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Note

(*B*)-Geometries and flocks of hyperbolic quadrics

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Abstract

We give a characteristic-free proof of the classification theorem for flocks of hyperbolic quadrics of PG$(3,q)$.

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1. Introduction

A *flock* of the hyperbolic quadric $\mathcal{Q}$ of the finite projective space PG$(3,q)$ is a partition of $\mathcal{Q}$ consisting of $q+1$ irreducible conics. In [12], Thas showed that all flocks of $\mathcal{Q}$ are linear if q is even, and that $\mathcal{Q}$ has non-linear flocks (called *Thas flocks*) if q is odd. Further, in [13], he showed that for $q = 3, 7$ and $q \equiv 1 \bmod 4$ $\mathcal{Q}$ has only (up to a projectivity) the linear flock and the Thas flock.

For $q = 11, 23, 59$ other flocks of $\mathcal{Q}$ were discovered, independently, by Bader [1], Baker and Ebert (for $q = 11, 23$) [4], Bonisoli [6] and Johnson [11]. Since these three flocks are related to exceptional near fields, these flocks are called *exceptional flocks*.

Finally, flocks of $\mathcal{Q}$, q odd, were classified by Bader and Lunardon [3]: *Every flock of* $\mathcal{Q}$, *q odd, is linear, a Thas flock or one of the exceptional flocks.*

Bonisoli and Korchmàros presented in [8] another proof of the above classification theorem, always for q odd. Both proofs use the following fundamental theorem by Thas in [13]: *if q is odd, each plane containing a conic of a flock of* $\mathcal{Q}$ *is the axis of an involutorial homology which preserves the flock.*

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The key result to prove Thas' Theorem is Theorem 1 in [13] which is actually an application of a generalization of Segre's Lemma of Tangents. The main ideas of Theorem 1, which holds for every q, are used in this paper to provide a geometric characteristic-free proof of the classification theorem of flocks of $\mathcal{Q}$.

A *maximal exterior set* (MES) of $\mathcal{Q}$ is a set of $q+1$ points of $\mathrm{PG}(3,q)$ such that the line joining any two of them has no point in common with $\mathcal{Q}$. The polar planes, with respect to $\mathcal{Q}$, of the points of a MES define a flock, and conversely.

In the recent paper [2], Bader et al. noted a correspondence between regular subgroups of $\mathrm{PGL}(2,q)$ and MES of $\mathcal{Q}$ by using a matrix model of $\mathrm{PG}(3,q)$, q odd. They got this result as a consequence of the result by Bonisoli and Korchmàros in [8, p. 296].

In [2] was also stated a close connection between flocks and (B)-geometries satisfying certain configurational properties, known as *rectangle conditions* [5,7].

In [5], Benz showed that each (B)-geometry can be associated with a suitable permutation set. In [7], Bonisoli stated a necessary and sufficient condition for such a permutation set to be a group.

The latter result is the starting point of our work. In this paper, we directly show that every flock of $\mathcal{Q}$ yields a (B)-geometry whose associated permutation set is actually a group.

2. Hyperbolic quadrics of $\mathrm{PG}(3,q)$ and (B)-geometries

Let $\mathbf{P}$ be a non-empty set and Σ, $\mathcal{L}^+$, $\mathcal{L}^-$ three non-empty sets of subsets of $\mathbf{P}$. The elements of $\mathbf{P}$ are called *points*, the element of Σ *sections* (or *circles*) and the elements of $\mathcal{L}^+$ resp. $\mathcal{L}^-$ are called *positive generators* resp. *negative generators*.

Point-section and point-generator incidence is simply given by $\in$ in the natural way.

For $P, Q \in \mathbf{P}$ we say that P and Q are *plus parallel* and write $P||_+Q$ if P and Q lie on the same positive generator; we say that P and Q are *minus parallel* and write $P||_-Q$ if P and Q lie on the same negative generator. The structure $\mathcal{B} = (\mathbf{P}, \mathcal{L}^+, \mathcal{L}^-; \Sigma)$ is called a (B)-*geometry* if the following properties hold:

(1) for any $P, Q \in \mathbf{P}$ there exists a unique $R \in \mathbf{P}$ with $P||_+R||_-Q$;
(2) for any $P \in \mathbf{P}$ and $s \in \Sigma$, there exist uniquely determined points $P_+, P_- \in s$ with $P_+||_+P||_-P_-$;
(3) there exist three pairwise non-parallel points.

A (B)-geometry $\mathcal{B}' = (\mathbf{P}', \mathcal{L}'^+, \mathcal{L}'^-; \Sigma')$ is a (B)-*subgeometry* of $\mathcal{B} = (\mathbf{P}, \mathcal{L}^+, \mathcal{L}^-; \Sigma)$ if $\mathbf{P}' \subseteq \mathbf{P}$ and the incidence relations are induced on $\mathbf{P}'$ by $\in$.

By starting with a set X with at least three elements and a permutation set G on X, we get a (B)-geometry $\mathcal{M}(G)$ as follows, see [5,7]: the points of $\mathcal{M}(G)$ are the elements of the cartesian product $X \times X$. The sections of $\mathcal{M}(G)$ are the elements of

G, i.e. for every $s \in G$ we define

$$s = \{(x, x^s) : x \in X\}.$$

If a is an element of X we define $(a)^+ = \{(a, y) : y \in X\}$ and $(a)^- = \{(x, a) : x \in X\}$; we set $\mathscr{L}^+ = \{(a)^+ : a \in X\}$ and $\mathscr{L}^- = \{(a)^- : a \in X\}$. Point-section incidence and point-generator incidence is simply given by $\in$ in natural way.

Conversely, let $\mathscr{B} = (\mathbf{P}, \mathscr{L}^+, \mathscr{L}^-; \Sigma)$ be a (B)-geometry and s be a section of $\mathscr{B}$. For any $t \in \Sigma$ and $P \in s$, let $T \in t$ such that $T||_+P$. Then $P^t \in s$ is defined to be the (unique) point of s with $P^t||\_T$. Hence, the set Σ defines a permutation set G on s. It was shown in [5] that $\mathscr{B}$ can be described as the (B)-geometry $\mathscr{M}(G)$ associated with G. Thus, it is always possible to identify sections of $\mathscr{B}$ with elements of G.

Two (B)-geometries $\mathscr{B} = (\mathbf{P}, \mathscr{L}^-, \mathscr{L}^-; \Sigma)$ and $\mathscr{B}' = (\mathbf{P}, \mathscr{L}'^+, \mathscr{L}'^-; \Sigma')$ are *isomorphic* if there is a bijection

$$\sigma: \ \mathbf{P} \to \mathbf{P}'$$

such that

$$P \|_\varepsilon Q \Leftrightarrow P^\sigma \|'_\varepsilon Q^\sigma,$$

$$s \in \Sigma \Leftrightarrow s^\sigma \in \Sigma'$$

for all $P \in \mathbf{P}$ and $s \in \Sigma$; here $\varepsilon = \pm$. Two (B)-geometries $\mathscr{M}(G)$ and $\mathscr{M}(G')$ defined by (X, G) and (X', G'), respectively, are isomorphic if and only if there are bijections

$$\alpha, \beta : X \to X'$$

such that

$$t \in G \Leftrightarrow \alpha^{-1} t \beta \in G',$$

see [5, Theorem 2].

Remark. If $\mathscr{B}$ is described as the (B)-geometry $\mathscr{M}(G)$ associated with G, we see that G contains the identity. For an arbitrary pair (X, G) it need not be true that G contains the identity permutation. However, the latter result implies that up to an isomorphism one can always assume $\mathrm{id} \in G$, as we do in the following.

An ordered quadruple (P_1, P_2, P_3, P_4) of points is a *rectangle* in $\mathscr{B}$ if the relations

$$P_1||\_P_2, \quad P_1||_+P_3, \quad P_2||_+P_4, \quad P_3||\_P_4$$

hold.

Let k be a non-negative integer. A (B)-geometry satisfies the kth *rectangle condition* if, for every $k+1$ rectangles $(P^j_1, P^j_2, P^j_3, P^j_4)$, $j = 1, \ldots, k+1$, with $P^1_i, \ldots, P^{k+1}_i$ pairwise distinct points, $i = 1, 2, 3$ and all lying on a common section s_i, then $P^1_4, \ldots, P^{k+1}_4$ are pairwise distinct points and there exists a section s_4 through all of them.

If this is the case, we say that the section s_4 closes the rectangles $(P^j_1, P^j_2, P^j_3, P^j_4)$, $j = 1, \ldots, k+1$.

Theorem 2.1 (Bonisoli [7]). *Let $\mathcal{B}$ be a (B)-geometry such that if P, Q and R are pairwise non-parallel points then there exists at most one section which is incident with all of them. Let G be the permutation set defined by $\mathcal{B}$ with $\mathrm{id} \in G$. Then G is a group if and only if the 3rd rectangle condition holds in $\mathcal{B}$.*

If G is such a group, for any $s_1, s_2 \in G$, we define the section $s_1 \circ s_2 \in G$ to be the section s_4 which closes the rectangles $(P_1^j, P_2^j, P_3^j, P_4^j)$, $j = 1, 2, 3, 4$ with $P_1^j \in \mathrm{id}$, $P_2^j \in s_1$, $P_3^j \in s_2$.

The hyperbolic quadric $\mathcal{Q}$ of $\mathrm{PG}(3, q)$ (or, more generally, of $\mathrm{PG}(3, K)$, K a field) defines a (B)-geometry in the following way:

Let $\mathbf{P}$ be the set of points of $\mathcal{Q}$, $\mathcal{R}^+$ and $\mathcal{R}^-$ the two reguli of $\mathcal{Q}$ and Σ the set of non-tangent plane sections of $\mathcal{Q}$. It is easily seen that $\mathcal{H} = (\mathbf{P}, \mathcal{R}^+, \mathcal{R}^-; \Sigma)$ turns out to be a (B)-geometry.

Denote by (X_0, X_1, X_2, X_3) the homogeneous projective coordinates in $\mathrm{PG}(3, q)$. Coordinates are chosen in such a way that the hyperbolic quadric $\mathcal{Q}$ of $\mathrm{PG}(3, q)$ has equation $X_0X_3 = X_1X_2$. So, the reguli of $\mathcal{Q}$ are $\mathcal{R}^+ = \{\ell^+_{(a,b)} : (a, b) \in K^2 \backslash \{(0, 0)\}\}$ and $\mathcal{R}^- = \{\ell^-_{(a,b)} : (a, b) \in K^2 \backslash \{(0, 0)\}\}$, where

$$\ell^+_{(a,b)} = \{(\lambda a, \mu a, \lambda b, \mu b) : (\lambda, \mu) \in \mathbf{F}_q^2 \backslash \{(0, 0)\}\}$$

and

$$\ell^-_{(a,b)} = \{(\lambda a, \lambda b, \mu a, \mu b) : (\lambda, \mu) \in \mathbf{F}_q^2 \backslash \{(0, 0)\}\}.$$

Denote by $\perp$ the polarity defined by $\mathcal{Q}$. In the matrix model of $\mathrm{PG}(3, q)$, any point $P = (p_0, p_1, p_2, p_3)$ will be represented by its 2×2 coordinates matrix $\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$. So, $\mathcal{Q} = \{P \in \mathrm{PG}(3, q) : \det(P) = 0\}$. If q is even, the set of points of $\mathrm{PG}(3, q)$ not on $\mathcal{Q}$ represents $\mathrm{PSL}(2, q)$. If q is odd, the set $\{P \in \mathrm{PG}(3, q) : \det(P)$ is a non-zero square$\}$ represents $\mathrm{PSL}(2, q)$ and $\{P \in \mathrm{PG}(3, q) : \det(P)$ is a non-square$\}$ represents the coset of $\mathrm{PSL}(2, q)$ in $\mathrm{PGL}(2, q)$ [2].

The point $I = (1, 0, 0, 1)$ gives the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $I^\perp \cap \mathcal{Q}$ is the identity in the permutation set G defined by Σ.

Further, any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PG}(3, q)$, $A \notin \mathcal{Q}$, defines the collineation

$$\begin{array}{rccc} \rho_A : & \mathrm{PG}(3, q) & \to & \mathrm{PG}(3, q) \\ & P & \mapsto & A \cdot P \end{array}$$

where $\cdot$ is the standard matrix product.

Proposition 2.2 (Bader et al. [2]). *The collineation ρ_A has the following properties*:

1. *$\rho(A)$ fixes $\mathcal{Q}$ setwise and $\mathcal{R}^-$ linewise.*

2. $\rho_A(P^\perp) = (A \cdot P)^\perp$, *for every point* P *of* $\mathrm{PG}(3,q)$, P *not on* $\mathcal{Q}$. *Further,* $\rho_A(P^\perp)$ *intersects* $\mathcal{Q}$ *in the section* $A^\perp \circ P^\perp$.

Proof. (1) [2, Lemma 4.1].

(2) The collineation ρ_A is defined by the matrix

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

of $\mathrm{PG}(3,q)$. Direct calculations show that $\rho_A(P^\perp) = (A \cdot P)^\perp$ for every $P \in \mathrm{PG}(3,q)$ not on $\mathcal{Q}$. In particular, $\rho_A(I^\perp) = A^\perp$. Since ρ_A fixes $\mathcal{R}^-$ linewise, it is easily seen that $\rho_A(P^\perp)$ meets $\mathcal{Q}$ in the section which closes all rectangles (P_1, P_2, P_3, P_4), with $P_1 \in I^\perp$, $P_2 \in A^\perp$ and $P_3 \in P^\perp$. By definition, this is the section $A^\perp \circ P^\perp$. □

Corollary 2.3. *The permutation set* G *is actually a group isomorphic to* $\mathrm{PGL}(2,q)$. *Such an isomorphism induces a 1–1 correspondence between subgroups of* $\mathrm{PGL}(2,q)$ *and* (B)-*subgeometries of* $\mathcal{H}$ *containing a fixed section corresponding to the identity and satisfying the* 3*rd rectangle condition.*

3. Flocks of $\mathcal{Q}$ and (B)-subgeometries of $\mathcal{H}$

It was noted in [2] that an exterior set of $\mathcal{Q}$ (i.e. a set of points of $\mathrm{PG}(3,q)$ such that a line joining any two of them misses $\mathcal{Q}$) is a subset of $\mathrm{PGL}(2,q)$ in the matrix model of $\mathrm{PG}(3,q)$. In particular, an MES is a regular subset of $\mathrm{PGL}(2,q)$, and conversely. Actually, the representation of the exceptional flocks in [10] gives the groups A_4, S_4, A_5 of $\mathrm{PGL}(2,q)$, for $q = 11, 23, 59$, respectively.

In this section, we give a geometric characteristic-free proof of the classification theorem of flocks of $\mathcal{Q}$ by proving that every flock of $\mathcal{Q}$ defines a (B)-subgeometry of $\mathcal{H}$ which satisfies the 3rd rectangle condition.

Let $F = \{C_1, \ldots, C_{q+1}\}$ be a flock of $\mathcal{Q}$ and $X = \{x_1, \ldots, x_{q+1}\}$ the MES defined by F. Consider the (B)-subgeometry $\mathcal{F} = (\mathbf{P}, \mathcal{R}^+, \mathcal{R}^-, F)$ of $\mathcal{H}$.

For each conic $C = \pi \cap \mathcal{Q}$ of F with $\pi : aX_0 + bX_1 + cX_2 + dX_3 = 0$, $\rho_{A^{-1}}(\pi)$ is the plane $\pi_0 = I^\perp$, where $A = \pi^\perp$. Thus, we can always suppose that $I \in X$.

Theorem 3.1. *The* (B)-*subgeometry* $\mathcal{F}$ *satisfies the* 3*rd rectangle condition.*

Proof. Let C_1, C_2 and C_3 be three sections of $\mathcal{F}$, with $C_2 \neq C_1 \neq C_3$. From Theorem 2.1 there exists a section $C = \pi \cap \mathcal{Q}$ of $\mathcal{H}$ which closes all rectangles (P_1, P_2, P_3, P_4), with $P_i \in C_i$, $i = 1, 2, 3$.

Since $|C_1| = q + 1$, $|F \backslash \{C_2\}| = q$ and $P_4 \notin C_2$ for all such rectangles, there are two distinct rectangles (P_1, P_2, P_3, P_4) and (P'_1, P'_2, P'_3, P'_4) with $P_i, P'_i \in C_i$, $i = 1, 2, 3$, such that P_4 and P_4' are on the same conic, say C_4, of F. To get the result it suffices to prove that the plane π_4 of C_4 shares one more point with the plane π of C.

Coordinates are chosen in such a way that $P_1 = (1,0,0,0)$, $P_2 = (0,0,1,0)$, $P_3 = (0,1,0,0)$, $P_4 = (0,0,0,1)$. Denote by L_1 the tangent line of C_1 at P_1 and by L_4 the tangent line of C_4 at P_4. By arguing as in the proof of [13, Theorem 1], with easy calculations, we get that either L_1 meets L_4 or L_4 meets the line P_2P_3. In both cases, the intersection point belongs to planes π and π_4 and the result follows. □

Corollary 3.2. *A MES of $\mathcal{Q}$ is, up to a collineation, a (regular) subgroup of order $q+1$ of* PGL$(2,q)$.

By arguing as in Proposition 18 in [8] (which holds for every q) it is possible to give another proof of the following classification theorem.

Theorem 3.3. *Every flock of $\mathcal{Q}$ is linear, a Thas flock or one of the exceptional flocks.*

Proof. Let X be the MES defined by a flock F of $\mathcal{Q}$ in the matrix model of PG$(3,q)$. By checking in the list of subgroups of PGL$(2,q)$ given in [9,14] one sees that the possibilities for X are:

(i) the cyclic group C_{q+1};
(ii) the dihedral group $D_{\frac{q+1}{2}}$, q odd;
(iii) A_4, $q = 11$;
(iv) S_4, $q = 23$;
(v) A_5, $q = 59$.

In each case, the subgroups form a single conjugacy class in PGL$(2,q)$ and so the corresponding MESs are equivalent. From Corollary 2.3, X acts on itself (viewed as a subset of points of PG$(3,q)$) via the matrix multiplication. In case (i), the orbits of X consist of the lines of a regular spread containing $\mathcal{R}^-$. In this case, the corresponding flock is linear.

In case (ii), q odd, the orbit of any point of PG$(3,q)$ not on $\mathcal{Q}$ consists of $(q+1)/2$ points on a line L and of $(q+1)/2$ points on $L^\perp$. In this case, the corresponding flock is a Thas flock.

From [6,10] in cases (iii), (iv) and (v), we get the exceptional MES. □

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