk}\left\{D^m[\mathbf{B}(s)]_{ki} - \sum_{n=0}^{\beta}\sum_{l=1}^{m}\binom{m}{l}D^l[\mathbf{A}(s)]_{kn}D^{m-l}[\mathbf{E}(s)]_{ni}\right\} \\ &= \sum_{k=0}^{\beta}h^{-j-k}\bar{a}_{jk}(s)\left\{h^{k-m}\rho_{I_i}^{k}C_{w_{I_i}}(s) - \sum_{n=0}^{\beta}\sum_{l=1}^{m}\binom{m}{l}h^{k+n-l}\sum_{I\in\wedge(s)}\rho_I^{k+n}C_{w_I}(s)D^{m-l}[\mathbf{E}(s)]_{ni}\right\} \\ &= b_j^0(s)h^{-j-m} - \sum_{l=1}^{m}\sum_{n=0}^{\beta}\binom{m}{l}d_{jn}(s)h^{-j+n-l}D^{m-l}[\mathbf{E}(s)]_{ni}, \end{aligned}$$

where $0 \le m \le \gamma$, $0 \le j \le \beta$, $1 \le i \le \kappa(s)$, and

$$b_j^0(s) := \sum_{k=0}^{\beta}\bar{a}_{jk}(s)\rho_{I_i}^{k}C_{w_{I_i}}(s), \quad 0 \le j \le \beta, \tag{19}$$

$$d_{jn}(s) := \sum_{I\in\wedge(s)}C_{w_I}(s)\sum_{k=0}^{\beta}\rho_I^{k+n}\bar{a}_{jk}(s), \quad 0 \le j, n \le \beta. \tag{20}$$

By mathematical induction, we can easily prove that

$$D^m[\mathbf{E}(s)]_{ji} = b_j^m(s)\,h^{-j-m}, \quad 0 \le m \le \gamma, \tag{21}$$

where

$$b_j^m(s) := b_j^0(s) - \sum_{l=1}^{m}\sum_{n=0}^{\beta}\binom{m}{l}d_{jn}(s)b_n^{m-l}(s), \quad 0 \le j \le \beta, \quad 1 \le m \le \gamma. \tag{22}$$

Since the evaluation point s^e is fixed on $\Re(s)$, there exist real numbers ρ^e such that $s - s^e = \rho^e h$. Hence from (5),

$$D^m P_j(s) = D^m\left(s-s^e\right)^j = \begin{cases}\binom{j}{m}\left(\rho^e h\right)^{j-m}, & j \ge m, \\ 0, & j < m,\end{cases} \quad 0 \le m \le \gamma. \tag{23}$$

Therefore, when $0 \le m \le \gamma$ and $1 \le i \le \kappa(s)$, from (11), (21) and (23) we have

$$\begin{aligned} D^m\psi_i(s) &= D^m\left(\sum_{j=0}^{\beta}P_j(s)[\mathbf{E}(s)]_{ji}\right) \\ &= \sum_{l=0}^{m}\binom{m}{l}\sum_{j=0}^{\beta}D^lP_j(s)D^{m-l}[\mathbf{E}(s)]_{ji} \\ &= \left[\sum_{l=0}^{m}\binom{m}{l}\sum_{j=l}^{\beta}\binom{j}{l}\left(\rho^e\right)^{j-l}b_j^{m-l}(s)\right]h^{-m}. \end{aligned}$$

From the discussion above we can find that $\bar{a}_{jk}(s)$, $C_{w_I}(s)$ and ρ_I are bounded, thus $b_j^0(s)$ given by (19) and $d_{jn}(s)$ defined in (20) are bounded. Hence by mathematical induction, $b_j^m(s)$ are bounded for any $0 \le j \le \beta$ and $1 \le m \le \gamma$. Besides, from Assumption 1, it follows that $|\rho^e| < 1$. Therefore, there are real numbers $C_{m1}(\mathbf{x})$ and $C_{m2}(\mathbf{x})$ independent with h such that

$$C_{m1}(\mathbf{x})\,h^{-m} \le \left|D^m\psi_i(\mathbf{x})\right| \le C_{m2}(\mathbf{x})\,h^{-m}, \quad \forall\,\mathbf{x}(s) \in \Gamma, \quad 0 \le m \le \gamma,$$

then we obtain

$$C_{\psi 1}h^{-m} \le \left\|D^m\psi_i(\mathbf{x})\right\|_{L^\infty(\Gamma)} \le C_{\psi 2}h^{-m}, \quad 0 \le m \le \gamma. \tag{24}$$

The conclusion of the lemma follows readily from (12) and (24).

Remark 3.1. The computable numbers $\bar{a}_{jk}(s)$ appearing in (17) are really measures of the geometrical quality of the distribution of nodes $\{\mathbf{x}_i\}_{i\in\wedge(s)}$ and values of weight functions $\{w_i(s)\}_{i\in\wedge(s)}$ around point $\mathbf{x}(s)$. As a result, the constants $C_{\Phi 1}$ and $C_{\Phi 2}$ in the inequality (14) are related to the good quality of the set of nodes.

Theorem 3.1. *Assume that* $v(\mathbf{x}) \in H^{m+1}(\Gamma)$, $0 \le m \le \gamma$. *Let*

$$\mathcal{M}v(\mathbf{x}) = \mathcal{M}v(\mathbf{x}(s)) = \sum_{i\in\wedge(\mathbf{x})}\Phi_i(\mathbf{x})\,v_i, \tag{25}$$

then

$$\|v(\mathbf{x}) - \mathcal{M}v\,(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m+1-k}\,\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}\,, \quad 0 \le k \le m \tag{26}$$

where C is a constant independent of h.

Proof. Since s is a curvilinear co-ordinate on Γ, we have $\mathbf{x}\left(s + \theta\left(\tilde{s} - s\right)\right) \in \Gamma$ for any $\mathbf{x}(s) \in \mathfrak{R}_j$ and $\mathbf{x}\left(\tilde{s}\right) \in \Gamma$, in which $0 < \theta < 1$ and $0 \le j \le N$. Thus $\mathfrak{R}_j$ is star-shaped with respect to Γ. Hence from Section 4 of [14], the Taylor polynomial of degree m of $v\left(\mathbf{x}\left(s\right)\right)$ averaged over $\mathfrak{R}_j$ is

$$Q_j^{m+1}v(s) := \sum_{l=0}^{m} \frac{1}{l!} \int_{\mathfrak{R}_j} D^l v\left(\tilde{s}\right)\left(s - \tilde{s}\right)^l \phi\left(\tilde{s}\right)\,\mathrm{d}\tilde{s}, \quad 0 \le j \le N, \tag{27}$$

where $\phi\left(\tilde{s}\right) \in C_0^\infty\left(\mathfrak{R}_j\right)$ and satisfying $\int_{\mathfrak{R}_j} \phi\left(\tilde{s}\right)\,\mathrm{d}\tilde{s} = 1$.

The residual term corresponding to (27) is defined as

$$R_j^{m+1}v(s) := v(s) - Q_j^{m+1}v\,(s)\,, \quad 0 \le j \le N,$$

satisfying [14]

$$\left\|R_j^{m+1}v(s)\right\|_{L^\infty(\mathfrak{R}_j)} \le Ch^{m+1/2}\,|v(s)|_{H^{m+1}(\mathfrak{R}_j)}\,, \quad 0 \le j \le N, \tag{28}$$

$$\left|R_j^{m+1}v(s)\right|_{H^k(\mathfrak{R}_j)} \le Ch^{m+1-k}\,|v(s)|_{H^{m+1}(\mathfrak{R}_j)}\,, \quad 0 \le k \le m, \quad 0 \le j \le N, \tag{29}$$

in which the constant C depends only on m, and is independent of h.

Besides, for any $s \in \mathfrak{R}_j$, $0 \le j \le N$, it follows from (25) that

$$v(s) - \mathcal{M}v(s) = Q_j^{m+1}v\,(s) + R_j^{m+1}v(s) - \sum_{i \in \wedge(s)} \Phi_i(s)\left(Q_j^{m+1}v\,(s_i) + R_j^{m+1}v(s_i)\right).$$

From Proposition 3.3,

$$\sum_{i \in \wedge(s)} \Phi_i\,(s)\left(s_i - \tilde{s}\right)^l = \sum_{j=0}^{l} \binom{l}{j}\left(s - \tilde{s}\right)^{l-j} \sum_{i \in \wedge(s)} \Phi_i(s)\,(s_i - s)^j = \left(s - \tilde{s}\right)^l,$$

then

$$\begin{aligned}\sum_{i \in \wedge(s)} \Phi_i\,(s)\,Q_j^{m+1}v(s_i) &= \sum_{i \in \wedge(s)} \Phi_i\,(s) \sum_{l=0}^{m} \frac{1}{l!} \int_{\mathfrak{R}_j} D^l v\left(\tilde{s}\right)\left(s_i - \tilde{s}\right)^l \phi\left(\tilde{s}\right)\,\mathrm{d}\tilde{s} \\ &= \sum_{l=0}^{m} \frac{1}{l!} \int_{\mathfrak{R}_j} \left(\sum_{i \in \wedge(s)} \Phi_i\,(s)\left(s_i - \tilde{s}\right)^l\right) D^l v\left(\tilde{s}\right) \phi\left(\tilde{s}\right)\,\mathrm{d}\tilde{s} \\ &= Q_j^{m+1}v(s), \quad 0 \le j \le N.\end{aligned}$$

Thus

$$|v(s) - \mathcal{M}v(s)|_{H^k(\mathfrak{R}_j)} \le \left|R_j^{m+1}v(s)\right|_{H^k(\mathfrak{R}_j)} + \left\|R_j^{m+1}v(s)\right\|_{L^\infty(\mathfrak{R}_j)} \sum_{i \in \wedge(s)} |\Phi_i(s)|_{H^k(\mathfrak{R}_j)}\,. \tag{30}$$

Furthermore, from Lemma 3.1, there is a constant $C_{\Phi 2}$ such that

$$|\Phi_i(s)|^2_{H^k(\mathfrak{R}_j)} = \int_{\mathfrak{R}_j} \left|D^k\Phi_i(s)\right|^2\,\mathrm{d}s \le \int_{\mathfrak{R}_j} \left(C_{\Phi 2}h^{-k}\right)^2\,\mathrm{d}s \le C_\Phi h^{1-2k}.$$

By Assumption 2, for any $\mathbf{x}(s) \in \Gamma$, there exist nonnegative integers $K_2(s)$ and a constant $I \in \wedge(s)$ such that

$$\sum_{i \in \wedge(s)} |\Phi_i(s)|_{H^k(\mathfrak{R}_j)} \le K_2(s)\,|\Phi_I(s)|_{H^k(\mathfrak{R}_j)} \le Ch^{1/2-k}, \quad 0 \le j \le N. \tag{31}$$

Substituting (28), (29) and (31) into (30), one gets

$$|v(s) - \mathcal{M}v(s)|_{H^k(\mathfrak{R}_j)} \le Ch^{m+1-k}\,|v(s)|_{H^{m+1}(\mathfrak{R}_j)}\,, \quad 0 \le k \le m, \quad 0 \le j \le N.$$

Hence, applying Assumption 2 yields,

$$|v\,(\mathbf{x}) - \mathcal{M}v\,(\mathbf{x})|_{H^k(\Gamma)} \le Ch^{m+1-k}\,|v\,(\mathbf{x})|_{H^{m+1}(\Gamma)}\,, \quad 0 \le k \le m.$$

Remark 3.2. Some related results have been obtained in [20] in the context of approximations based on h–p cloud functions. With the help of the celebrated Jackson-type inequalities, Zuppa have established the error estimates for h–p approximations. Although the h–p cloud function considered by Zuppa is different from the MLS shape function, the convergence order of h–p approximations given in [20] is the same as that of MLS approximations presented in Theorem 3.1.

Remark 3.3. From Remark 3.1, we can conclude that constants appearing in error estimates of MLS approximations depend at some extend of the good quality of the covering of weight functions [19].

4. The Galerkin boundary node method (GBNM)

4.1. Galerkin procedures and numerical implementation

Since in BEM, the variational formulation on boundary and coerciveness properties provide the basic mathematical foundation for rigorous error and convergence analysis, we start with a general boundary integral equation of the form

$$Av = f, \quad \text{on } \Gamma, \tag{32}$$

where $f \in H^{\tau-2\alpha}(\Gamma)$, $\tau \in \mathbb{R}$, is the given data, 2α is a fixed constant, and A: $H^{\tau}(\Gamma) \to H^{\tau-2\alpha}(\Gamma)$ is a pseudo differential operator of real order 2α.

In the classical Galerkin method, the weak solution of (32) is determined in [21–23]:

$$\begin{cases} \text{find } v \in H^{\alpha}(\Gamma) \quad \text{such that } \forall v' \in H^{\alpha}(\Gamma), \\ \langle Av, v' \rangle_{L^2(\Gamma)} = \langle f, v' \rangle_{L^2(\Gamma)}. \end{cases} \tag{33}$$

In the MLS method, the numerical approximations start from scattered nodes instead of elements. Thus, if the MLS approximation scheme is used to obtain the approximate solution of problem (33), we can develop a meshless method in the following way.

Proposition 3.4 indicates that $\Phi_i(\mathbf{x}) \in C_0^{\gamma}\left(\Re^i\right)$, $\mathbf{x} \in \Gamma$, $i \in \wedge(\mathbf{x})$, thus $\Phi_i(\mathbf{x}) \in C^{\gamma}(\Gamma)$, $1 \le i \le N$. Let

$$V_h(\Gamma) := \text{span}\{\Phi_i, 1 \le i \le N\}, \tag{34}$$

where the basis functions Φ_i are defined in (12).

Clearly, $\Phi_i(\mathbf{x}) \in H^m(\Gamma) \subset H^{\alpha}(\Gamma)$ provided that $\alpha \le m \le \gamma$. Thus, the variational problem (33) can be approximated by

$$\begin{cases} \text{find } v_h \in V_h(\Gamma) \quad \text{such that } \forall v' \in V_h(\Gamma), \\ \langle Av_h, v' \rangle_{L^2(\Gamma)} = \langle f, v' \rangle_{L^2(\Gamma)}. \end{cases} \tag{35}$$

On $V_h(\Gamma)$, the Galerkin approximation v_h of the real solution v may be written as a linear combination

$$v_h(\mathbf{x}) = \sum_{i \in \wedge(\mathbf{x})} \Phi_i(\mathbf{x})\, v_i, \tag{36}$$

where the coefficients v_i are determined by

$$\sum_{i=1}^{N} a_{ij} v_i = f_j, \quad 1 \le j \le N, \tag{37}$$

with

$$\begin{cases} a_{ij} = \langle A\Phi_i, \Phi_j \rangle_{L^2(\Gamma)} = \displaystyle\int_{\Gamma} A\Phi_i(\mathbf{x}) \cdot \Phi_j(\mathbf{x})\, \mathrm{d}S_{\mathbf{x}}, \\ f_j = \langle f, \Phi_j \rangle_{L^2(\Gamma)} = \displaystyle\int_{\Gamma} f(\mathbf{x})\Phi_j(\mathbf{x})\, \mathrm{d}S_{\mathbf{x}}. \end{cases} \tag{38}$$

From Proposition 3.4, (38) can be rewritten as

$$\begin{cases} a_{ij} = \displaystyle\int_{\Re^j} A\Phi_i(\mathbf{x}) \cdot \Phi_j(\mathbf{x})\, \mathrm{d}S_{\mathbf{x}}, \\ f_j = \displaystyle\int_{\Re^j} f(\mathbf{x})\Phi_j(\mathbf{x})\, \mathrm{d}S_{\mathbf{x}}. \end{cases} \tag{39}$$

Eqs. (39) will now be applied on $\Re^j$, which is defined by (3) and is a part of the boundary Γ. The integrations can be numerically calculated by employing a cell structure as in EFGM and BNM.

The kernel of the operator A may be singular, strongly singular or hypersingular. There are many regularization procedures to integrate and are available in the literature [24]. Commonly, the first case can be overcome by using special Gauss points and weights; and the second case may be simplified as in [25] with the help of a linear geometric representation for each cell. While for the last case, the special strategies proposed in [25] or [26] can be applied.

Remark 4.1. Note that the cell can be of any shape and the only restriction is that the unions of all cells be equal the integral area, thus the concept of cell is quite different from that of an element in BEM. As a consequence, GBNM is a boundary-type meshless method.

Remark 4.2. By employing the variational formulation of (32), the boundary function f is multiplied by a test function and integrated on Γ. Thus in GBNM, boundary conditions are implemented exactly despite the MLS shape functions lacking the delta function property.

4.2. Error estimates

The approximation estimate obtained in Section 3.2 can be used to establish the abstract error bound for the approximate solutions obtained by using GBNM. In this section, the error estimates of GBNM for solving the boundary integral formula (32) are presented. For the sake of proving some theorems, we need some lemmas and assumptions.

Lemma 4.1 (*Inverse Property*). *For any $v_h(\mathbf{x}) \in V_h(\Gamma)$, we have a constant C independent with the parameter h such that*

$$\|v_h(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m-k} \|v_h(\mathbf{x})\|_{H^m(\Gamma)}, \quad -\gamma \le m \le k, \quad 0 \le k \le \gamma. \tag{40}$$

Proof. For any $\mathbf{x} \in \Gamma$, it follows from Lemma 3.1 that

$$|\Phi_i(\mathbf{x})|^2_{H^k(\Gamma)} = \int_\Gamma \sum_{|\lambda|=k} \left|D^\lambda \Phi_i(\mathbf{x})\right|^2 \, \mathrm{d}S_{\mathbf{x}} \le \int_\Gamma \left(C_{k2}(\mathbf{x})h^{-k}\right)^2 \, \mathrm{d}S_{\mathbf{x}}, \quad 0 \le k \le \gamma,$$

and

$$|\Phi_i(\mathbf{x})|^2_{H^m(\Gamma)} = \int_\Gamma \sum_{|\lambda|=m} \left|D^\lambda \Phi_i(\mathbf{x})\right|^2 \, \mathrm{d}S_{\mathbf{x}} \ge \int_\Gamma \left(C_{m1}(\mathbf{x})h^{-m}\right)^2 \, \mathrm{d}S_{\mathbf{x}}, \quad 0 \le m \le \gamma.$$

Hence

$$|\Phi_i(\mathbf{x})|_{H^k(\Gamma)} \le C_\psi h^{m-k} |\Phi_i(\mathbf{x})|_{H^m(\Gamma)}, \quad i \in \wedge(\mathbf{x}), \quad 0 \le m \le k \le \gamma.$$

Therefore

$$\|v_h(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m-k} \|v_h(\mathbf{x})\|_{H^m(\Gamma)}, \quad 0 \le m \le k \le \gamma, \tag{41}$$

and

$$\|v_h(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{-k} \|v_h(\mathbf{x})\|_{H^0(\Gamma)}, \quad 0 \le k \le \gamma. \tag{42}$$

On the other hand,

$$\begin{aligned} \|v_h(\mathbf{x})\|^2_{H^0(\Gamma)} &\le \|v_h(\mathbf{x})\|_{H^m(\Gamma)} \|v_h(\mathbf{x})\|_{H^{-m}(\Gamma)} \\ &\le Ch^m \|v_h(\mathbf{x})\|_{H^m(\Gamma)} \|v_h(\mathbf{x})\|_{H^0(\Gamma)}, \quad -\gamma \le m \le 0. \end{aligned} \tag{43}$$

Thus, from (42) and (43) one gets

$$\|v_h(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m-k} \|v_h(\mathbf{x})\|_{H^m(\Gamma)}, \quad -\gamma \le m \le 0, \quad 0 \le k \le \gamma. \tag{44}$$

Consequently, the conclusion of the lemma follows from (41) and (44).

Lemma 4.2. *Let S_h be a projection from $L^2(\Gamma)$ onto V_h, then if $v(\mathbf{x}) \in H^{m+1}(\Gamma)$, there holds*

$$\|v(\mathbf{x}) - S_h v(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m+1-k} \|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \tag{45}$$

where $-(\gamma+1) \le k \le m$, $0 \le m \le \gamma$, and C is a constant independent of h.

Proof. From Theorem 3.1, we have

$$\|v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\|_{H^k(\Gamma)} \le Ch^{m+1-k} \|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \quad 0 \le k \le m \le \gamma,$$

and

$$\|v(\mathbf{x}) - S_h v(\mathbf{x})\|_{H^0(\Gamma)} \le \|v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\|_{H^0(\Gamma)} \le Ch^{m+1} \|v(\mathbf{x})\|_{H^{m+1}(\Gamma)},$$

hence using Lemma 4.1 yields

$$\begin{aligned} \|v(\mathbf{x}) - S_h v(\mathbf{x})\|_{H^k(\Gamma)} &\le \|v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\|_{H^k(\Gamma)} + \|\mathcal{M}v(\mathbf{x}) - S_h v(\mathbf{x})\|_{H^k(\Gamma)} \\ &\le \|v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\|_{H^k(\Gamma)} + Ch^{-k} \|\mathcal{M}v(\mathbf{x}) - S_h v(\mathbf{x})\|_{H^0(\Gamma)} \\ &\le \|v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\|_{H^k(\Gamma)} + Ch^{-k} \left\{\|\mathcal{M}v - v\|_{H^0(\Gamma)} + \|v - S_h v\|_{H^0(\Gamma)}\right\} \\ &\le Ch^{m+1-k} \|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \quad 0 \le k \le m \le \gamma, \end{aligned} \tag{46}$$

and by a classical duality argument we deduce

$$\begin{aligned}\|v(\mathbf{x})-S_hv(\mathbf{x})\|_{H^{-k}(\Gamma)} &= \sup_{\varphi(\mathbf{x})\in H^k(\Gamma)} \frac{\langle v(\mathbf{x})-S_hv(\mathbf{x}),\varphi(\mathbf{x})\rangle_{L^2(\Gamma)}}{\|\varphi(\mathbf{x})\|_{H^k(\Gamma)}}\\ &= \sup_{\varphi(\mathbf{x})\in H^k(\Gamma)} \frac{\langle v(\mathbf{x})-S_hv(\mathbf{x}),\varphi(\mathbf{x})-S_h\varphi(\mathbf{x})\rangle_{L^2(\Gamma)}}{\|\varphi(\mathbf{x})\|_{H^k(\Gamma)}}\\ &\le Ch^{m+1+k}\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)},\end{aligned} \tag{47}$$

for $1\le k\le \gamma+1$ and $0\le m\le\gamma$.

When $-1\le k\le 0$, from an interpolation theorem of Sobolev spaces [15] we get

$$\begin{aligned}\|v(\mathbf{x})-S_hv(\mathbf{x})\|_{H^k(\Gamma)} &\le \|v(\mathbf{x})-S_hv(\mathbf{x})\|_{H^{-1}(\Gamma)}^{-k}\|v(\mathbf{x})-\mathcal{M}v(\mathbf{x})\|_{H^0(\Gamma)}^{1+k}\\ &\le Ch^{m+1-k}\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)},\quad 0\le m\le\gamma.\end{aligned} \tag{48}$$

The proof is completed via (46)–(48). □

Assumption 4. When $\alpha\le\tau\le\gamma+1$, the boundary integral operator $A\colon H^{\tau}(\Gamma)\to H^{\tau-2\alpha}(\Gamma)$ is a continuous isomorphism.

Assumption 5. The operator A satisfies the following Gårding inequality

$$\operatorname{Re}\langle (A+K)v,v\rangle_{L^2(\Gamma)}\ge C\|v\|_{H^\alpha(\Gamma)}^2,\quad \forall v\in H^\alpha(\Gamma),$$

in which $K\colon H^\alpha(\Gamma)\to H^{-\alpha}(\Gamma)$ is a compact operator, and the constant $C>0$.

Via Theorem 3.1, we have the following lemma [21].

Lemma 4.3. *Suppose that the variational problem (33) has a unique solution $v\in H^\alpha(\Gamma)$, then under the conditions of Assumption 5, we have*

$$\sup_{0\ne v'\in V_h}\frac{\left|\langle Av_h,v'\rangle\right|}{\|v'\|_{H^\alpha(\Gamma)}}\ge C_B\|v_h\|_{H^\alpha(\Gamma)},\quad C_B>0,\quad \forall v_h\in V_h. \tag{49}$$

Besides, there is a constant C_c independent with v and h such that

$$\|v-v_h\|_{H^\alpha(\Gamma)}\le C_c\inf_{v'\in V_h}\left\|v-v'\right\|_{H^\alpha(\Gamma)}. \tag{50}$$

We are now in a position to establish the following theorem regarding the convergence of the solution of the approximate problem (35) to the exact solution of the variational problem (33).

Theorem 4.1 (*Asymptotic Error Estimates*). *Let $v(\mathbf{x})$ and $v_h(\mathbf{x})$ be the solutions, respectively, of variational problems (33) and (35). Then if $v(\mathbf{x})\in H^{m+1}(\Gamma)$, we have*

$$\|v(\mathbf{x})-v_h(\mathbf{x})\|_{H^k(\Gamma)}\le Ch^{m+1-k}\|v(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \tag{51}$$

where $2\alpha-\gamma-1\le k\le m$, $\max\{\alpha,0\}\le m\le\gamma$, and C is a constant independent of h.

Proof. Since $S_hv(\mathbf{x})\in V_h$, it follows from (50) and Lemma 4.2 that

$$\|v-v_h\|_{H^\alpha(\Gamma)}\le C_c\|v-S_hv\|_{H^\alpha(\Gamma)}\le Ch^{m+1-\alpha}\|v\|_{H^{m+1}(\Gamma)},\ \max\{\alpha,0\}\le m\le\gamma.$$

Let $\phi(\mathbf{x})$ be the solution of

$$A\phi(\mathbf{x})=\varphi(\mathbf{x}),\quad \mathbf{x}\in\Gamma,$$

then by Assumption 4,

$$\|\phi(\mathbf{x})\|_{H^{\tau+2\alpha}(\Gamma)}\le C\|\varphi(\mathbf{x})\|_{H^\tau(\Gamma)},\quad -\alpha\le\tau\le\gamma+1-2\alpha.$$

Besides, from (33) and (35) we obtain

$$\langle A(v-v_h),v'\rangle_{L^2(\Gamma)}=0,\quad \forall v'\in V_h.$$

Thus, by the duality argument we get

$$\begin{aligned}\|v-v_h\|_{H^{-k}(\Gamma)} &= \sup_{\varphi\in H^k(\Gamma)}\frac{\langle v-v_h,\varphi\rangle_{L^2(\Gamma)}}{\|\varphi\|_{H^k(\Gamma)}}\le C\sup_{\phi\in H^{k+2\alpha}(\Gamma)}\frac{|\langle A(v-v_h),\phi-S_h\phi\rangle|}{\|\phi\|_{H^{k+2\alpha}(\Gamma)}}\\ &\le Ch^{k+\alpha}\|v-v_h\|_{H^\alpha(\Gamma)}\le Ch^{m+1+k}\|v\|_{H^{m+1}(\Gamma)},\end{aligned} \tag{52}$$

for $\max\{-\alpha,0\}\le k\le\gamma+1-2\alpha$, $\max\{\alpha,0\}\le m\le\gamma$.

On the other hand, applying Lemmas 4.1 and 4.2 yields

$$\begin{aligned}\|v-v_h\|_{H^k(\Gamma)} &\le \|v-S_hv\|_{H^k(\Gamma)}+\|S_hv-v_h\|_{H^k(\Gamma)}\\ &\le \|v-S_hv\|_{H^k(\Gamma)}+Ch^{\alpha-k}\|S_hv-v_h\|_{H^\alpha(\Gamma)}\\ &\le \|v-S_hv\|_{H^k(\Gamma)}+Ch^{\alpha-k}\left(\|S_hv-v\|_{H^\alpha(\Gamma)}+\|v-v_h\|_{H^\alpha(\Gamma)}\right)\\ &\le Ch^{m+1-k}\|v\|_{H^{m+1}(\Gamma)}, \quad \max\{\alpha,0\}\le k\le m\le\gamma. \end{aligned} \tag{53}$$

Let $\tau_1=-\max\{-\alpha,0\}$, $\tau_2=\max\{\alpha,0\}$, then when $\tau_1\le k\le\tau_2$, we have

$$\begin{aligned}\|v-v_h\|_{H^k(\Gamma)} &\le \left(\|v-v_h\|_{H^{\tau_1}(\Gamma)}^{(k-\tau_2)}\|v-v_h\|_{H^{\tau_2}(\Gamma)}^{(\tau_1-k)}\right)^{\frac{1}{\tau_1-\tau_2}}\\ &\le Ch^{m+1-k}\|v\|_{H^{m+1}(\Gamma)}, \quad \max\{\alpha,0\}\le m\le\gamma. \end{aligned} \tag{54}$$

Consequently, the conclusion of the theorem follows from (52)–(54).

4.3. Dirichlet problems

Consider the interior and exterior Dirichlet problem

$$\begin{cases}\Delta u=0, & \text{in } \Omega\cup\Omega',\\ u=u_0, & \text{on } \Gamma,\end{cases} \tag{55}$$

where u_0 is the given boundary data, and u satisfies at infinity the decay condition: $|u(\mathbf{x})|=O\left(|\mathbf{x}|^{-1}\right)$ as $|\mathbf{x}|\to\infty$.

Let σ be the jump through Γ of the flux $\frac{\partial u}{\partial \mathbf{n}}$, where $\mathbf{n}$ is the outward normal to the boundary. Then by a generalized Green's formula, the solution of problem (55) corresponds to

$$\int_{\mathbb{R}^2}\nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x}=\langle\sigma,v\rangle_\Gamma, \quad \forall v\in W_0^1(\mathbb{R}^2). \tag{56}$$

Besides, the solution can be represented by a simple layer potential

$$u(\mathbf{x})=\int_\Gamma\sigma(\mathbf{y})\,u^*(\mathbf{x},\mathbf{y})\;\mathrm{d}S_{\mathbf{y}}, \quad \mathbf{x}\in\mathbb{R}^2, \tag{57}$$

where $u^*(\mathbf{x},\mathbf{y})=-\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|$ is the fundamental solution of a Laplace operator.

Let

$$A\sigma(\mathbf{x}):=\int_\Gamma\sigma(\mathbf{y})\,u^*(\mathbf{x},\mathbf{y})\;\mathrm{d}S_{\mathbf{y}}, \quad \mathbf{x}\in\Gamma, \tag{58}$$

then the order of the operator A is -1. From [21–23,27], the density function σ of (57) can be determined by

$$\begin{cases}\text{find } \sigma\in H^{-1/2}(\Gamma) \;\text{ such that } \forall\sigma'\in H^{-1/2}(\Gamma),\\ \langle A\sigma,\sigma'\rangle_{L^2(\Gamma)}=\langle u_0,\sigma'\rangle_{L^2(\Gamma)}.\end{cases} \tag{59}$$

From (35), the approximate problem of (59) is

$$\begin{cases}\text{find } \sigma_h\in V_h(\Gamma) \;\text{ such that } \forall\sigma'\in V_h(\Gamma),\\ \langle A\sigma_h,\sigma'\rangle_{L^2(\Gamma)}=\langle u_0,\sigma'\rangle_{L^2(\Gamma)},\end{cases} \tag{60}$$

in which $V_h(\Gamma)$ is defined by (34).

From [23,27,28], we have the following theorems:

Theorem 4.2. *If $u_0\in H^k(\Gamma)$, $k\ge 1/2$, then by the Lax–Milgram theorem, the variational problems (59) and (60) have, respectively, one and only one solutions $\sigma\in H^{k-1}(\Gamma)$ and $\sigma_h\in V_h(\Gamma)$, and problem (55) has a unique solution $u\in H^{k+1/2}(\Omega)\times W_{k-1/2}^{k+1/2}(\Omega')$.*

Theorem 4.3. *If $\sigma\in H^{k-1}(\Gamma)$, $k\ge 1/2$, let $u\in W_0^1(\mathbb{R}^2)$ be the solution of (56), then $u|_\Gamma\in H^k(\Gamma)$. Besides, the mapping $\sigma\to u|_\Omega$ defined by (56) is linear and continuous from $H^{k-1}(\Gamma)$ to $H^{k+1/2}(\Omega)$, and $\sigma\to u|_{\Omega'}$ is a linear and continuous mapping from $H^{k-1}(\Gamma)$ to $W_{k-1/2}^{k+1/2}(\Omega')$.*

From Theorems 4.2 and 4.3, it is true that the boundary integral operator A given by (58) satisfies Assumptions 4 and 5. Thus by Theorem 4.1,

$$\|\sigma(\mathbf{x})-\sigma_h(\mathbf{x})\|_{H^k(\Gamma)}\le Ch^{m+1-k}\|\sigma(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \tag{61}$$

where $-(\gamma+2)\le k\le m$, $0\le m\le\gamma$, and C is a constant independent of h.

In the following, the convergence order of solving the problem (55) by the presented meshless method will be derived in energy norms and global maximum norm. Besides, we will deduce the error inside the neighborhood of Γ.

Theorem 4.4 (*Energy Norms*). *Let $u(\mathbf{x})$ given by (57) be the solution of the problem (55), and*

$$u_h(\mathbf{x}) = \int_\Gamma \sigma_h(\mathbf{y})\, u^*(\mathbf{x}, \mathbf{y})\, \mathrm{d}S_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^2, \tag{62}$$

then there is a constant C independent with h such that

$$\|u - u_h\|_{W_0^1(\mathbb{R}^2)} \le Ch^{m+3/2} \|\sigma\|_{H^{m+1}(\Gamma)}, \quad 0 \le m \le \gamma, \tag{63}$$

$$\|u - u_h\|_{H^2(\Omega)} + \|u - u_h\|_{W_1^2(\Omega')} \le Ch^{m+1/2} \|\sigma\|_{H^{m+1}(\Gamma)}, \quad 1/2 \le m \le \gamma. \tag{64}$$

Proof. The proof follows immediately from Theorem 4.3 with (61). □

Theorem 4.5 (*Global Maximum Norm*). *Let $u(\mathbf{x})$ and $u_h(\mathbf{x})$ be given by (57) and (62), respectively. If there exists a constant $\delta > 0$ such that $d(\mathbf{x}, \Gamma) = \min_{\mathbf{y}\in\Gamma} \{|\mathbf{x} - \mathbf{y}|\} \ge \delta$, then*

$$|u(\mathbf{x}) - u_h(\mathbf{x})| \le C \left(|\ln d(\mathbf{x}, \Gamma)| + \sum_{l=1}^{\gamma+2} (d(\mathbf{x}, \Gamma))^{-l} \right) h^{m+\gamma+3} \|\sigma(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \tag{65}$$

$$\left|\nabla^\lambda u(\mathbf{x}) - \nabla^\lambda u_h(\mathbf{x})\right| \le C \left(\sum_{l=0}^{\gamma+2} (d(\mathbf{x}, \Gamma))^{-l-|\lambda|} \right) h^{m+\gamma+3} \|\sigma(\mathbf{x})\|_{H^{m+1}(\Gamma)}, \tag{66}$$

where $0 \le m \le \gamma$, $\lambda = (\lambda_1, \lambda_2)$, $|\lambda| = \lambda_1 + \lambda_2 \ge 1$, and the constant C is independent of h.

Proof. From (61), we have

$$\begin{aligned}
|u(\mathbf{x}) - u_h(\mathbf{x})| &= \frac{1}{2\pi} \left| \int_\Gamma (\sigma(\mathbf{y}) - \sigma_h(\mathbf{y})) \ln|\mathbf{x} - \mathbf{y}|\, \mathrm{d}S_{\mathbf{y}} \right| \\
&\le C \|\sigma - \sigma_h\|_{H^{-(\gamma+2)}(\Gamma)} \|\ln|\mathbf{x} - \mathbf{y}|\|_{H^{\gamma+2}(\Gamma)}, \\
&\le C \left(|\ln d(\mathbf{x}, \Gamma)| + \sum_{l=1}^{\gamma+2} (d(\mathbf{x}, \Gamma))^{-l} \right) h^{m+\gamma+3} \|\sigma(\mathbf{x})\|_{H^{m+1}(\Gamma)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\nabla u(\mathbf{x}) - \nabla u_h(\mathbf{x})| &= \frac{1}{2\pi} \left| \int_\Gamma (\sigma(\mathbf{y}) - \sigma_h(\mathbf{y})) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{x} - \mathbf{y}|^2}\, \mathrm{d}S_{\mathbf{y}} \right| \\
&\le C \left(\sum_{l=0}^{\gamma+2} (d(\mathbf{x}, \Gamma))^{-l-1} \right) h^{m+\gamma+3} \|\sigma(\mathbf{x})\|_{H^{m+1}(\Gamma)}.
\end{aligned}$$

The proof is now completed via the same type of estimate as for the other derivatives. □

Remark 4.3. Theorem 4.5 obtained the error of u and its derivatives outside the neighborhood of Γ, which shows extremely high accuracy can be achieved not only for the primary field variable u but also for its derivatives. Contrary to the case of the domain type methods, such as the FEM, Theorem 4.5 indicates that the errors of the field function u and its derivatives in our GBNM are all of the same order.

The following theorem will give the error inside the neighborhood of Γ.

Theorem 4.6. *Let $u(\mathbf{x})$ and $u_h(\mathbf{x})$ be given by (57) and (62), respectively. There exists $\delta > 0$, for any $\mathbf{x} \in \mathbb{R}^2$ with $d(\mathbf{x}, \Gamma) < \delta$ and for given $\varepsilon > 0$, we have*

$$|u(\mathbf{x}) - u_h(\mathbf{x})| \le C(\delta)\, h^{m+1-\varepsilon} \|\sigma\|_{H^{m+1}(\Gamma)}, \quad \varepsilon \le m \le \gamma, \tag{67}$$

where the constant C is independent of h and ε.

Proof. If $\varepsilon > 0$, one gets

$$\begin{aligned}
|u(\mathbf{x}) - u_h(\mathbf{x})| &= \frac{1}{2\pi} \left| \int_\Gamma (\sigma(\mathbf{y}) - \sigma_h(\mathbf{y})) \ln|\mathbf{x} - \mathbf{y}|\, \mathrm{d}S_{\mathbf{y}} \right| \\
&\le C \|\sigma - \sigma_h\|_{H^\varepsilon(\Gamma)} \|\ln|\mathbf{x} - \mathbf{y}|\|_{H^{-\varepsilon}(\Gamma)}.
\end{aligned} \tag{68}$$

Table 1
Problems and solutions.

	Example 1	Example 2
Problems	$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = \sin\left(x_1^2 - x_2^2\right)\exp\left(2x_1x_2\right), & \text{on } \Gamma \end{cases}$	$\begin{cases} \Delta u = 0, & \text{in } \Omega' \\ u = \dfrac{x_1}{x_1^2 + x_2^2}, & \text{on } \Gamma \end{cases}$
Exact solutions	$u = \sin\left(x_1^2 - x_2^2\right)\exp\left(2x_1x_2\right)$	$u = \frac{x_1}{x_1^2+x_2^2}$

For any $\varepsilon > 0$, it follows from (61) that

$$\|\sigma - \sigma_h\|_{H^{\varepsilon}(\Gamma)} \leq Ch^{m+1-\varepsilon}\, \|\sigma\|_{H^{m+1}(\Gamma)}, \quad 0 < \varepsilon \leq m \leq \gamma. \tag{69}$$

Besides, let $k = \frac{2}{1+\varepsilon}$, then $0 < k < 2$. Thus according to the Sobolev imbedding theorem,

$$L^2(\Gamma) \hookrightarrow L^k(\Gamma) \hookrightarrow H^{-\varepsilon}(\Gamma). \tag{70}$$

Let $\Gamma^* := \{\mathbf{y} \in \Gamma \,||\mathbf{x} - \mathbf{y}| < \delta\}$ and $\ell_{\mathbf{x}} := \max_{\mathbf{y}\in\Gamma}\{|\mathbf{x} - \mathbf{y}|\}$, then

$$\begin{aligned} \|\ln|\mathbf{x} - \mathbf{y}|\|^2_{L^2(\Gamma)} &= \int_{\Gamma/\Gamma^*} |\ln|\mathbf{x} - \mathbf{y}||^2\, \mathrm{d}S_{\mathbf{x}} + \int_{\Gamma^*} |\ln|\mathbf{x} - \mathbf{y}||^2\, \mathrm{d}S_{\mathbf{x}} \\ &\leq \int_{\Gamma/\Gamma^*} \max\left\{\left|\ln \ell_{\mathbf{y}}\right|, |\ln\delta|\right\}^2\, \mathrm{d}S_{\mathbf{x}} + \delta\,|\ln\delta|^2 + 2\delta\,|\ln\delta| + 2\int_{\Gamma^*} \mathrm{d}S_{\mathbf{x}} \\ &\leq \mathrm{mes}(\Gamma)\left(\max\left\{\left|\ln \ell_{\mathbf{y}}\right|, |\ln\delta|\right\}^2\right) + \delta\,|\ln\delta|^2 + 2\delta\,|\ln\delta| + 2\,\mathrm{mes}(\Gamma). \end{aligned}$$

Hence using (70) yields

$$\|\ln|\mathbf{x} - \mathbf{y}|\|_{H^{-\varepsilon}(\Gamma)} \leq C(\delta). \tag{71}$$

Substituting (69) and (71) into (68), the proof is completed. □

Remark 4.4. Since the flux $\partial u/\partial \mathbf{n}$ is discontinuous through Γ, we cannot obtain the error of $|\nabla u(\mathbf{x}) - \nabla u_h(\mathbf{x})|$ inside the neighborhood of Γ.

5. Numerical experiments

To demonstrate the validity of our method, we present some numerical experiments. In order to compare the results, the solutions of the examples under consideration can be found explicitly.

In all examples, the polynomial basis is chosen as a quadratic basis, that is $\beta = 2$ in (4). Besides, the weight function is chosen as the following spline,

$$w(d) = \begin{cases} 1 - 6d^2 + 8d^3 - 3d^4, & d \leq 1, \\ 0, & d > 1, \end{cases}$$

where $d = |\mathbf{x} - \mathbf{x}_i|\, h$, h is the radius of the influence domain of boundary points. In all examples, h is taken to be $2.5\bar{d}$, with $\bar{d}$ as the nodal spacing.

Assume that $\Omega = [-1, 1] \times [-1, 1]$, in the following we consider an interior problem and an exterior problem. The problems and the explicit expressions of the solutions are given in Table 1.

The exact and numerical solutions for u and its derivatives are presented in Figs. 1 and 2. In this analysis, we employed 32 regular distributed nodes on the boundary. Results for potentials are accurate for both examples. The derivatives, however, indicate considerable error for points close to the boundary. This is the main pitfall of the proposed method and other methods based on BIEs, which may be overcame by increasing the number of boundary nodes in the vicinity of the calculation point. Anyway, this remains a subject of the further studies.

Numerical results with both random and uniformly distributed nodes are also performed for the two examples. In the former case, nodes are generated by adding a random perturbation of value $0.30\bar{d}$ to a uniform grid with $\bar{d}$-spacing with $\bar{d} = 0.125$. Maximal absolute errors in both cases are listed in Tables 2 and 3. Comparing two cases for random and uniformly spaced nodes, we can see that difference between their results is very small.

To study the convergence of the method, four different regular nodes arrangements of 2, 4, 8 and 16 nodes on each edge have been used. The convergence rates are plotted with respect to different Sobolev norms in Figs. 3 and 4, which show the numerical computational is doing better than the estimate.

For investigating the behavior of points far away from the boundary and near the boundary, the values of the numerical approximations of the potential u and its derivatives at some inner points are given in Tables 4 and 5. In this analysis, the boundary nodes are equally spaced on the boundary. From the two tables, it is true that the error decreases with the decrease

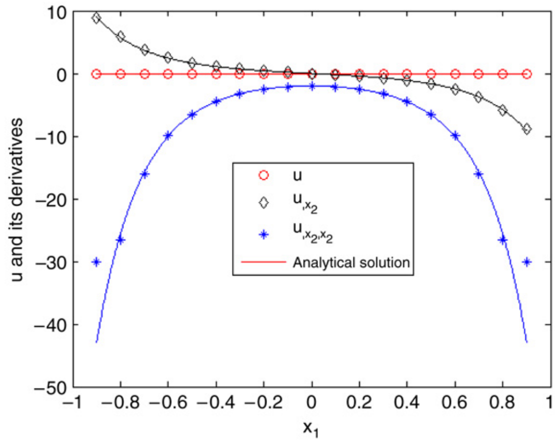

Fig. 1. Results of u and its derivatives for example 1 along the line $x_1 = x_2$.

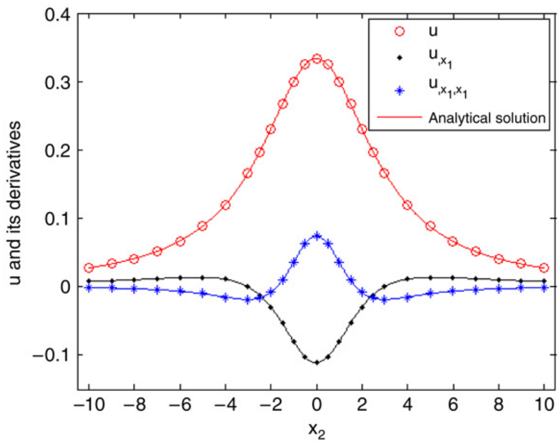

Fig. 2. Results of u and its derivatives for example 2 along the line $x_1 = 3$.

Table 2
Maximal absolute errors for example 1.

Node distribution	$\max \lvert u - u_h \rvert$	$\max \lvert u_{,x_1} - u_{h,x_1} \rvert$	$\max \lvert u_{,x_2} - u_{h,x_2} \rvert$	$\max \lvert u_{,x_1x_1} - u_{h,x_1x_1} \rvert$	$\max \lvert u_{,x_2x_2} - u_{h,x_2x_2} \rvert$
Uniform	8.6625E−5	2.4062E−3	2.4062E−3	3.7285E−2	3.7285E−2
Random	1.5924E−4	3.0456E−3	3.0456E−3	6.9859E−2	6.9859E−2

Table 3
Maximal absolute errors for example 2.

Node distribution	$\max \lvert u - u_h \rvert$	$\max \lvert u_{,x_1} - u_{h,x_1} \rvert$	$\max \lvert u_{,x_2} - u_{h,x_2} \rvert$	$\max \lvert u_{,x_1x_1} - u_{h,x_1x_1} \rvert$	$\max \lvert u_{,x_2x_2} - u_{h,x_2x_2} \rvert$
Uniform	2.2056E−5	1.8465E−5	9.3716E−6	8.5251E−6	8.5251E−6
Random	2.9503E−5	2.5503E−5	1.3046E−5	1.3897E−5	1.3897E−5

of the radii of the influence domain of boundary point. The numerical convergence orders of u and its derivatives match our theoretical results for points far away from the boundary. While points lie in the neighborhood of Γ, the numerical results of u also confirm the theoretical error statements.

6. Conclusions

A GBNM is presented in this paper, which based on BIEs and scattered nodes on the boundary for boundary value problems. One of the difficulties encountered in using the MLS approximation is the shape function lacking the delta function property. Thus many special strategies are introduced in many meshless methods, such as BNM and EFGM, for satisfying boundary conditions. In this paper, an equivalent variational form of BIE is used for the solution of boundary value

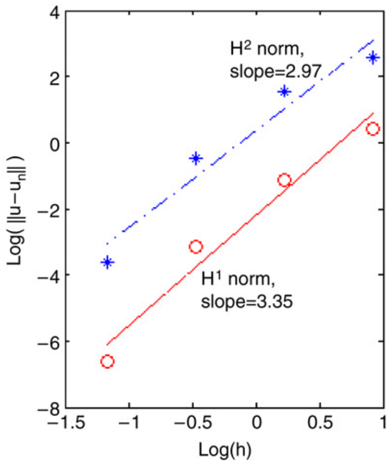

Fig. 3. The convergence rates for example 1.

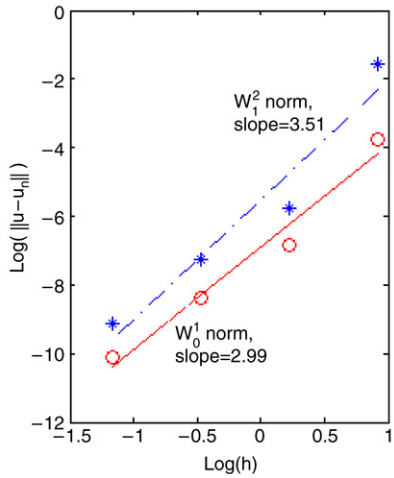

Fig. 4. The convergence rates for example 2.

Table 4
Approximations and convergence rates of the solutions for example 1.

x_1, x_2		Numerical solutions				Exact solutions	Rates
		$N = 8$	$N = 16$	$N = 32$	$N = 64$		
	u	−0.0476595679	−0.0394598707	−0.0399896993	−0.0399900080	−0.0399893342	5.10
0.0, 0.2	$u_{,x_1}$	−0.0198990473	−0.0156664816	−0.0160031771	−0.0159963861	−0.0159957337	4.31
	$u_{,x_1x_1}$	2.3844803869	1.9624989965	1.9919674706	1.9920350074	1.9920019199	5.03
	u	−0.1889036355	−0.2635511950	−0.2432428611	−0.2434937070	−0.2434926608	5.33
0.4, −0.8	$u_{,x_2}$	0.1457467610	0.6955418577	0.5494645852	0.5535411882	0.5535349960	5.31
	$u_{,x_2x_2}$	2.4990594433	−0.2180482372	0.7913095332	0.72923627196	0.7294211025	4.36
0.99, 0.0	u	1.1762456938	0.7341990809	0.8211109666	0.8304866846	0.8305530686	4.04
0.999, 0.0	u	1.1822859530	0.7383169664	0.8275294068	0.8389278100	0.8403892400	2.66
0.9999, 0.0	u	1.1826498692	0.7386949890	0.8279365790	0.8393737270	0.8413629129	2.52

problems. A key advantage with the variational formulation is the system matrices are symmetric. Besides, via multiplying the boundary function by a test function and integrating over the boundary, boundary conditions can be implemented exactly. Moreover, as in FEM and BEM for elliptic boundary value problems, also in GBNM, the variational formulation provides the basic mathematical foundation for rigorous error and convergence analysis.

The error estimates for GBNM in Sobolev spaces by means of pseudo-differential operator theory have been presented, which show that the error bound of the numerical solution is directly related to the radii of the influence domain of boundary point. Besides, far away from the boundary, the error of potential u and its derivatives possess L^∞-superconvergence. While

Table 5
Approximations and convergence rates of the solutions for example 2.

x_1, x_2		Numerical solutions				Exact solutions	Rates
		$N = 8$	$N = 16$	$N = 32$	$N = 64$		
	u	0.0062732269	0.0061083390	0.0060997912	0.0060979588	0.0060975610	2.91
2.0, 2.0	$u_{,x_1}$	0.0030600571	0.0029796733	0.0029755072	0.0029746140	0.0029744200	2.94
	$u_{,x_1x_1}$	−0.0001129655	−0.0001099272	−0.0001097683	−0.0001097344	−0.0001097271	3.39
	u	0.0197964409	0.0192657095	0.0192379720	0.0192320482	0.0192307692	2.86
18.0, 10.0	$u_{,x_2}$	−0.0038099896	−0.0037051958	−0.0036996549	−0.0036984772	−0.0036982249	2.86
	$u_{,x_2x_2}$	0.0010853732	0.0010546443	0.0010529946	0.0010526458	0.0010525717	2.89
1.01, 0.0	u	0.8919230023	0.9883560086	0.9910803471	0.9906429660	0.9900990099	2.33
1.001, 0.0	u	0.8909427271	0.9893444106	0.9945783365	0.9970827688	0.9990009990	1.86
1.0001, 0.0	u	0.8907458298	0.9891803973	0.9944463632	0.9970128455	0.9999000100	1.67

inside the neighborhood of Γ, the error of u are also established. Some numerical experiments have been given to confirm the theoretical results.

Much additional work remains to be done, such as numerical computations that involve engineering problems and complex geometrical objects and adaptively procedures, but the results presented in this article show that the proposed method has a great potential to become a very competitive method for the solution of a wide range of boundary value problems.

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