Scattering and the Levandosky–Strauss conjecture for fourth-order nonlinear wave equations

Benoît Pausader

Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
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Abstract

We investigate scattering theory in the energy space for fourth-order nonlinear defocusing wave equations and prove the Levandosky–Strauss conjecture stating that scattering holds true for such equations and arbitrary initial data.
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0. Introduction

There has been an increasing activity in recent years on models involving nonlinear fourth-order partial differential equations. We investigate in the sequel scattering theory for nonlinear wave equations of fourth order in $\mathbb{R}^n$, $n \geq 1$. The fourth-order nonlinear wave equation we discuss in this paper is often referred to in the mathematics and physics literature as the nonlinear beam equation but also, see, for instance, the book by Peletier and Troy [28], as the Bretherton equation. It is written as

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = \lambda |u|^{p-1}u,$$

where $m > 0$ is a positive real number, $\Delta = \text{div} \nabla$ is the classical Laplace operator, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. The equation is said to be defocusing when $\lambda < 0$ and focusing when $\lambda > 0$. At a first
glance, (0.1) is a formal fourth-order extension of the classical Klein–Gordon equation, but it also inherits a Schrödinger structure because of the decomposition \( \partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta) \). However, it can be noted that the equation satisfies neither finite speed propagation nor mass conservation, and this turns out to be a painful source of difficulties. The original Bretherton equation, written down for \( n = 1 \) by Bretherton [5], arose in the study of weak interactions of dispersive waves. A similar equation for \( n = 2 \) was proposed in Love [23] for the motion of a clamped plate. The equation was discussed in Levine [20]. Recent developments on (0.1) were established by Levandosky [17,18], and Levandosky and Strauss [19]. We also refer to Berger and Milewski [2], Berloff and Howard [3], Holm and Lynch [11], Lazer and McKenna [16], Lin [21], and McKenna and Walter [24,25] for closely related references.

As already mentioned, we are concerned in this paper with scattering theory for the fourth-order wave equation (0.1). A rough definition of scattering is that solutions of the equation can be approximated by solutions of a model equation, in our case the linear equation, when time becomes infinite. A more precise definition is in Section 1. Abstract scattering theory, in the semigroup setting, was developed in Strauss [29,30]. In what follows we let \( H^2 \) be the Sobolev space of functions in \( L^2 \) with two derivatives in \( L^2 \). Also we let \( 2^\circ \) be given by

\[
2^\circ = +\infty \quad \text{if } n \leq 4 \quad \text{and} \quad 2^\circ = \frac{2n}{n-4} \quad \text{if } n \geq 5.
\]

As is well known, \( 2^\circ \) is the critical exponent for the embedding of \( H^2 \) into Lebesgue’s spaces when \( n \geq 5 \). Scattering for low energy initial data, arbitrary \( \lambda \), and when \( 1 + \frac{3}{n} \leq p < 2^\circ - 1 \) was established by Levandosky [18]. Levandosky and Strauss [19] then conjectured that scattering should also hold true for such \( p \) and arbitrary initial data in the defocusing case. We prove the Levandosky–Strauss conjecture when \( n \geq 5 \).

Our paper is organized as follows. We state our result in Section 1 and fix notations in Section 2. We prove local and global Strichartz estimates in Section 3. While local Strichartz estimates can be obtained by exploiting the sole Schrödinger structure of the equation, we get the global estimates by using recent advances in Levandosky [18] and material about oscillatory integrals in Kenig, Ponce and Vega [15]. A general scattering criterion, in the spirit of the one in Tao and Visan [33], is developed in Section 4. Frequency localization is proved in Section 5. What we refer to as almost finite speed propagation is established in Section 6. At last we prove the Levandosky–Strauss conjecture in Section 7 by using the material in the preceding sections and a Morawetz type estimate established in Levandosky and Strauss [19].

1. Statement of the result

We let \( \mathcal{E} = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) be the energy space associated with (0.1), and for \( I \) an interval, we let

\[
\mathcal{E}_I = C(I,H^2) \cap C^1(I,L^2) \cap C^2(I,H^{-2}).
\]  

We say that \( u \) is a solution in \( I \) of the nonlinear fourth-order equation (0.1) if \( u \in \mathcal{E}_I \) and \( u \) solves (0.1) in \( H^{-2} \). The linear equation associated to (0.1) is written as

\[
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = 0.
\]  

(1.2)
Let \((u_0, u_1) \in \mathcal{E}\). Then there exists a unique solution \(\omega \in \mathbb{R}_+\) of (1.2) with Cauchy data \((u_0, u_1)\). We let \(E_0\) be the linear energy associated with the linear equation (1.2), and \(E\) be the energy associated with the nonlinear equation (0.1). For \((u, v) \in \mathcal{E}\) we then have that

\[
E_0(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} (v^2 + (\Delta u)^2 + mu^2) \, dx, \quad \text{and}
\]

\[
E(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} (v^2 + (\Delta u)^2 + mu^2) \, dx - \frac{\lambda}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx. \tag{1.3}
\]

We equip \(\mathcal{E}\) with the scalar product whose polar form is \(E_0\). This gives the usual Hilbert structure on \(\mathcal{E}\). In what follows we say that there is scattering in forward time for \((u_0, u_1)\) if the two following conditions hold true:

(i) the solution \(u\) of (0.1) with Cauchy data \((u_0, u_1)\) is defined on the whole of \(\mathbb{R}_+\), and
(ii) there exists a unique couple \((u_0^+, u_1^+) \in \mathcal{E}\) such that

\[
\| (u(t), u_t(t)) - (\omega(t), \omega_t(t)) \|_E \to 0 \tag{1.4}
\]

as \(t \to +\infty\), where \(\omega(t)\) is the solution of the linear equation with Cauchy data \((u_0^+, u_1^+)\).

In the sequel we refer to \((u_0^+, u_1^+)\) as the scattering pair associated to \((u_0, u_1)\). Given a set \(\mathcal{F} \subset \mathcal{E}\) such that scattering in forward time holds true for any initial data in \(\mathcal{F}\), we define the wave operator \(W_+ : \mathcal{F} \to \mathcal{E}\) by

\[
W_+(u_0, u_1) = (u_0^+, u_1^+), \tag{1.5}
\]

where \((u_0^+, u_1^+)\) is such that (1.4) holds. Note that \(W_+\) is often referred to in the mathematical literature as \(W_+^{-1}\). Similarly, we say that there is scattering in backward time for \((u_0, u_1)\) if there is scattering in forward time for \((u_0, -u_1)\). At last, we refer to scattering without any specificity when scattering holds true both in backward and forward time. The main result of this paper is concerned with the Levandosky–Strauss conjecture [19]. As already mentioned, the Levandosky–Strauss conjecture asserts that scattering holds true when (0.1) is defocusing, in other words when \(\lambda < 0\) in (0.1), and when \(1 + \frac{8}{n} < p < 2^* - 1\). We prove the conjecture when \(n \geq 5\).

**Theorem.** Let \(n \geq 5\), \(\lambda < 0\), and \(1 + \frac{8}{n} < p < 2^* - 1\). Scattering for (0.1) holds true for any initial data \((u, v) \in \mathcal{E}\), and \(W_+\) in (1.5) realizes an homeomorphism from \(\mathcal{F}_R\) onto \(\mathcal{B}_R\) for all \(R > 0\), where \(\mathcal{F}_R\) consists of the \((u, v) \in \mathcal{E}\) such that \(E(u, v) \leq R\), and \(\mathcal{B}_R\) consists of the \((u, v) \in \mathcal{E}\) such that \(E_0(u, v) \leq R\).

The rest of the paper is devoted to the proof of the above theorem. We roughly follow the approach developed by Lin and Strauss [22] for the Schrödinger equation. However, a major difficulty with (0.1) is that it does not satisfy mass conservation. It neither satisfies finite speed propagation. Finite speed propagation is traditionally used to prove scattering for the nonlinear
Klein–Gordon equation as, for instance, in Brenner [4], and Morawetz and Strauss [27]. We overcome the difficulty by using recent ideas of Tao [31] about frequency localization. A brief sketch of the proof is as follows. We prove local and global in time Strichartz estimates in Section 3. We prove in Section 4 that, as one would have expected, strong decay implies scattering. A key point we establish in Sections 5 and 6 is that, in the subcritical case, (0.1) satisfies almost finite speed propagation. We prove in Section 7 that almost finite speed propagation, combined with the Morawetz type estimates in Levandosky and Strauss [19], provides strong decay of the solutions. Then it remains to remember that, as already mentioned, strong decay of the solutions implies scattering.

2. Notations

We introduce notations we use in the sequel. Given \((u_0, u_1) \in \mathcal{E}\), there exists a unique solution \(u \in \mathcal{E}\) of (1.2) such that \((u(t), u(t)) = (u_0, u_1)\). We define \(\mathcal{W}(t)\) by \((u(t), u(t)) = \mathcal{W}(t)(u_0, u_1)\) for all \(t\). In other words, \(\mathcal{W}(t)\) is the isometry semigroup associated to the skew-adjoint operator \((D(A), A)\) with \(D(A) = H^4 \times H^2 \subset \mathcal{E}\), \(A(u, v) = (v, -\Delta^2 u - mu)\). We let \(\pi_1 : \mathcal{E} \to H^2\) and \(\pi_2 : \mathcal{E} \to L^2\) be the first and second projections. We let also \(Ff = \hat{f}\) be the Fourier transform of \(f\) given by

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{i \langle y, \xi \rangle} \, dy
\]

for all \(\xi \in \mathbb{R}^n\). Let \(\psi \in C_c^\infty(\mathbb{R}^n)\) be supported in the ball \(B_0(2)\) such that \(\psi = 1\) in \(B_0(1)\), and \(0 \leq \psi \leq 1\). For any dyadic number \(N = 2^k, k \in \mathbb{Z}\), we define the following Littlewood–Paley operators:

\[
P_{\leq N} f(\xi) = \psi(\xi/N) \hat{f}(\xi),
\]

\[
P_{> N} f(\xi) = (1 - \psi(\xi/N)) \hat{f}(\xi),
\]

\[
P_N f(\xi) = (\psi(\xi/N) - \psi(2\xi/N)) \hat{f}(\xi).
\]

Similarly we define \(P_{< N}\) and \(P_{\geq N}\) by the equations \(P_{< N} = P_{\leq N} - P_N\) and \(P_{\geq N} = P_{> N} + P_N\). We adopt the convention that these operators act on couples of functions by \(P_{\leq N}(u, v) = (P_{\leq N}u, P_{\leq N}v)\), and similarly for the other operators \(P_{> N}, P_N, P_{< N},\) and \(P_{\geq N}\). These operators commute one with another. They also commute with derivative operators and with the semigroup \(\mathcal{W}(t)\). In addition they are self-adjoint and bounded on \(L^p\) for all \(1 \leq p \leq \infty\). Moreover, they enjoy the following Bernstein properties:

\[
(i) \quad \|P_{\geq N} f\|_{L^p} \leq C N^{-s} \|\nabla|^s P_{\geq N} f\|_{L^p} \leq C N^{-s} \|\nabla|^s f\|_{L^p},
\]

\[
(ii) \quad \|\nabla|^s P_{\leq N} f\|_{L^p} \leq C N^s \|P_{\leq N} f\|_{L^p} \leq C N^s \|f\|_{L^p},
\]

\[
(iii) \quad \|\nabla|^{\pm s} P_N f\|_{L^p} \leq C N^{\pm s} \|P_N f\|_{L^p} \leq C N^{\pm s} \|f\|_{L^p}
\]

for all \(s \geq 0\), and all \(1 \leq p \leq \infty\), where \(|\nabla|^s\) is the classical fractional differentiation operator, and \(C > 0\) is independent of \(f, N,\) and \(p\). When \(N = 1\), these estimates follow from straightforward computations on the convolution kernels of the operators. We recover the case of \(N\)}
arbitrary by considering the effect of dilations on these estimates. We refer to Tao [32] for more
details. Given \( a \geq 1 \), we let \( a' \) be the conjugate of \( a \), so that \( \frac{1}{a} + \frac{1}{a'} = 1 \). For short, we adopt the
convention that \( x^p = |x|^{p-1}x \).

3. Strichartz estimates

We discuss Strichartz estimates for (0.1) and start with local in time estimates in Lemma 3.1. Global in
time estimates are discussed in Lemma 3.2. Local in time estimates follow from the
sole Schrödinger structure of the equation. Following standard terminology we say that a pair
\((q, r)\) is Schrödinger admissible, for short S-admissible, if
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \quad (3.1)
\]
and \( r \) is such that \( 2 \leq r \leq +\infty \) if \( n = 1 \), \( 2 \leq r < +\infty \) if \( n = 2 \), and \( 2 \leq r \leq 2^* \) if \( n \geq 3 \), where
\( 2^* = \frac{2n}{n-2} \). Now we introduce various notions of admissible and controlling pairs.

**Definition 3.1.** For \( 2 \leq q \leq +\infty \), a pair \((q, r)\) is said to be Bretherton or beam admissible, for
short B-admissible, if \( 2 \leq r \leq +\infty \) when \( n = 1 \), \( 2 \leq r < +\infty \) when \( n = 2 \), \( 2 \leq r \leq 2^* \) when \( n \geq 3 \), and
\[
\frac{2}{q} + \frac{n}{r} = \frac{n-4}{2} \quad (3.2)
\]
with \( 0 < r < +\infty \) when \( n \geq 5 \). A pair \((p, q)\) is said to be Bretherton or beam low-admissible, for short Bl-admissible, if \( p, q \geq 2 \),
\[
\frac{4}{p} + \frac{n}{q} \leq \frac{n}{2} \quad (3.3)
\]
and \((p, q, n) \neq (2, \infty, 4)\). A pair \((p, q)\) is Bretherton or beam controlling, for short B-controlling,
if \((p, q)\) is Bl-admissible, \( q \neq \infty \), and \((p, q)\) satisfies
\[
\frac{2}{p} + \frac{n}{q} = \sigma \quad (3.4)
\]
for some \( \sigma \) such that \((n-4)/2 \leq \sigma \leq n/2\).

As a remark, if \((q, r)\) is S-admissible in the sense of (3.1) and \( 2r < n \), then \((q, r')\) is
B-admissible for \( r' = \frac{nm}{n-2r} \). Note that \( s = r' \) is the critical Sobolev exponent for the embedding
of \( H^{2,r} \) into \( L^s \), where \( H^{2,r} \) stands for the Sobolev space of functions in \( L^r \) with two
derivatives in \( L^s \). More generally, given \( s \in \mathbb{R} \) and \( p \geq 1 \), we let \( H^{s,p} = H^{s,p}(\mathbb{R}^n) \) be the usual
fractional Sobolev spaces in \( \mathbb{R}^n \). Following standard notations we let also \( H^1 = H^{1,2} \). Local in
time Strichartz estimates for (0.1) are as follows.

**Lemma 3.1.** Let \( I \subset \mathbb{R} \) be a bounded interval such that \( 0 \in I \), \( u_0 \in H^2 \), \( u_1 \in L^2 \), and \( h \in C(I, H^{-2}) \cap L^a(I, L^{b'}) \) for some S-admissible pair \((a, b)\). There exists a unique \( u \in \mathbb{E}_I \) which
solves the linear equation
\[
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = h
\]  

(3.5) in \( C(I, H^{-2}) \) with Cauchy data \( u|_{t=0} = u_0 \) and \( u_t|_{t=0} = u_1 \). Moreover it holds that \( u \in L^q(I, L^r) \) for any \( B \)-admissible pair \((q, r)\), and that

\[
\| (u, u_t) \|_{C(I, \mathcal{E})} + \| u \|_{L^q(I, L^r)} \leq C \left( 1 + |I|^{3/2} \left( \sqrt{E_0(u_0, u_1)} + \| h \|_{L^s'(I, L^{s'})} \right) \right),
\]

(3.6) where \( |I| \) is the length of \( I \), \( E_0 \) is as in (1.3), and \( C \geq 1 \) does not depend on \( u_0, u_1, h, \) and \( I \).

**Proof.** We let \( v \) solve (3.5) in \( C(I, H^{-4}) \) with Cauchy data \((0, 0)\). We let also \( w \) be such that for all \( t \), \((w(t), w_t(t)) = \mathcal{W}(t)(u_0, u_1)\). Then \( v \in C(I, L^2) \cap C^1(I, H^{-2}) \cap C^2(I, H^{-4}) \) and \( w \in \mathbb{E}_I \).

Let \( \tilde{v} = -i v_t + \Delta v \) and \( \tilde{w} = -i w_t + \Delta w \). We consider the linear Schrödinger equation

\[
iu_t + \Delta u = h.
\]

(3.7) As is easily checked, \( \tilde{v} \) solves (3.7) in \( C(I, H^{-4}) \) with Cauchy data \( \tilde{v}|_{t=0} = 0 \), and \( \tilde{w} \) solves (3.7) in \( C(I, H^{-2}) \) when \( h \equiv 0 \) with Cauchy data \( \tilde{w}|_{t=0} = -iu_1 + \Delta u_0 \). We may then apply the standard Strichartz estimates for the Schrödinger equation, as stated for instance in Cazenave [6], to \( \tilde{v} \) and \( \tilde{w} \). We refer also to Keel and Tao [14]. The Strichartz estimates for \( \tilde{v} \) give that \( \tilde{v} \in C(I, L^2) \cap L^q(I, L^r) \) for any \( S \)-admissible pair \((q, r)\), and that the \( L^q L^r \)-norm of \( \tilde{v} \) is controlled by the \( L^{a'} L^{b'} \)-norm of \( h \). This includes the choice of \((q, s)\) given by \( q = +\infty \) and \( s = 2 \). In particular, it follows that \( v \in \mathbb{E}_I \), and by considering the real and imaginary parts of \( \tilde{v} \) we also get that for any \( S \)-admissible pair \((q, s)\),

\[
\| \Delta v \|_{C(I, L^2)} + \| v_t \|_{C(I, L^2)} + \| \Delta v \|_{L^q(I, L^r)} + \| v_t \|_{L^q(I, L^r)} \leq C \| h \|_{L^{a'}(I, L^{b'})},
\]

(3.8) where \( C > 0 \), independent of \( I \), depends only on \( n \), \((a, b)\), and \((q, s)\). As a remark this implies that \( v \) solves (3.5) in \( C(I, H^{-2}) \) and not only in \( C(I, H^{-4}) \). By the control on the norm of \( v_t \) in (3.8), we can write that

\[
\| v \|_{C(I, H^2)} + \| v_t \|_{C(I, L^2)} + \| v \|_{L^q(I, H^2)} \leq C \left( 1 + |I| \right) \| h \|_{L^{a'}(I, L^{b'})},
\]

(3.9) where \( C > 0 \), independent of \( I \), depends only on \( n \), \((a, b)\), and \((q, s)\). Let \((q, r)\) be a \( B \)-admissible pair as in the statement of Lemma 3.1. When \( n \leq 4 \), by the Sobolev embedding theorem,

\[
\| v \|_{L^q(I, L^r)} \leq C |I|^{1/q} \| v \|_{C(I, H^2)} \leq C \left( 1 + |I|^{1/2} \right) \| v \|_{C(I, H^2)},
\]

(3.10) where \( C > 0 \) depends only on \( n \) and \((q, r)\). When \( n \geq 5 \), we let \( s \) be given by \( s = nr/(n + 2r) \). Then \((q, s)\) is \( S \)-admissible and \( s^2 = r \). Combining (3.9) and the Sobolev embedding theorem, we get that

\[
\| v \|_{C(I, H^2)} + \| v_t \|_{C(I, L^2)} + \| v \|_{L^q(I, L^r)} \leq C \left( 1 + |I|^{3/2} \right) \| h \|_{L^{a'}(I, L^{b'})},
\]

(3.11) where \( C > 0 \), independent of \( I \), depends only on \( n \), \((a, b)\), and \((q, r)\). Similarly, the Strichartz’s estimates for \( \tilde{w} \) give that
\[
\|w\|_{C(I,H^2)} + \|w_t\|_{C(I,L^2)} + \|w\|_{L^q(I,L^r)} \leq C(1 + |I|^{3/2})(\|u_1\|_{L^2} + \|u_0\|_{L^2} + \|\Delta u_0\|_{L^2}) \\
\leq C(1 + |I|^{3/2})\sqrt{E_0(u_0,u_1)},
\]

(3.12)

where \( C \geq 1 \), independent of \( I \), depends only on \( n, m, \) and \((q, r)\). By (3.11) and (3.12), letting \( u = v + w \), we get a solution of (3.5) in \( C(I, H^{-2}) \) with Cauchy data \( u_{t=0} = u_0 \) and \( u_{t=0} = u_1 \) which satisfies (3.6) for any \( B \)-admissible pair \((q, r)\). Uniqueness of \( u \) follows from the remark that if \( u_1 \) and \( u_2 \) are two such solutions, then \( \tilde{u} = u_2 - u_1 \) solves (3.5) with \( h = 0 \) and Cauchy data \( \tilde{u}_{t=0} = 0 \) and \( \tilde{u}_{t=0} = 0 \) so that \( \tilde{u} = 0 \). This proves Lemma 3.1. \( \square \)

As a remark, the proof of Lemma 3.1 also gives that \( u_t \in L^q(I, L^s) \) for any \( S \)-admissible pair \((q, s)\). Since \( 2 \leq s \leq 2^* \) for such pairs, and \( u \in C(I, H^2) \), we also get from the Sobolev embedding theorem that \( u \in L^q(I, L^s) \).

Local well-posedness in the energy-subcritical and in the energy-critical case for (0.1) follows from Lemma 3.1 by the standard methods developed for semilinear Schrödinger equations by Ginibre and Velo [10], Kato [12,13], and Cazenave and Weissler [8,9]. Unconditional uniqueness also holds true for (0.1). We refer to Cazenave [6] for an excellent exposition in book form on such methods. Let \( p \) be such that \( 1 \leq p \leq 2^* - 1 \) if \( n \geq 5 \), any \( 1 \leq p < \infty \) if \( n \leq 4 \). With only slight and obvious modifications with respect to the proofs in Cazenave [6], it follows from the estimates in Lemma 3.1 that for any \((u_0, u_1) \in \mathcal{E} \), there exists a unique solution \( u \in \mathbb{E}_I \) of (0.1) defined on some maximal interval \( I = (-T^-, T^+) \). For any \( B \)-admissible pair \((q, r)\), \( u \in L^q_{\text{loc}}(I, L^r) \), and the solution satisfies conservation of the energy:

\[
E(u(t), u_t(t)) = E(u_0, u_1)
\]

(3.13)

for all \( t \in I \). Moreover, we also have that if \( T^+ \neq \infty \) and \( p < 2^* - 1 \), then \( \|u(t)\|_{H^2} \to \infty \) as \( t \to T^+ \), while if \( n \geq 5 \) and \( p = 2^* - 1 \), then the blow-up arises in mixed norms and

\[
\|u\|_{L^{\frac{n+2}{n-4}}(0,T^+) \times \mathbb{R}^n} = +\infty.
\]

(3.14)

Similar statements hold true for \( T^- \). At last, well-posedness holds true in the sense that if \((u_0^k, u_1^k)\) is a sequence in \( \mathcal{E} \) that converges to \((u_0, u_1) \in \mathcal{E} \), and if \( u^k \) denotes the corresponding solution of (0.1) with maximal interval \((-T^{-,k}, T^{+,k})\), then \( \lim \inf T^{+,k} \geq T^+ \), \( \lim \inf T^{-,k} \geq T^- \), and for any finite interval \( I' \subset (-T^-, T^+) \), and any \( B \)-admissible pair \((q, r)\),

\[
u^k \to u \quad \text{in } C(I', H^2) \cap C^1(I', L^2) \cap L^q(I', L^r)
\]

(3.15)
as \( k \to +\infty \). As a remark, local well-posedness has already been established by Levandosky [18] in the energy-subcritical case of (0.1). The approach in Levandosky [18] was based on the system representation of (0.1).

Local in time Strichartz estimates, as in Lemma 3.1, are powerful enough to deal with local existence. Scattering requires global in time estimates. We prove such global in time estimates in what follows. In order to do this we need to deal with a degenerate critical point in the low frequency mode. The critical point is responsible for slow decay as time goes to infinity. We overcome the difficulty thanks to a powerful estimate in Levandosky [18] for the Fourier transform of radial functions. A similar idea for fourth-order Schrödinger equations was later on used...
in Ben-Artzi, Koch, and Saut [1]. High frequencies are treated via standard stationary phase estimates from Kenig, Ponce, and Vega [15]. For \( h \in C(I, H^{-2}) \) we consider the linear equation with forcing term

\[
\frac{\partial^2 u}{\partial t^2} + \Delta u + mu = h.
\] (3.16)

The global in time Strichartz estimates we prove state as follows.

**Lemma 3.2.** Let \( I \subset \mathbb{R} \) be an interval such that \( 0 \in I \). Let \((p, q)\) be any B-controlling pair, and \((a, b)\) be any \( B\)-admissible pair as in (3.3) and (3.4). Let also \((c, d)\) be any \( S\)-admissible pair, \((u_0, u_1) \in E\), and \( h \in C(I, H^{-2}) \cap L^a(I, L^b) \cap L^c(I, L^d) \). There exists a unique \( u \in E_I \) such that \( u \) solves the linear equation (3.16) with Cauchy data \((u_0, u_1)\), and

\[
\| (u, u_t) \|_{C(I, E)} + \| u \|_{L^p(I, L^q)} \leq C \left( \| (u_0, u_1) \|_E + \| h \|_{L^a(I, L^b)} + \| h \|_{L^c(I, L^d)} \right),
\] (3.17)

where \( C \) is independent of \( u_0, u_1, \) and \( h \). Moreover, for any \( \alpha \geq 2 \), if \( u_0 \in L^a \) and \( u_1 \in H^{-2, \alpha} \), then

\[
\| u \|_{L^a} \leq C \left( |t|^{-\frac{\alpha}{2}(1-\frac{1}{a})} + |t|^{-\frac{\alpha}{2}(1-\frac{1}{a})} \right) \left( \| u_0 \|_{L^a} + \| (1 + \Delta)^{-1/2} u_1 \|_{L^a} \right),
\] (3.18)

for all \( t \neq 0 \) when \( h = 0 \), where \( C \) is independent of \( u_0 \) and \( u_1 \).

**Proof.** In order to prove the lemma, we define a “half-wave” operator \( u \mapsto T_t u \) for \( u \) in \( L^1 + L^2 \) by

\[
\mathcal{F}(T_t u)(\xi) = \exp(\sqrt[4]{1 + |\xi|^4}) \mathcal{F}(u)(\xi)
\] (3.19)

for \( \xi \in \mathbb{R}^n \), and \( t \in \mathbb{R} \). Also we define \( T^l_t \) and \( T^h_t \), the low and high frequency parts of \( T_t \), by

\[
T^l_t u = P_{\leq 2} T_t u \quad \text{and} \quad T^h_t u = P_{>1/2} T_t u.
\] (3.20)

As is easily checked, \( T_t = P_{\leq 1} T^l_t + P_{>1} T^h_t \) for all \( t \). Now we claim that there exists \( C > 0 \) depending only on \( n \) such that for any \( \alpha \geq 2 \), and any \( u \in L^a \),

\[
\| T^l_t u \|_{L^a} \leq C \left( 1 + |t|^{-\frac{\alpha}{2}(1-\frac{1}{a})} \right) \| u \|_{L^a}
\] (3.21)

for all \( t \in \mathbb{R} \). We prove (3.21) in what follows.

Let \( u \in C_c^\infty(\mathbb{R}^n) \) be a smooth function with compact support. By a crude estimate, we see that

\[
|T^l_t u(x)| \leq C \left| \int \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(y-x,\xi)} e^{it\sqrt{1+|\xi|^4}} \psi(\xi/2) u(y) d\xi dy \right|
\]

\[
\leq C \| u \|_{L^1},
\] (3.22)

where \( \psi \) is as in (2.2). It is clear that
\[ T^l_t u = (2\pi)^{-\frac{n}{2}} \left( T^l_t F^{-1} \psi(\cdot/2) \right) * u \]  

(3.23)

for all \( t \) and all \( u \). By Levandosky [18, Lemma 2.3], combined with (3.23), we then get that for \( t \) such that \( |t| \geq 1 \),

\[ \| T^l_t u \|_{L^\infty} \leq C |t|^{-\frac{n}{4}} \| u \|_{L^1}, \]  

(3.24)

where \( C > 0 \) is independent of \( t \) and \( u \). Independently, Plancherel’s theorem asserts that \( T^l_t \) is bounded \( L^2 \to L^2 \). Hence \( T^l_t \) extends to an operator \( L^1 + L^2 \to L^2 + L^\infty \) and, by (3.22) and (3.24), we then get that

\[ \left\| T^l_t \right\|_{L^1 \to L^\infty} \leq C \left( 1 + |t| \right)^{-\frac{n}{4}}, \]  

and

\[ \left\| T^l_t \right\|_{L^2 \to L^2} \leq C \]  

(3.25)

for all \( t \), where \( C > 0 \) is independent of \( t \). Then (3.21) follows from (3.25) by the Riesz–Thorin theorem. This proves the above claim that (3.21) holds true.

Now that (3.21) is proved we continue with the proof of the lemma. Let \((p, q)\) and \((a, b)\) be \( B_\ell\)-admissible pairs as in (3.3). By the definition of \( P \leq N \) in (2.2), and the definition of \( T^l_t \) in (3.19), we can write that \( T^l_s T^l_t \ast f = P \leq 2 T^l_{s-t} \) and also that \( P \leq 1 T^l_s T^l_t \ast f = P \leq 1 T^l_{s-t} \). Since \( P \leq N \) is bounded on \( L^p \) for \( 1 \leq p \leq \infty \), we get with (3.25) and the \( TT^\ast\)-method of Keel and Tao [14] that there exists \( C > 0 \), independent of \( u \), such that

\[ \| P \leq 1 T^l_t u \|_{L^p(\mathbb{R}, L^q)} \leq C \| u \|_{L^2} \]  

(3.26)

for all \( u \in L^2 \), and that

\[ \left\| \int_0^t P \leq 1 T_{s-t} u(s) \, ds \right\|_{L^p(\mathbb{R}, L^q)} \leq C \| u \|_{L^2(\mathbb{R}, L^q)}, \]  

\[ \left\| \int_{\mathbb{R}} P \leq 1 T_{s-t} u(s) \, ds \right\|_{L^2} \leq C \| u \|_{L^2(\mathbb{R}, L^q)} \]  

(3.27)

for all \( u \in L^2(\mathbb{R}, L^q) \). For the reader’s convenience we briefly recall the result in Keel and Tao [14]. Let \( H \) be an Hilbert space and \( U(t) : H \to L^2 \) be such that for any \( s, t \), and any \( f \in L^1 \),

\[ \| U(t) \|_{H \to L^2} \leq C, \]  

(3.28)

and one of the two following decay estimates holds true

\[ \| U(s) U(t)^\ast f \|_{L^\infty} \leq C |t - s|^{-\sigma} \| f \|_{L^1}, \]  

or

\[ \| U(s) U(t)^\ast f \|_{L^\infty} \leq C (1 + |t - s|)^{-\sigma} \| f \|_{L^1}, \]  

(3.29)
where \( C > 0 \) and \( \sigma > 0 \) do not depend on \( s, t, \) and \( f \). Following Keel and Tao [14], define \( \sigma \)-admissible pairs \( (q, r) \) by the relations \( q, r \geq 2, (q, r, \sigma) \neq (2, \infty, 1) \), and

\[
\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}, \tag{3.30}
\]

and say that the pair is sharp \( \sigma \)-admissible if equality holds in (3.30). The result in Keel and Tao [14] then states that for any \( f \in H \) and any \( F \in L^q(\mathbb{R}, L^r) \),

\[
\| U(t)f \|_{L^q(\mathbb{R}, L^r)} \leq C \| f \|_H,
\]

\[
\left\| \int_{\mathbb{R}} U(s)^* F(s) \, ds \right\|_H \leq C \| F \|_{L^q(\mathbb{R}, L^r)}, \quad \text{and}
\]

\[
\left\| \int_{s < t} U(t)U(s)^* F(s) \, ds \right\|_{L^\tilde{q}(\mathbb{R}, L^\tilde{r})} \leq C \| F \|_{L^q(\mathbb{R}, L^r)}. \tag{3.31}
\]

for all sharp \( \sigma \)-admissible pairs \( (q, r) \) and \( (\tilde{q}, \tilde{r}) \), where \( C > 0 \) does not depend on \( f \) and \( F \), and for all \( \sigma \)-admissible pairs \( (q, r) \) and \( (\tilde{q}, \tilde{r}) \) if the second condition in (3.29) holds true. In our case we let \( H = L^2 \), \( U(t) = T^t_l \), and \( \sigma = n/4 \). Then (3.28) and the second equation in (3.29) follow from (3.25), the boundedness of \( P \leq N \), and the identity \( T^t_l T^s_l = P \leq 2 T^t_l s - t \). Then (3.26) follows from the first equation in (3.31), and by noting that \( P \leq 1 \| P \leq 2 = P \leq 1 \). The second equation in (3.27) follows from the second equation in (3.31) and again by noting that \( P \leq 1 \| P \leq 2 = P \leq 1 \). The first equation in (3.27), when the \( L^p L^q \)-norm is restricted to \( \mathbb{R}_+ \), follows from the third equation in (3.31) that we apply to \( F = 1_{\mathbb{R}_+} P \leq 1 u \), where \( 1_{\mathbb{R}_+} \) is the characteristic function of \( \mathbb{R}_+ \), and from the identity \( P \leq 1 T^t_l T^s_l = P \leq 1 T^t_l s - t \). Then we get the global \( L^p L^q \)-norm, and so the first equation in (3.27), by writing that for \( t < 0 \),

\[
\int_0^t P \leq 1 T^t_l - s u(s) \, ds = \int_{s < t} P \leq 1 T^t_l - s u(s) \, ds - T_t \int_{\mathbb{R}} P \leq 1 T^t_l - s 1_{\mathbb{R}_-} u(s) \, ds
\]

and thus, thanks to the three equations in (3.31), that

\[
\left\| \int_0^t P \leq 1 T^t_l - s u(s) \, ds \right\|_{L^p(\mathbb{R}_-, L^q)} \leq C \| u \|_{L^q(\mathbb{R}_-, L^r)} + C \left\| \int_{s < t} P \leq 2 T^t_l - s 1_{\mathbb{R}_-} u(s) \, ds \right\|_{L^2} \leq C \| u \|_{L^q(\mathbb{R}_-, L^r)}.\]
This proves (3.26) and (3.27).

In parallel to (3.21), we claim now that there exists $C > 0$ depending only on $n$ such that for $\alpha \geq 2$,

$$\| T_t^h u \|_{L^\alpha} \leq C |t|^{-\frac{n}{2} - \frac{2}{\alpha}} \| u \|_{L^\alpha'}$$

(3.32)

for all $t \in \mathbb{R} \setminus \{0\}$. We prove (3.32) in what follows.

Let $u \in C^\infty_c (\mathbb{R}^n)$ be a smooth function with compact support. We clearly have that

$$T_t^h u (x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \frac{1 - \psi(2\xi)}{\sqrt{H_\phi(\xi)}} \sqrt{H_\psi(\xi)} e^{i \xi \cdot (x - y)} \, d\xi \, dy$$

(3.33)

for all $t \in \mathbb{R}$, and all $x \in \mathbb{R}^n$, where $\phi(\xi) = \sqrt{1 + |\xi|^4}$ and $H_\psi(\xi) = |\det(\partial_{ij} \psi)|$. The phase function $\phi$ in (3.33) satisfies the assumptions of Kenig, Ponce, and Vega [15, Lemma 3.4]. With respect to the notation in Kenig, Ponce, and Vega [15], $m = 2$ and $\Omega$ is the complement of the ball of radius $1/2$. This gives that

$$\| T_t^h u \|_{L^\infty} \leq |t|^{-\frac{n}{2}} \| u \|_{L^1}$$

(3.34)

for all $t \in \mathbb{R} \setminus \{0\}$. By Plancherel’s theorem, we also have

$$\| T_t^h u \|_{L^2} \leq \| u \|_{L^2}$$

(3.35)

for all $t$. We get (3.32) from (3.34) and (3.35) by the Riesz–Thorin theorem. This proves the above claim that (3.32) holds true.

We continue with the proof of the lemma. Let $(p, q)$ and $(a, b)$ be S-admissible pairs. By noting that $T_s^h T_t^{h*} = P_{s1/2} T_s^h T_{s-t}^{h*}$ and $P_{s1/2} T_s^h T_{s-t}^{h*} = P_{s1} T_{s-t}$, and since $P_{sN}$ is bounded on $L^p$ for $1 \leq p \leq \infty$, we get with (3.34), (3.35), and the $T T^*$-method of Keel and Tao [14], that there exists $C > 0$, independent of $u$, such that

$$\| P_{s1} T_t u \|_{L^p(\mathbb{R}, L^q)} \leq C \| u \|_{L^2}$$

(3.36)

for all $u \in L^2$, and that

$$\left\| \int_0^t P_{s1} T_{s-t} u(s) \, ds \right\|_{L^p(\mathbb{R}, L^q)} \leq C \| u \|_{L^{a'}(\mathbb{R}, L^{b'})},$$

$$\left\| \int_0^t P_{s1} T_{s-t} u(s) \, ds \right\|_{L^2} \leq C \| u \|_{L^{a'}(\mathbb{R}, L^{b'})}$$

(3.37)

for all $u \in L^{a'}(\mathbb{R}, L^{b'})$. Here we proceed as above, when proving (3.26) and (3.27), with the slight differences that we only have the first equation in (3.29), that $\sigma$ needs to be changed into $\sigma = n/2$, and that we have to restrict ourselves to sharp $\sigma$-admissible pairs in the sense of Keel and Tao [14].
Now we enter more specifically into the proof of Lemma 3.2. The existence and uniqueness of the solution $u$ follow from straightforward semigroup techniques (see e.g. Cazenave and Haraux [7]). For the moment we assume that $m = 1$ and prove (3.17) and (3.18). In order to do this we use the explicit representation formula for solutions of (3.16). We compute

\[
\hat{u}(t) = \frac{e^{it\rho} + e^{-it\rho}}{2} \hat{u}_0 + \frac{e^{it\rho} - e^{-it\rho}}{2i} \hat{u}_1 + \int_0^t \frac{e^{i(t-s)\rho} - e^{-i(t-s)\rho}}{2i} \hat{h}(s) \, ds,
\]

and

\[
\partial_t \hat{u}(t) = -\frac{e^{it\rho} - e^{-it\rho}}{2i} \rho \hat{u}_0 + \frac{e^{it\rho} + e^{-it\rho}}{2} \hat{u}_1 + \int_0^t \frac{e^{i(t-s)\rho} + e^{-i(t-s)\rho}}{2} \hat{h}(s) \, ds,
\]

where $\rho = \sqrt{1 + |\xi|^4}$. As a consequence,

\[
\hat{u}(t) = \frac{1}{2}(T_t + T_{-t})u_0 + \frac{1}{2i}(1 + \Delta^2)^{-1/2}(T_t - T_{-t})u_1
\]

\[
+ \frac{1}{2i}(1 + \Delta^2)^{-1/2} \int_0^t (T_{t-s} - T_{s-t})h(s) \, ds
\]

(3.38)

and

\[
u(t) = \frac{T_t - T_{-t}}{2i}(1 + \Delta^2)^{1/2}u_0 + \frac{T_t + T_{-t}}{2}u_1 + \int_0^t \frac{T_{t-s} + T_{s-t}}{2}h(s) \, ds
\]

(3.39)

for all $t$, where $T_t$ is as in (3.19). By the decay estimates (3.21) and (3.32) we get from (3.20) and (3.38) that, in case $h = 0$, and for $\alpha \geq 2$,

\[
\|u(t)\|_{L^\alpha} \leq \|P_{\leq 1}u(t)\|_{L^\alpha} + \|P_{> 1}u(t)\|_{L^\alpha}
\]

\[
\leq C(|t|^{-\frac{\alpha}{2}}(1 - \frac{\alpha}{2}) + |t|^{-\frac{\alpha}{2}}(1 - \frac{\alpha}{2}))\|u_0\|_{L^\alpha'} + \|1 + \Delta^2\|^{-1/2}\|u_1\|_{L^\alpha'}).
\]

This proves (3.18).

By (3.26), (3.27), and (3.38), (3.39), we then get that for any Bl-admissible pairs $(p, q)$ and $(a, b)$,

\[
\|P_{\leq 1}(u, u_t)\|_{L^p(I, L^q)} = \|P_{\leq 2}P_{\leq 1}(u, u_t)\|_{L^p(I, L^q)}
\]

\[
\leq C\left(\|(u_0, u_1)\|_{C} + \|(1 + \Delta^2)^{-1/2}P_{\leq 2}h\|_{L^{a'}(I, L^{b'})} + \|P_{\leq 2}h\|_{L^{a'}(I, L^{b'})}\right)
\]

\[
\leq C\left(\|(u_0, u_1)\|_{C} + \|h\|_{L^{a'}(I, L^{b'})}\right).
\]

(3.40)

We used in (3.40) that $P_{\leq 2}P_{\leq 1} = P_{\leq 1}$ and that the kernels of the operators $P_{\leq 2}$ and $(1 + \Delta^2)^{-1/2}P_{\leq 2}$ lie in $L^1$. Similarly, by (3.36), (3.37), and (3.38), (3.39), we get that for any S-admissible pairs $(p, r)$ and $(c, d)$,
\[ P_{>1}(1 + \Delta^2)^{1/2} u, u_t) \leq C \left( \parallel (1 + \Delta^2)^{1/2} u_0 \parallel_{L^2} + \parallel u_1 \parallel_{L^2} + \parallel h \parallel_{L^c(I, L^{d'})} \right). \] (3.41)

Now, we just remark that if \((p, q)\) is B-controlling, then there exists \(r \leq q\) such that \((p, r)\) is S-admissible, and \(H^{2,r} \subset L^q\). Since \(\frac{1 - \Delta}{\sqrt{1 + \Delta^2}}\) is bounded \(L^p \rightarrow L^p\) for \(1 < p < \infty\), we get from Bessel’s potential theory that

\[ \parallel P_{>1} u \parallel_{L^p(I, L^q)} \leq C \parallel (1 - \Delta) P_{>1} u \parallel_{L^p(I, L^r)} \leq C \parallel (1 + \Delta^2)^{1/2} P_{>1} u \parallel_{L^p(I, L^r)}. \] (3.42)

By (2.3), equations (i) and (ii), and (3.40)–(3.42), we get that (3.17) holds true. At this stage we proved (3.17) and (3.18) when \(m = 1\).

In case \(m \neq 1\), we remark that if \(u\) solves (3.16) with Cauchy data \((u_0, u_1)\), then \(v(t, x) = u(\lambda^2 t, \lambda x)\) solves (3.16) with \(\lambda^4 m\) in place of \(m\) and \(\tilde{h}\) in place of \(h\), where \(\tilde{h}(t, x) = \lambda^4 h(\lambda^2 t, \lambda x)\). Moreover \(v\) satisfies the Cauchy data \((v(0), v_t(0)) = (\tilde{u}_0, \lambda^2 \tilde{u}_1)\), where \(\tilde{u}_0(x) = u_0(\lambda x)\) and \(\tilde{u}_1(x) = u_1(\lambda x)\). This ends the proof of the lemma.

As a remark, combining the second inequality in (3.27), the second inequality in (3.37), and the explicit formula for \(W(t)\) in (3.38) and (3.39), we get the estimate that for any S-admissible pair \((a, b)\), any Bl-admissible pair \((c, d)\), and any \(u \in L^a(R, L^b) \cap L^c(R, L^d)\),

\[ \parallel \int R W(-t) (0, u(t)) dt \parallel_{E} \leq C \left( \parallel u \parallel_{L^a(R, L^b)} + \parallel u \parallel_{L^c(R, L^d)} \right), \] (3.43)

where \(C > 0\) does not depend on \(u\). Indeed,

\[ \parallel \int R W(-t) (0, u(t)) dt \parallel_{E}^2 = \parallel \int R \frac{1}{2t} (1 + \Delta)^{-1/2} (T_t - T_{-t}) u(t) dt \parallel_{H^2}^2 + \parallel \int R \frac{T_t + T_{-t}}{2} u(t) dt \parallel_{L^2}^2 \]

\[ \leq \parallel \int R T_t u(t) dt \parallel_{L^2}^2 + \parallel \int R T_{-t} u(t) dt \parallel_{L^2}^2 \]

\[ \leq C \left( \parallel u \parallel_{L^a(R, L^b)}^2 + \parallel u \parallel_{L^c(R, L^d)}^2 \right). \]

Also we get that for any S-admissible pairs \((a, b)\) and \((c, d)\), and for any \(u \in L^c(R, L^d)\),

\[ \parallel \int_{0 < s < t} \pi_2 P_{>1} W(t - s) (0, u(s)) ds \parallel_{L^a(R, L^b)} \leq C \parallel u \parallel_{L^c(R, L^d)}, \] (3.44)

and, when \(q \geq 2\), that
We mainly use the first bound in the right-hand side of (3.46) for \( t \) large, and the second bound in the right-hand side of (3.46) for \( t \) small. The function of \( t \) in the second bound is integrable around 0 when \( q < 2^{\frac{p}{4}} \). As a remark, \( q = p + 1 \) is an important example, where \( p \) is the exponent in (0.1). At last we mention that (3.38) can be rewritten as

\[
(u(t), u_t(t)) = W(t)(u_0, u_1) + \int_0^t W(t-s)(0, h(s)) \, ds
\]

for all \( t \), and all solution \( u \) of (3.16). Equation (3.47) is referred to as the Duhamel formula for (3.16).

4. A general criterion for scattering

We prove a general result for scattering in the spirit of the one in Tao and Visan [33] concerning the Schrödinger equation. As one can check, by our assumptions on \( p \), the pairs

\[
\left( \frac{2n+4}{n+8p}, \frac{2n+4}{n+8p} \right) \quad \text{and} \quad \left( \frac{2n+2}{n+4p}, \frac{2n+2}{n+4p} \right)
\]

are B-controlling in the sense of Definition 3.1. Our result is stated as follows.

**Lemma 4.1.** Let \( u \in E_{R^+} \) be a strong solution of (0.1) with \( 1 + \frac{8}{n} \leq p \leq 2^{\frac{p}{4}} - 1 \) when \( n \geq 5 \), and \( 1 + \frac{8}{n} \leq p < \infty \) when \( n \leq 4 \). Suppose that

\[
u \in L^2 \frac{4n+4}{n+8p}(R_+ \times R^n) \cap L^2 \frac{4n+2}{n+4p}(R_+ \times R^n).
\]  

Then there is scattering in forward time for \((u_0, u_1) = (u(0), u_t(0))\) and

\[
E(u(0), u_t(0)) = E_0(u_0^+, u_1^+),
\]

where \((u_0^+, u_1^+)\) is the scattering pair associated to \((u(0), u_t(0))\) as in (1.4). Furthermore, \( W_+ \), as defined in (1.5), is continuous at \((u_0, u_1)\) in the sense that if \( u_k \) is the solution of the nonlinear problem (0.1) corresponding to an initial data \((u_0^k, u_1^k)\) such that \((u_0^k, u_1^k) \rightarrow (u_0, u_1)\) in \( E \) as \( k \rightarrow + \infty \), then \( u_k \) is defined on \( R_+ \) for \( k \) sufficiently large, and there is scattering in forward time.
for \((u^k_0, u^k_1)\) with scattering associated pair \((u^{+,k}_0, u^{+,k}_1)\) satisfying that \((u^{+,k}_0, u^{+,k}_1) \to (u^+_0, u^+_1)\) in \(E\) as \(k \to +\infty\).

**Proof.** First, we prove that if \(u\) solves (0.1) with \(1 + \frac{8}{n} \leq p \leq 2^\sharp - 1\) and (4.1) holds true, then there exists a couple \((u^+_0, u^+_1)\) \(\in E\) such that

\[
\left\| (u(t), u_t(t)) - \mathcal{W}(t)(u^+_0, u^+_1) \right\|_E \to 0 \quad \text{as } t \to +\infty,
\]

(4.3)

where \((u^+_0, u^+_1)\) is uniquely defined by

\[
(u^+_0, u^+_1) = (u_0, u_1) + \lambda \int_0^\infty \mathcal{W}(-s)(0, u^p(s)) \, ds,
\]

(4.4)

and \(u^p = |u|^{p-1}u\) is as defined in Section 2. We prove (4.3) and (4.4) in what follows. Let

\[
\bar{v}(t) = (v_0(t), v_1(t)) = \mathcal{W}(-t)(u(t), u_t(t))
\]

(4.5)

be the value at time \(-t\) of the solution \(v\) of the Cauchy problem (1.2) with initial data \((v(0), v_t(0)) = (u(t), u_t(t))\). In order to prove (4.3) it suffices to prove that \((v_0(t), v_1(t))\) converges in \(E\) as \(t \to +\infty\). It follows from Duhamel’s formula (3.47) and the semigroup property that

\[
(v_0(t), v_1(t)) = \mathcal{W}(-t)\left(\mathcal{W}(t)(u_0, u_1) + \lambda \int_0^t \mathcal{W}(t-s)(0, u^p(s)) \, ds\right)
\]

\[
= (u_0, u_1) + \lambda \int_0^t \mathcal{W}(-s)(0, u^p(s)) \, ds.
\]

(4.6)

Hence

\[
\bar{v}(t + s) - \bar{v}(t) = \lambda \int_t^{t+s} \mathcal{W}(-t')(0, u^p(t')) \, dt',
\]

where \(\bar{v}\) is as in (4.5), and if \(s \geq 0\), by the Strichartz estimates (3.43) with \((a, b) = (2(n + 2)/n, 2(n + 2)/n)\) and \((c, d) = (2(n + 4)/n, 2(n + 4)/n)\), we get that

\[
\left\| \bar{v}(t + s) - \bar{v}(t) \right\|_E \leq C\left(\left\| u^p \right\|_{L^\infty([t,t+s] \times \mathbb{R}^n)} + \left\| u^p \right\|_{L^c([t,t+s] \times \mathbb{R}^n)}\right).
\]

(4.7)

By (4.1), given \(\epsilon > 0\), there exists \(t_0\) sufficiently large such that

\[
\left\| u \right\|_{L^{\frac{n+4}{n-2}}(\{0, \infty\} \times \mathbb{R}^n)} + \left\| u \right\|_{L^{\frac{n+2}{n-4}}(\{0, \infty\} \times \mathbb{R}^n)} \leq \epsilon.
\]

As a consequence, by (4.7), for \(t \geq t_0\) and \(s \geq 0\),
and we get that \( \bar{v}(t) \) converges to some limit \( \bar{u}^+ \) as \( t \to +\infty \). Since \( \mathcal{W}(t) \) is a unitary operator,

\[
\| (u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+) \|_E = \| \mathcal{W}(-t)(u(t), u_t(t)) - (u_0^+, u_1^+) \|_E \to 0
\]
as \( t \to +\infty \). By Duhamel’s formula we then get that

\[
(u_0^+, u_1^+) = (u_0, u_1) + \lambda \int_0^t \mathcal{W}(-s)(0, u^p(s)) \, ds + o(1), \quad (4.8)
\]

where \( \|o(1)\|_E \to 0 \) as \( t \to +\infty \), and letting \( t \to +\infty \) in (4.8), we get (4.4). This ends the proof of (4.3) and (4.4). In what follows, we let

\[
\mathcal{W}(t)(u_0^+, u_1^+) = (u^+(t), u_t^+(t)) \quad (4.9)
\]

for \( t \geq 0 \), and we note that by the Strichartz estimates (3.17), \( u^+ \in L^{p+1}(\mathbb{R}, L^{p+1}) \). Here we use (3.17) with \( h = 0 \) and the pair \( (p+1, p+1) \) which turns out to be B-controlling because of the assumptions on \( p \). In particular, there exists a sequence of positive times \( t_k \to \infty \) such that

\[
\| u^+(t_k) \|_{L^{p+1}} \to 0. \quad (4.10)
\]

By conservation of the energy for \( u \) and of the linear energy for \( u^+ \), and since \( \|u^+(t_k) - u(t_k)\|_{H^2} + \|u_t^+(t_k) - u_t(t_k)\|_{L^2} \to 0 \) by (4.3), we can write with (4.10) that

\[
E(u_0, u_1) = E(u(t_k), u_t(t_k)) = E(u^+(t_k), u_t^+(t_k)) + o(1)
= E_0(u^+(t_k), u_t^+(t_k)) - \frac{\lambda}{p+1} \int_{\mathbb{R}^n} |u^+(t_k)|^{p+1} + o(1)
= E_0(u_0^+, u_1^+) + o(1).
\]

Letting \( k \to +\infty \), it follows that \( E(u_0, u_1) = E_0(u_0^+, u_1^+) \). This proves (4.2). In order to end the proof of Lemma 4.1 it remains to prove the continuity of \( W_+ \) as defined in the lemma. Let \( (u_k^0, u_k^1) \in \mathcal{E} \) be such that \( (u_k^0, u_k^1) \to (u_0, u_1) \) in \( \mathcal{E} \) as \( k \to +\infty \). Let \( u^k \) be the solution of the nonlinear problem (0.1) associated to the Cauchy data \( (u_k^0, u_k^1) \) and, when it exists, \( \bar{u}^+_s = (u^+_0, u^+_1) \) be the associated scattering pair. Let \( w = u - u^k \). Then \( w \) solves the equation

\[
\frac{\partial^2 w}{\partial t^2} + \Delta^2 w + mw = \lambda u^p - \lambda (u - w)^p \quad (4.11)
\]

with Cauchy data \( (w(0), w_t(0)) = (u_0 - u_0^k, u_1 - u_1^k) \). Let \( T > 0 \) be such that
\[ \|u\|_{L^{\frac{n+4}{n+2}} \cap L^2} + \|u\|_{L^{\frac{n+2}{n+2}} \cap L^2} < \epsilon, \tag{4.12} \]

where \( \epsilon > 0 \) is to be chosen later on. We know by the local theory, see the discussion after Lemma 3.1, that \( w \to 0 \) in \( C([0,T], H^2) \cap C^1([0,T], L^2) \cap L^{\frac{n+2}{n+2}} ([0,T] \times \mathbb{R}^n) \). For \( t \geq T \), we let

\[ g(t) = \|w\|_{L^{\frac{n+4}{n+2}} ([T,t] \times \mathbb{R}^n)} + \|w\|_{L^{\frac{n+2}{n+2}} ([T,t] \times \mathbb{R}^n)} + \|(w,w_t)\|_{C([T,t], \mathcal{E})}. \tag{4.13} \]

By the Strichartz estimates (3.17) that we consider for (4.11), we get that

\[
\begin{align*}
g(t) &\leq C \left( \sqrt{E_0(w(T), w_t(T))} + \sum_{\rho} \|u^\rho - (u - w)^\rho\|_{L^p([T,t] \times \mathbb{R}^n)} \right) \\
&\leq C \left( \sqrt{E_0(w(T), w_t(T))} + \sum_{\rho} |u^\rho - (u - w)^\rho| + |w^\rho| \right)_{L^p([T,t] \times \mathbb{R}^n)} \\
&\leq C \left( \sqrt{E_0(w(T), w_t(T))} + \sum_{\rho} (\|w\|_{L^{p\rho}([T,t] \times \mathbb{R}^n)} + \|w\|_{L^{p\rho}([T,t] \times \mathbb{R}^n)}) \right) \\
&\leq C \left( \sqrt{E_0(w(T), w_t(T))} + \sum_{\rho} (\epsilon^{p-1} h(t) + h(t)^p) \right),
\end{align*}
\]

where \( \epsilon \) and \( T \) are as in (4.12), \( g \) is as in (4.13), and \( \sum_{\rho} \) stands for the summation over the two values \( \rho = 2(n+4)/(n+8) \) and \( \rho = 2(n+2)/(n+4) \). Now we let \( \epsilon \in (0,1) \) be such that

\[ 4C \epsilon^\frac{2}{7} < 1 \]

and we choose \( k \) sufficiently large such that

\[ C \sqrt{E_0(w(T), w_t(T))} \leq \min \left( \frac{1}{6(24C)^{\frac{1}{7}}}, \frac{1}{6} \right). \]

Then

\[ g(t) \leq 4C \sqrt{E_0(w(T), w_t(T))} \to 0 \tag{4.14} \]

as \( k \to +\infty \), where \( w \) is as in (4.11). In particular, for \( k \) sufficiently large, \( u^k \) exists globally. Indeed, the \( u^k \)'s are bounded in \( \mathcal{E} \) by (4.14). As already mentioned, this ensures global existence when \( p < 2^\frac{d}{2} - 1 \). By noting that the \( u^k \)'s are also bounded in \( L^{\frac{2n+4}{n+2}}(\mathbb{R}^n) \) when \( p = 2^\frac{d}{2} - 1 \) and \( n \geq 5 \), we get global existence in that case from (3.14). Still by (4.14), now with \( t = +\infty \), we get that \( u^k \to u \) in \( L^{\frac{2n+4}{n+2}}(\mathbb{R}^n) \cap L^{\frac{n+2}{n+2}}(\mathbb{R}^n) \) as \( k \to +\infty \). By (4.3) there is scattering in forward time for \( u^k \) and by (4.4), the convergence of \( u^k \), and Strichartz estimates (3.43), we get that

\[ \|\bar{u}_s^+ - \bar{u}_s^+|_{\mathcal{E}} = |\lambda| \left\| \int_0^\infty \mathcal{W}(-s)(0, u^p(s) - (u(s) + w(s))^p) \right\|_{\mathcal{E}} \to 0 \]

as \( k \to +\infty \). This ends the proof of Lemma 4.1. \( \square \)
The following result is a useful corollary to Lemma 4.1. It will be used in the proof of our theorem in Section 1.

**Corollary 4.1.** Let \( n \geq 5 \) and \( u \in E_{\mathbb{R}^+} \) be a strong solution of Eq. (0.1) with \( 1 + 8/n < p < n/4 + 4/n - 4 \).
Assume \((u, u_t)\) is uniformly bounded with respect to \( t \) and that for some \( \gamma \geq 1 \),
\[
\|u(t)\|_{L^\gamma} \to 0 \quad (4.15)
\]
as \( t \to +\infty \). Then there is scattering in forward time for \((u(0), u_t(0))\), (4.1) holds true, and the conclusion of Lemma 4.1 also holds true.

**Proof.** By assumption \( u \) is uniformly bounded in \( L^2 \cap L^{2^*} \). By (4.15) and Hölder’s inequality we then get that \( u \) converges to 0 in \( L^q \) at least for \( 2 < q < 2^* \). In view of Lemma 4.1, and since, by the local theory discussed after Lemma 3.1,
\[
u \in C\left(\mathbb{R}^+, H^2 \right) \cap L^{2 + 2/4 + 4/p}_{\text{loc}} \left(\mathbb{R}^+, L^{2 + 2/4 + 4/p} \right),
\]
the corollary reduces to proving that there exists \( T_0 \geq 0 \) such that
\[
\|u\|_{L^{2 + 4/8 + 4/p}(T_0, \infty) \times \mathbb{R}^n} + \|u\|_{L^{2 + 2/4 + 4/p}(T_0, \infty) \times \mathbb{R}^n} \leq C \quad (4.16)
\]
for some constant \( C > 0 \). Let \( 2 < r = 2np/(n + 8) \), \( \rho = 2np/(n + 4) < 2^* \), and \( \epsilon > 0 \) be some positive constant to be chosen later on. Let \( T_0 > 0 \) be such that
\[
\sup_{t \geq T_0} \left( \|u(t)\|_{L^r} + \|u(t)\|_{L^\rho} \right) \leq \epsilon \quad (4.17)
\]
and, for \( t \geq T_0 \), let
\[
g(t) = \max \left( \|u\|_{L^{2 + 4/8 + 4/p}(T_0, t) \times \mathbb{R}^n}, \|u\|_{L^{2 + 2/4 + 4/p}(T_0, t) \times \mathbb{R}^n} \right).
\]

By Duhamel’s formula (3.47),
\[
(u(t), u_t(t)) = W(t - T_0)(u(T_0), u_t(T_0)) + \lambda \int_{T_0}^t W(t - s)(0, u^p(s)) \, ds
\]
for all \( t \geq T_0 \). By the Strichartz estimates (3.17) in Lemma 3.2, and (4.17), using Hölders’ inequalities, and since \((2n + 4/8 + 4/p, 2n + 4/n - 4)\) and \((2n + 2/4 + 4/p, 2n + 2/n + 4)\) are B-controlling pairs, we then get that
\[
g(t) \leq C \sqrt{E_0(u(T_0), u_t(T_0))} + C \left( \|u^p\|_{L^2(T_0, t), L^{2p/4 + 2}} + \|u^p\|_{L^2(T_0, t), L^{2p/4 + 4}} \right)
\]
\[
\leq C \sqrt{E_0(u(T_0), u_t(T_0))}
\]
\[
+ C \left( \|u\|_{L^\infty([T_0,t],L^p)}^{2p/n} \|u\|_{L^{n+2}/n+2}^{(n+2)p/n+2} \right) \leq C \left( E_0(u(T_0),u_t(T_0)) + \varepsilon \left( g(t) \|u\|_{L^\infty([T_0,t],L^p)}^{(n+2)p/n+2} \right) \right)
\]
\[
\leq C \left( E_0(u(T_0),u_t(T_0)) + \varepsilon \left( g(t) \|u\|_{L^\infty([T_0,t],L^p)}^{(n+2)p/n+2} \right) \right). \tag{4.18}
\]

It can be noted here that \((2, 2^*)\) is S-admissible and that \((2, 2^2)\) is Bl-admissible. The first inequality in (4.18) is by (3.17), the second inequality is by Hölder’s inequality, and the third inequality is by (4.17). Now we remark that \(g\) is continuous, that \(g(T_0) = 0\), and that for any \(t > T_0\),
\[
g(t) \leq C' + \epsilon' \left( g(t) + \epsilon(t) \right), \tag{4.19}
\]
where \(C' = C \sqrt{E(u_0, u_1)}\) does not depend on \(t\), and \(\epsilon' = C \left( \frac{2p}{n+4} + \frac{4p}{n+8} \right)\) can be made as small as we want when \(\varepsilon\) is sufficiently small. In particular, we can choose \(\epsilon\) such that
\[
\epsilon' < \frac{C'}{(2C')^{(n+2)p/n+4} + (2C')^{(n+4)p/n+8}}. 
\]
Since the two powers in (4.19) are greater than 1 by our assumptions on \(p\), we get that \(g(t) \leq 2C'\) for all \(t \geq T_0\). This proves (4.16), and thus also the corollary.

By standard arguments the counterpart to Lemma 4.1 holds true. To make a precise statement, it follows from standard arguments that when \(1 + \frac{4}{n} \leq p \leq 2^* - 1\) (respectively \(1 + \frac{4}{n} \leq p < \infty\) when \(n \leq 4\), given any solution of the linear equation (1.2), written as \((\omega, \omega_t) = \mathcal{W}(\cdot)(u_0^+, u_1^+)\), there exist \(T\) and a unique solution \(u\) of the nonlinear equation (0.1), defined on \([T, \infty)\), such that (1.4) holds true and
\[
u \in L^{\frac{2n+4}{n+2}}(\mathbb{T}, \mathbb{R}^n) \cap L^{\frac{2n+4}{n+8}}(\mathbb{T}, \mathbb{R}^n). 
\]
Furthermore, one has a continuity property in the sense that if
\[
u(u_0^+, u_1^+) \to (u_0^+, u_1^+) 
\]
holds true for any initial data \((u, v) \in \mathcal{E}\) of energy \(E_0(u, v) \leq \varepsilon_0\). Moreover, \(E \geq 0\) for such initial data, and \(W_+\) in (1.5) realizes an homeomorphism from \(\mathcal{E}\) onto \(\mathcal{B}_\varepsilon\) for all \(\varepsilon \in (0, \varepsilon_0]\), where \(\mathcal{E}\) consists of the \((u, v) \in \mathcal{E}\) such that \(E_0(u, v) \leq \varepsilon_0\) and \(E(u, v) \leq \varepsilon\), and \(\mathcal{B}_\varepsilon\) consists of the \((u, v) \in \mathcal{E}\) such that \(E_0(u, v) \leq \varepsilon_0\) and \(E(u, v) \leq \varepsilon\). The case \(p < 2^\# - 1\) in this statement was proved by Levandosky [18], as well as it was proved by Levandosky [18] that the equation possesses travelling waves of arbitrarily low energy when \(\lambda > 0\) and \(p < 1 + \frac{4}{n}\). Travelling waves cannot scatter since their \(L^q(\mathbb{R}^n)\)-norms, \(2 \leq q \leq 2^\#,\) are constant, whereas, by Strichartz estimates, solutions of the linear equations have powers of their \(L^q\)-norm integrable in time. If we accept complex-valued functions, then, based on material in Levandosky [17], we can construct standing waves with arbitrarily small energy when \(p < 1 + \frac{8}{n}\), contradicting once again scattering in the small energy setting.

5. Frequency localization

We prove frequency localization for solutions of the nonlinear equation (0.1). We assume in what follows that \(p\) is such that

\[
1 + \frac{8}{n} < p < 2^\# - 1, \tag{5.1}
\]

and that \(\lambda < 0\). We prove the following frequency localization result in this section, using ideas recently introduced by Tao [31] for the Schrödinger equation.

**Lemma 5.1.** Let \(n \geq 5\), and \(u \in \mathcal{E}_{\mathbb{R}^+}\) be a forward global solution of the nonlinear equation (0.1) with \(\lambda < 0\) and \(p\) such that (5.1) holds true. There exist a couple \((u_0^+, u_1^+) \in \mathcal{E}, \eta > 0,\) and a function \(w \in \mathcal{E}_{\mathbb{R}^+}\) such that

\[
(u, ut) = \mathcal{W}(\cdot)(u_0^+, u_1^+) + (w, wt),
\]

\[
\mathcal{W}(-t)(w(t), wt(t)) \to (0, 0) \quad \text{in } \mathcal{E} \text{ as } t \to +\infty, \quad \text{and}
\]

\[
\sup_{N \geq 1} \limsup_{t \to \infty} N^\eta E_0(P_{\geq N}(w(t), wt(t))) \leq C, \tag{5.2}
\]

where \(C > 0\) depends only on \(E(u(0), ut(0)), m, \lambda,\) and \(n\).

As a consequence of this lemma we get that the following corollary holds true. We prove the corollary in what follows and then prove the lemma in several steps.

**Corollary 5.1.** Let \(n \geq 5\), \(u \in \mathcal{E}_{\mathbb{R}^+}\) be a forward global solution of the nonlinear equation (0.1) with \(\lambda < 0\), and \(p\) such that (5.1) holds true, and \(\varepsilon > 0\). There exist \(t_0\) and \(N\) such that

\[
E_0(P_{\geq N}(u(t), ut(t))) \leq \varepsilon^2 \tag{5.3}
\]

for all time \(t \geq t_0\).
**Proof.** Since \((u_0^+, u_1^+) \in \mathcal{E}\), there exists \(N_0\) such that
\[
E_0(P \geq N_0(u_0^+, u_1^+)) \leq \frac{\epsilon^2}{4}.
\]  
(5.4)

Since \(\mathcal{W}\) is a unitary operator and since \(\mathcal{W}\) commutes with \(P \geq N\) for any \(N\), we get by (5.4) that for any time \(t\), and for any \(N > N_0\),
\[
E_0(P \geq N \mathcal{W}(t)(u_0^+, u_1^+)) = E_0(\mathcal{W}(t)P \geq N(u_0^+, u_1^+))
\]
\[
= E_0(P \geq N(u_0^+, u_1^+))
\]
\[
= E_0(P \geq N P \geq N_0(u_0^+, u_1^+)) \leq \frac{\epsilon^2}{4}.
\]  
(5.5)

Independently, by (5.2), there exists \(N_1\) such that
\[
E_0(P \geq N(w(t), w_t(t))) \leq \frac{\epsilon^2}{4} \tag{5.6}
\]
for all \(N \geq N_1\), and all \(t \geq t_N\), where \(t_N\) depends only on \(N\). Let \(N > \max(N_0, N_1)\), and \(t \geq t_N\).

By (5.2), (5.5), and (5.6) we then get that
\[
E_0(P \geq N(u(t), u_t(t))) \leq 2(E_0(P \geq N(w(t), w_t(t))) + E_0(P \geq N \mathcal{W}(t)(u_0^+, u_1^+)))
\]
\[
\leq \epsilon^2.
\]

This proves the corollary. \(\square\)

Now it remains to prove Lemma 5.1. We proceed in several steps. As a first remark, we note that, when \(p\) satisfies (5.1), there always exist an S-admissible pair \((a, b)\), \(d \geq 2\), \(\kappa \in (0, 1)\),
\[
\frac{2}{p} < \alpha < \frac{2n}{n + 4},
\]  
(5.7)

\(\alpha\) close to \(2n/(n + 4)\), and \(\theta \in (0, 1)\) such that \(a > 2\) and

(i) \[
\frac{1}{b'} = \frac{p - \kappa}{d} + \frac{\kappa}{2},
\]

(ii) \[
a'(p - \kappa) \geq 2,
\]

(iii) \[
\frac{n - 4}{2} < \frac{2}{a'(p - \kappa)} + \frac{n}{d} < \frac{n}{2},
\]

(iv) \[
\frac{1}{\alpha p} = \frac{1 - \theta}{2} + \frac{\theta}{\alpha'}, \quad \text{and} \quad p\theta > 1.
\]  
(5.8)

Now Step 5.1 states as follows. Without loss of generality, we assume in the sequel that \(m = 1\) and \(\lambda = -1\).
Step 5.1. Let $I \subset \mathbb{R}$ be an interval, and $u \in E_I$ be a solution of (0.1) with $\lambda = -1$ and $p$ such that (5.1) holds true. Let also $E > 0$ be such that $E(u, u_t) \leq E$. For any $B$-admissible pair $(q, r)$,

$$\|u\|_{L^q(I, L^r)} \leq C \left(1 + |I| \right)^{\frac{1}{q}}$$  \hspace{2cm} (5.9)

where $C$ depends only on $E$, $q$, and $n$.

**Proof.** Step 5.1 follows from the Strichartz estimates in Lemma 3.1. First we assume that $|I| \leq 1$ is small enough, $I = [t_0, t_1]$. We write (0.1) as a superposition of two linear beam equations as in (3.5), with forcing term $h_1 = -u^p$ and $h_2 = -u$. Suppose first that $p > (n + 2)/(n - 4)$, then there exists $\delta > 0$ such that $(2p + \delta, \frac{2np}{n+2}) = (\gamma, \rho)$ is $B$-admissible. Let $\mu > 0$ be such that $\frac{1}{2p} = \frac{1}{2p + \delta} + \frac{1}{\mu}$. By the Strichartz estimates (3.6) in Lemma 3.1, that we apply to the two linear beam equations with forcing terms $h_1$ and $h_2$,

$$\|u\|_{L^\gamma(I, L^\rho)} \leq C \left(\sqrt{E(u, u_t)} + \|u^p\|_{L^2(I, L^{\frac{2n}{n+2}})} + \|u\|_{L^1(I, L^n)} \right) \leq C \left(\sqrt{E(u, u_t)} + \|u^p\|_{L^2(I, L^{\frac{2n}{n+2}})} + \|u\|_{L^1(I, L^n)} \right) \leq C \left(\sqrt{E(u, u_t)} + |I|^{\frac{p}{2}} \|u\|_{L^\gamma(I, L^\rho)} \right),$$

where $u^p = |u|^{p-1}u$. Besides, $h(t) = \|u\|_{L^\gamma([t_0, t], L^\rho)}$ is continuous and $h(0) = 0$. It follows that if $|I| \leq \varepsilon_0$ is sufficiently small, then

$$\|u\|_{L^\gamma(I, L^\rho)} \leq 2C \sqrt{E(u, u_t)}. \hspace{2cm} (5.10)$$

Applying the Strichartz estimates (3.6), with (3.13) if $p \leq (n + 2)/(n - 4)$, or (5.10) if $p > (n + 2)/(n - 4)$, since $(q, r)$ is $B$-admissible and $(2, 2^*)$ is $S$-admissible, we get that

$$\|u\|_{L^q(I, L^r)} \leq C \left(\sqrt{E(u, u_t)} + \|u^p\|_{L^2(I, L^{\frac{2n}{n+2}})} \right) \leq C'. \hspace{2cm} (5.11)$$

Now, if $I$ is of arbitrary length, we decompose $I = \bigcup_{j=1}^k I_j$ with the $I_j$’s such that their interiors are disjoint and such that $|I_j| = \varepsilon_0$, except maybe for the last interval which can be of a smaller length. Then $k \leq \frac{|I|}{\varepsilon_0} + 1$ and

$$\|u\|_{L^q(I, L^r)}^q = \sum_{j=1}^k \|u\|_{L^q(I_j, L^r)}^q \leq C \left(|I| + 1 \right).$$

This ends the proof of Step 5.1. 

The next step in the proof of Lemma 5.1 is stated as follows.

**Step 5.2.** Let $u \in E_I$ be a forward solution of (0.1) with $\lambda = -1$ and $p$ such that (5.1) holds true. For $(a, b)$ an $S$-admissible pair like in (5.8), there exist $\eta > 0$, and $C > 0$ depending only on $n$ and $E = E(u(0), u_t(0))$, such that
\[ \| P_{\geq N} u^p \|_{L^{d'}(I,L^{d'})} \leq C N^{-\eta} (1 + |I|)^{\frac{1}{d'}} \]  

(5.12)

for all finite interval \( I \subset \mathbb{R}_+ \).

**Proof.** Again we may assume that \( |I| \leq 1 \). The case of intervals of arbitrary length follows from the case \( |I| \leq 1 \) as in the proof of Step 5.1. Let \( u_h = P_{\geq N} u \) and \( u_l = u - u_h \). Then

\[ |u^p - u_l^p| \leq C |u_h| (|u|^p - 1 + |u_l|^p - 1), \]

and we get with Hölder’s inequality, (2.3), (3.13), (5.8) equation (i), and (5.9), that

\[
\begin{align*}
\| P_{\geq N} (u^p - u_l^p) \|_{L^{d'}(I,L^{d'})} & \leq C \| u_h \|^\kappa |u_h|^{1-\kappa} (|u|^p - 1 + |u_l|^p - 1) \| u \|_{L^{d'}(I,L^{d'})} \\
& \leq C \| u_h \|_{L^\infty(I,L^2)}^\kappa \| u_h \|^{1-\kappa} (|u|^p - 1 + |u_l|^p - 1) \| u \|_{L^{d'}(I,L^{d'})} \\
& \leq C \| u_h \|_{L^\infty(I,L^2)}^\kappa \| u_h \|^{1-\kappa} \| u \|_{L^{d'}(p-\kappa)(I,L^{d'})}^p + \| u_l \|_{L^{d'}(p-\kappa)(I,L^{d'})}^p \\
& \leq C N^{-\kappa} \| u_h \|_{L^\infty(I,H^2)} \\
& \leq C N^{-\kappa},
\end{align*}
\]  

(5.13)

where \( C \) depends only on \( n \) and \( E \), where \( \kappa > 0 \) by (5.8), and where we used the fact that the norm of \( u \) with respect to the pair \( (a'(p-\kappa),d) \) can be controlled thanks to (3.13) and Step 5.1. The middle inequalities in (5.13) are because of Hölder’s inequality and (5.8) equation (i). The last inequality in (5.13) is by (2.3) equation (i), and (5.9). Independently, still by (2.3), (3.13), (5.8), and (5.9), we can write that

\[
\begin{align*}
\| P_{\geq N} u_l^p \|_{L^{d'}(I,L^{d'})} & \leq C N^{-1} \| \nabla |u|^p \|_{L^{d'}(I,L^{d'})} \\
& \leq C N^{-1} \| \nabla u_l^p \|_{L^{d'}(I,L^{d'})} \\
& \leq C N^{-1} \| \nabla u_l \|_{L^\infty(I,L^2)}^\kappa \| \nabla u_l \|^{1-\kappa} \| u_l \|_{L^{d'}(I,L^{d'})} \\
& \leq C N^{-1} \| u_l \|_{L^\infty(I,H^1)}^\kappa N^{1-\kappa} \| u_l \|_{L^{d'}(p-\kappa)(I,L^{d'})}^{p-1} \| u_l \|_{L^{d'}(p-\kappa)(I,L^{d'})} \\
& \leq C N^{-\kappa}.
\end{align*}
\]  

(5.14)

The first inequality in (5.14) is by (2.3) equation (i). The second inequality is by boundedness of Riesz transforms. The third inequality is by direct computations. The fourth inequality is by (5.8) equation (i). The last inequality in (5.14) is by (2.3) equation (ii). By letting \( \eta = \kappa \), (5.12) in Step 5.2 follows from (5.13) and (5.14) when \( |I| \leq 1 \). As already mentioned, this ends the proof of Step 5.2. \( \Box \)

The last step before the proof of Lemma 5.1 is stated as follows.
Step 5.3. Let \( u \in \mathbb{E}_{\mathbb{R}_+} \) be a forward global solution of (0.1) with \( \lambda = -1 \) and \( p \) such that (5.1) holds true. Let also \( E > 0 \) be such that \( E(u, u_t) \leq E \). Then, there exist a couple \( \bar{u}_s = (u_0^+, u_1^+) \in \mathcal{E} \), and a function \( w \in \mathbb{E}_{\mathbb{R}_+} \), such that

\[
(\bar{u}_s) = \mathcal{W}(t) \bar{u}_s + (w(t), w_t(t)) \]

\[
E(\bar{u}_s) \leq E, \quad E_0(w(t), w_t(t)) \leq 4E, \quad \text{and} \quad \mathcal{W}(-t)(w(t), w_t(t)) \rightharpoonup (0, 0) \quad \text{in} \quad \mathcal{E}
\]  

(5.15)
as \( t \to +\infty \), where the first two equations hold true for all \( t \geq 0 \). Furthermore,

\[
(w(t), w_t(t)) = \mathcal{W}(t)(u_0^+ - u_0^+, u_1^+ - u_1^+) - \int_0^t \mathcal{W}(t-s)(0, u_p(s)) \, ds
\]

\[
= w-\lim_{T \to \infty} \int_t^T \mathcal{W}(t-s)(0, u_p(s)) \, ds
\]  

(5.16)for all \( t \geq 0 \), where the notation \( w \)-lim stands for the weak limit.

**Proof.** By conservation of the energy (3.13), and since \( \mathcal{W} \) is a unitary operator, we get that \( \bar{v}(t) \) is uniformly bounded in \( \mathcal{E} \), where for any time \( t \geq 0 \),

\[
\bar{v}(t) = \mathcal{W}(-t)(u(t), u_t(t)).
\]  

(5.17)

Hence, up to a subsequence, \( \bar{v}(t) \) converges weakly in \( \mathcal{E} \) as \( t \to +\infty \).

We claim that the limit is unique. In order to prove the claim it suffices to prove that

\[
\lim_{t_1, t_2 \to +\infty} \left\langle \bar{v}(t_1) - \bar{v}(t_2), \bar{\phi} \right\rangle_{\mathcal{E}} = 0
\]  

(5.18)

for all \( \phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n) \), where \( \bar{\phi} = (\phi_0, \phi_1) \), and \( \langle \cdot, \cdot \rangle_{\mathcal{E}} \) stands for the scalar product in \( \mathcal{E} \). Let \( t_2 \leq t_1 \in \mathbb{R}_+ \), and \( \phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n) \). By Duhamel’s formula (3.47), the semigroup property of \( \mathcal{W} \), and since \( \mathcal{W} \) is a unitary operator, we have

\[
\left| \left\langle \bar{v}(t_1) - \bar{v}(t_2), \bar{\phi} \right\rangle_{\mathcal{E}} \right| = \left| \left\langle \int_{t_2}^{t_1} \mathcal{W}(-s)(0, u_p(s)) \, ds, \bar{\phi} \right\rangle_{\mathcal{E}} \right|
\]

\[
\leq \int_{t_2}^{t_1} \left| \left\langle (0, u_p(s)), \mathcal{W}(s)\bar{\phi} \right\rangle_{\mathcal{E}} \right| \, ds
\]
\[ \leq \int_{t_2}^{t_1} \| u^p(s) \|_{L^\alpha} \| \pi_2 \mathcal{W}(s) \bar{\phi} \|_{L^{\alpha'}} ds \]
\[ \leq C \| u \|_{L^{\infty}(\mathbb{R}, H^2)}^p \int_{t_2}^{t_1} \| \pi_2 \mathcal{W}(s) \bar{\phi} \|_{L^{\alpha'}} ds, \quad (5.19) \]

where \( \alpha \) is as in (5.7), so that \( H^2 \subset L^{\alpha p} \). Now, since \( \alpha' > 2^p \), by (3.45), equation (i), we get that there exist \( \delta > 0 \) and \( C > 0 \) such that for any \( s > 0 \),

\[ \| \pi_2 \mathcal{W}(s) \bar{\phi} \|_{L^{\alpha'}} \leq C s^{-1-\delta} \quad (5.20) \]

and from (5.19), (5.20), we deduce that (5.18) holds true. This implies uniqueness and the above claim.

By (5.18) we also get that there exists a pair \((u_0^+, u_1^+) \in \mathcal{E}\) such that

\[ \bar{v}(t) \rightharpoonup (u_0^+, u_1^+) \quad (5.21) \]

weakly in \( \mathcal{E} \) as \( t \to +\infty \). Besides, since \( \mathcal{W} \) is a unitary operator, and by conservation of the energy as in (3.13), we have that

\[ \| \bar{v}(t) \|_{\mathcal{E}} = \| (u(t), u_t(t)) \|_{\mathcal{E}} \leq \sqrt{E} \quad (5.22) \]

while, by weak lower semicontinuity of the norm, we get from (5.22) that

\[ \| (u_0^+, u_1^+) \|_{\mathcal{E}} \leq \sqrt{E}. \quad (5.23) \]

In what follows we let

\[ (w(t), w_t(t)) = (u(t), u_t(t)) - \mathcal{W}(t) (u_0^+, u_1^+). \quad (5.24) \]

Then the first equation in (5.15) holds true. By conservation of the energy (3.13), and (5.23), we can write that \( \| (w, w_t) \|_{\mathcal{E}} \leq 2 \sqrt{E} \). Together with (5.21), (5.23), and (5.24), this proves that the second and third equations in (5.15) also hold true. Now it remains to prove (5.16). By Duhamel’s formula (3.47), we have

\[ (w(t), w_t(t)) = \mathcal{W}(t) (u_0 - u_0^+, u_1 - u_1^+) - \int_0^t \mathcal{W}(t-s) (0, u^p(s)) ds. \quad (5.25) \]

This proves the first equation in (5.16). We fix \( T > 0 \). By Duhamel’s formula (3.47) with initial time \( T \),

\[ (u(t), u_t(t)) = \mathcal{W}(t) \mathcal{W}(-T) (u(T), u_t(T)) + \int_T^t \mathcal{W}(t-s) (0, u^p(s)) ds. \]
As a consequence,
\[
(w(t), w_t(t)) = \mathcal{W}(t)(\mathcal{W}(-T)(u(T), u_t(T)) - (u_0^+, u_1^+)) + \int_t^T \mathcal{W}(t-s)(0, u^p(s)) \, ds
\]
for all \( t \leq T \). Using (5.21), and letting \( T \to +\infty \) in (5.26), we obtain that the second equation in (5.16) holds true. This ends the proof of Step 5.3.

Thanks to Steps 5.1–5.3 we are in position to prove our frequency localization Lemma 5.1. We prove the lemma in the sequel.

**Proof of Lemma 5.1.** We suppose \( N \geq 8 \). We let \( \varepsilon = N^{-\eta_0} > 0 \) where \( \eta_0 \) is to be defined later on. By density of smooth functions in the energy space, we can find an element \( \tilde{\phi} = (\phi_0, \phi_1) \in C_\infty_c(\mathbb{R}^n) \times C_\infty_c(\mathbb{R}^n) \) such that
\[
\| (u_0 - u_0^+, u_1 - u_1^+) - \tilde{\phi} \|_E \leq \varepsilon,
\]
where \( u_0 = u(0) \), and \( u_1 = u_t(0) \). Applying \( P_{\geq N} \) to the two equations in (5.16), we get
\[
P_{\geq N}(w(t), w_t(t)) = \mathcal{W}(t)P_{\geq N}(\tilde{\phi} + \tilde{e}) - \int_0^t \mathcal{W}(t-s)(0, P_{\geq N}u^p(s)) \, ds
\]
\[
= \text{w-lim}_{T \to \infty} \int_t^T \mathcal{W}(t-s)(0, P_{\geq N}u^p(s)) \, ds,
\]
where \( \tilde{e} = (u_0 - u_0^+, u_1 - u_1^+) - \tilde{\phi} \). By Step 5.3, \( E_0(\bar{w}) \leq 4E \), where \( \bar{w} = (w, w_t) \). Then, with (5.27) and (5.28), since \( \mathcal{W} \) is a unitary operator and \( P_{\geq N} \) is bounded on \( E \), we get that for \( t \geq 0 \),
\[
E_0(P_{\geq N} \bar{w}) = \| P_{\geq N} \bar{w}, P_{\geq N} \bar{w} \|_E
\]
\[
\leq \left| \left| P_{\geq N} \bar{w}, \mathcal{W}(t)P_{\geq N}\tilde{\phi} - \int_0^t \mathcal{W}(t-s)(0, P_{\geq N}u^p(s)) \, ds \right|_E \right| + 2\sqrt{E}\varepsilon
\]
\[
\leq \left| \left| \text{w-lim}_{T \to \infty} \int_t^T \mathcal{W}(t-t')(0, P_{\geq N}u^p(t')) \, dt', \mathcal{W}(t)P_{\geq N}\tilde{\phi} \right|_E \right| + C\varepsilon
\]
\[
+ \left| \left| \text{w-lim}_{T \to \infty} \int_t^T \mathcal{W}(t-t')(0, P_{\geq N}u^p(t')) \, dt', \int_0^t \mathcal{W}(t-s)(0, P_{\geq N}u^p(s)) \, ds \right|_E \right|
\]
where \( \langle \cdot, \cdot \rangle_E \) stands for the scalar product in \( E \). Then we get that
\[ E_0(P_{\geq N} \overline{w}) \leq \int_t^\infty U_N(t') \, dt' + \left| \int_0^t \int_s^{t'} V_N(s, t') \, ds \, dt' \right| + C\epsilon, \tag{5.29} \]

where, by semigroup property, and since \( \mathcal{W} \) is a unitary operator,

\[
U_N(t') = \left\| \mathcal{W}(t - t')(0, P_{\geq N} u^p(t')), \mathcal{W}(t) P_{\geq N} \bar{\phi} \right\|_{\mathcal{E}}
\]

\[
V_N(s, t') = \left\| \mathcal{W}(t - t')(0, P_{\geq N} u^p(t')), \mathcal{W}(t - s)(0, P_{\geq N} u^p(s)) \right\|_{\mathcal{E}}
\]

and \( T = T(t) \) is taken sufficiently large. Now we estimate each term in (5.29) and split the integral into several parts. First, using the fact that \( H^2 \subset L^{\alpha p} \), which follows from (5.7), conservation of the energy, (2.3) equation (i), and the fast decay of \( P_{\geq N} \mathcal{W} \) as in (3.45), equation (ii), we observe that

\[
U_N(t') = \left\| (0, P_{\geq N} u^p(t')), \mathcal{W}(t') P_{\geq N} \bar{\phi} \right\|_{\mathcal{E}} \leq \left\| u^p \right\|_{L^\infty(\mathbb{R}^+, L^\alpha)} \left\| \pi_2 \mathcal{W}(t') P_{\geq N} \bar{\phi} \right\|_{L^{\alpha'}} \leq C t'^{-2 - \delta} \tag{5.31} \]

for \( \delta = \frac{n}{2} \left( 1 - \frac{2}{\alpha'} \right) - 2 \). It turns out that \( \delta > 0 \) since, by (5.7), we have \( \alpha < \frac{2n}{n+4} \). It follows from (5.31) that

\[
\int_t^\infty U_N(t') \, dt' \leq \epsilon \tag{5.32} \]

for \( t > 0 \) sufficiently large. Now, from (2.3), equation (i), and (3.45), equation (ii), we observe that

\[
V_N(s, t') \leq \left\| (0, P_{\geq N} u^p(t')), \mathcal{W}(t' - s)(0, P_{\geq N} u^p(s)) \right\|_{L^{\alpha'}} \leq C |t' - s|^{-2 - \delta} \left\| u^p \right\|_{L^\infty(\mathbb{R}^+, L^\alpha)}. \]

Hence, by conservation of the energy (3.13), for \( 0 < \eta_1 = \frac{a'}{4} \eta \), where \( a, \eta \) are as in Step 5.2, we get that

\[
\left| \int_{t' \geq t} \int_{s \leq t} V_N(s, t') \, ds \, dt' \right| \leq C N^{-\delta \eta_1}. \tag{5.33} \]

Similarly, assuming \( t \geq N^{\eta_1} \),

\[
\left| \int_{t \geq t} \int_{s \leq t - N^{\eta_1}} V_N(s, t') \, ds \, dt' \right| \leq C N^{-\delta \eta_1}. \tag{5.34} \]
Writing \( I = \{ t - N^{\eta_1} \leq s \leq t \} \), and using (3.44) and (5.30), we get that
\[
\left| \int_{t}^{t+N^{\eta_1}} \int_{t-N^{\eta_1}}^{t} V_N(s, t') \, ds \, dt' \right| = \left| \int_{t}^{t+N^{\eta_1}} \left( \int_{0}^{t'} \mathbb{W}(t' - s) \left( 0, 1_{I}(s) P_{N^{\eta_1}}(s) \right) \, ds \right) \, dt' \right| \\
\leq \left\| P_{N^{\eta_1}} \right\|_{L^a \left( \mathbb{R}, L^{a'} \right)} \left\| 1_{I}(s) P_{N^{\eta_1}}(s) \right\|_{L^{a'} \left( \mathbb{R}, L^{a'} \right)},
\]
where \((a, b)\) is as in (5.8), equations (i)–(iii). Hence, using (5.9) in Step 5.1, and (5.12) in Step 5.2, we get that
\[
\left| \int_{t}^{t+N^{\eta_1}} \int_{t-N^{\eta_1}}^{t} V_N(s, t') \, ds \, dt' \right| \leq C |I|^{\frac{2}{\alpha} N^{-\eta}} \leq C N^{\frac{2}{\alpha} \eta_1} N^{-\eta}.
\]
Now, since \( \frac{2}{\alpha_1} \eta_1 - \eta = -\frac{1}{2} \eta \), we deduce from (5.29) and (5.32)–(5.35) that
\[
E_0(P_{N} \mathbb{W}) \leq C \epsilon + C \epsilon + C N^{-\delta \eta_1} + C N^{-\frac{\eta}{2}}
\]
for \( t \) sufficiently large. The last inequality in (5.2) follows from (5.36) if we take \( \eta_0 < \min \left( \frac{1}{2} \eta, \delta \eta_1 \right) \). Together with (5.15) this ends the proof of Lemma 5.1.

6. Almost finite speed propagation

We prove what we referred to as almost finite speed propagation in the introduction. Equation (6.1) in Lemma 6.1 basically states that solutions almost live in cones like \(|x| \leq R(2 + Kt)|\) for \( R \) sufficiently large. Lemma 6.1 states as follows.

**Lemma 6.1.** Let \( E > 0 \) and \( \alpha \) be as in (5.7). We consider (0.1) with \( \lambda < 0 \) and \( p \) as in (5.1). There exist \( \epsilon' > 0 \) and \( M > 1 \) such that for any \( N \geq 1 \), \( t_0 \geq 0 \), and \( \epsilon \leq \epsilon' \), if \( u \in \mathbb{E}_{\mathbb{R}^+} \) is a forward global solution of (0.1) of energy less than or equal to \( E \) satisfying (5.3) as in Corollary 5.1, then
\[
\int_{|x| \geq R(2 + Kt)} \left| u(t, x) \right|^p dx \leq (4M \epsilon)^{\frac{p}{\alpha}}
\]
for all \( t \geq t_0 \), where \( R, K \geq 0 \) do not depend on \( t \).

A useful corollary to Lemma 6.1 is as follows.

**Corollary 6.1.** Let \( n \geq 5 \), and let \( u \in \mathbb{E}_{\mathbb{R}^+} \) be a forward global solution of (0.1) with \( \lambda < 0 \) and \( p \) such that (5.1) holds true. Given \( \epsilon \), there exist \( T > 0 \) and \( R_1 > 0 \) such that
\[
\int_{|x| \geq R_1(1+t)} |u(t,x)|^{p+1} \, dx \leq \epsilon
\]  
(6.2)

for all \( t \geq T \).

**Proof.** Let \( E = E(u, u_t) \). For \( \epsilon' \) as in Lemma 6.1 we let also \( \epsilon_0 \leq \epsilon' \) to be chosen later on. By Corollary 5.1 there exist \( N > 0 \) and \( T > 0 \) such that for \( t \geq T \), \( E_0(P_{\geq N}(u(t), u_t(t))) \leq \epsilon_0 \). We may then apply Lemma 6.1, and we see that there exist \( R, K \geq 0 \) such that for \( t \geq T \), (6.1) holds true with \( \epsilon_0 \) in place of \( \epsilon \) and \( T \) in place of \( t_0 \). Independently, by conservation of the energy as in (3.13), and the Sobolev embedding theorem, we know that

\[
\int_{\mathbb{R}^n} |u(t,x)|^2 \, dx \leq C \sqrt{E}
\]  
(6.3)

for all \( t \). Then, by Hölder’s inequality, choosing \( \epsilon_0 \) to be sufficiently small, depending only on \( E \) and \( \epsilon \), we get from (6.3) that

\[
\int_{|x| \geq R(2+Kt)} |u(t)|^{p+1} \leq \left( \int_{|x| \geq R(2+Kt)} |u(t)|^{p\alpha} \right)^{\frac{2^p-(p+1)}{2^p-p\alpha}} \left( \int_{\mathbb{R}^n} |u(t)|^2 \right)^{\frac{p+1-p\alpha}{2^p-p\alpha}} \leq (4M\epsilon_0)^{\frac{p\alpha}{2^p-p\alpha}} (C\sqrt{E})^{\frac{p+1-p\alpha}{2^p-p\alpha}} \leq \epsilon.
\]

This proves (6.2), and thus the corollary, with \( R_1 = (2 + K)R \). \( \square \)

Now we prove Lemma 6.1 by splitting \( u \) into several parts as in (6.27) and (6.38). In view of time translation invariance, we can suppose \( t_0 = 0 \). Without loss of generality, we may also assume that \( m = 1 \) and \( \lambda = -1 \). We proceed in several steps. We let \( \alpha \) be as in (5.7) and let \( M > 0 \) be the sharp constant for the embedding of \( H^2 \) into \( L^{p\alpha} \). Then

\[
\|v\|_{L^{pa}} \leq M \|v\|_{H^2}
\]  
(6.4)

for all \( v \in H^2 \). Let \( u \) solve (0.1) and \( p \) be as in (5.1). We set \( u_0 = u(0), u_1 = u_t(0) \), and define \( \omega \) by

\[
(\omega(t), \omega_t(t)) = \mathcal{W}(t)(u_0, u_1),
\]  
(6.5)

where \( \mathcal{W}(t) \) is the isometry semigroup in Section 2. We let \( \varphi \) be given by

\[
\varphi(t, \xi, x) = t \sqrt{1 + |\xi|^4 - \langle x, \xi \rangle}
\]  
(6.6)

for all \( t \in \mathbb{R} \), and all \( \xi, x \in \mathbb{R}^n \). We also define
\[ K = \sup_{\xi \in B_0(N)} \frac{2|\xi|^3}{\sqrt{1 + |\xi|^4}}. \]  

(6.7)

Given \( e \in \mathbb{R}^n \), the notation \( \partial_e \varphi \) refers to \( \langle \nabla_\xi \varphi, e \rangle \). As a remark, for any \( i \geq 2 \), there exists \( M_i > 0 \) such that \( |\partial^i \varphi| \leq M_i t \), where \( \partial^i \) stands for iterations of length \( i \) of the derivatives \( \partial_e \varphi \) for \( e \) in the canonical basis of \( \mathbb{R}^n \).

**Step 6.1.** Let \( \epsilon > 0 \) and \( N \geq 1 \). There exists \( R_0 > 0 \) depending on \( \epsilon, N, n, p, u_0, \) and \( u_1 \), such that for any \( R \geq R_0 \) and any \( t \geq 0 \),

\[ \| r_2(t) \|_{L^{pa}} \leq M \epsilon, \]  

(6.8)

where \( r_2(t) = 1_{S_t} P_{< N} \omega(t) \), \( S_t = \{|x| \geq R(2 + Kt)\} \), \( K \) is as in (6.7), \( 1_{S_t} \) is the characteristic function of \( S_t \), \( \omega \) is as in (6.5), and \( M \) is an in (6.4).

**Proof.** In order to prove this step, we cut off the initial data at infinity and use a high-frequency cut-off to estimate the solution in the exterior of a cone. First, by density, we may find \( \bar{\varphi} = (\varphi_0, \varphi_1) \) for \( \varphi_0, \varphi_1 \in C^\infty_c (\mathbb{R}^n) \), such that

\[ E_0(u_0 - \varphi_0, u_1 - \varphi_1) \leq \epsilon^2 / 16. \]  

(6.9)

Let \( \chi \in C^\infty_c (\mathbb{R}^n) \) be a smooth function supported in \( B_0(\frac{3}{2}) \) and such that \( \chi = 1 \) in \( B_0(1) \). We let for any \( x \in \mathbb{R}^n \),

\[ \bar{w}_c(x) = \left( \chi \left(R^{-1} x\right) \varphi_0(x), \chi \left(R^{-1} x\right) \varphi_1(x) \right) \]  

(6.10)

and we let also

\[ \left(w_c(t), \partial_t w_c(t)\right) = \mathcal{W}(t) \bar{w}_c. \]  

(6.11)

Then, by dominated convergence, \( \bar{\varphi} - \bar{w}_c \) converges to 0 in \( \mathcal{E} \) as \( R \to +\infty \) and it follows that there exists \( R_0 \geq 2 \) depending on \( \varphi_0, \varphi_1, \epsilon \) such that

\[ E_0(\bar{\varphi} - \bar{w}_c) \leq \epsilon^2 / 16 \]  

(6.12)

for any \( R \geq R_0 \). From now on we assume that \( R \geq R_0 \). Then, by (6.4), the boundedness of \( P_{< N} \) on \( \mathcal{E} \), unitarity of \( \mathcal{W} \), (6.9), and (6.12), we get that for any \( R \geq R_0 \) and any \( t \geq 0 \),

\[ \| P_{< N} \left( \omega(t) - w_c(t)\right) \|_{L^{pa}} \leq \frac{M}{2} \epsilon. \]  

(6.13)

Now we estimate the norm of \( 1_{S_t} P_{< N} w_c(t) \). We do it through nonstationary phase estimates. We know from the explicit formula (3.38) that \( w_c \) will be a linear combination of terms like

\[ \Phi(x) = 1_{S_t}(x) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\psi(t, \xi, x-y)} \tilde{s}(\xi) \tilde{\phi}(y) \, dy \, d\xi, \]  

(6.14)
where $S_t$ is as in (6.8),

$$\tilde{s}(\xi) = \psi\left(\frac{2\xi}{N}\right) \text{ or } \tilde{s}(\xi) = \frac{\psi\left(\frac{2\xi}{N}\right)}{\sqrt{1 + |\xi|^4}}.$$  

where $\psi$ is as in (2.2), and $\tilde{\phi}(y) = \chi(R^{-1}y)\phi_j(y)$, $j = 0, 1$. Now we remark that, by (6.10), given $x \in S_t$, the expression in the integrand in (6.14) vanishes when $|x - y| \leq R^2 + Kt(R - 1)$. Hence, by (6.15) gives that for any $t \geq 0$ and any $\xi \in \mathbb{R}^n$,

$$|\partial e\varphi(t, \xi, x - y)| = \frac{2t|\xi|^2 \langle \xi, e \rangle}{\sqrt{1 + |\xi|^4}} - |x - y| \geq \frac{R}{2} + Kt(R - 1) \geq (R - 1)\left(\frac{1}{2} + Kt\right).$$ (6.15)

For $x, y$ such that $|x - y| > R^2 + Kt$, we consider the operator $L_{x,y}$ given by

$$L_{x,y}(h) = \frac{1}{\partial e\varphi(t, \cdot, x - y)} \partial e h$$ (6.16)

for $h \in C^\infty(\mathbb{R}^n)$. Integrating by parts $n$ times we get that

$$\left| \int_{\mathbb{R}^n} e^{i\varphi(t, \xi, x - y)} \tilde{s}(\xi) d\xi \right| = \left| \int_{\mathbb{R}^n} e^{i\varphi(t, \xi, x - y)} (L_{x,y}^*)^n \tilde{s} d\xi \right|,$$ (6.17)

where $L_{x,y}^*$, the adjoint operator of $L_{x,y}$ in (6.16), is defined for all $h \in C^\infty(\mathbb{R}^n)$ by the formula $L_{x,y}^* h = -\partial e \frac{\partial}{\partial e \varphi}$. Now, (6.15) gives that for any $\xi \in \mathbb{R}^n$,

$$\left| (L_{x,y}^*)^n \tilde{s}(\xi) \right| \leq C \frac{||\tilde{s}||}{R^n} \leq CR^{-n},$$ (6.18)

where $C$ does not depend on $R \geq R_0$, $N$, $t$, $x$, and $y$ such that $|x - y| > R^2 + Kt$. Hence, by (6.14), (6.17), and (6.18), we get that

$$\|\Phi\|_{L^\infty} \leq C \frac{B_0(N)}{R^n} \|\tilde{\phi}\|_{L^1} \leq C \frac{N^n}{R^2} \|\tilde{\phi}\|_{L^2},$$ (6.19)

where $\Phi$ is as in (6.14). On the other hand, it is clear from (6.14) and Parseval’s theorem that

$$\|\Phi\|_{L^2} \leq C \|\psi\|_{L^\infty} \|\tilde{\phi}\|_{L^2},$$ (6.20)

where $C$ depends only on $n$. Combining (6.19) and (6.20), we deduce by Hölder’s inequality that

$$\|\Phi\|_{L^{2n}} \leq \|\Phi\|_{L^\infty}^{1 - \frac{2}{2n}} \|\Phi\|_{L^2}^{\frac{2}{2n}} \leq C (N^n R^{-\frac{5}{2}})^{1 - \frac{2}{2n}},$$ (6.21)
where $C$ is independent of $R$ and $t$. In particular, we see with (6.21) that for $R \geq R_0$ sufficiently large, depending only on $u_0$, $u_1$, $N$, and $\epsilon$, for any $t \geq 0$,

$$\|1_S P_{<N} w_c(t)\|_{L^{p\alpha}} \leq \frac{M}{2} \epsilon$$

(6.22)

and consequently, by combining (6.13) and (6.22), we see that (6.8) holds true. This ends the proof of Step 6.1 \footnote{\textbf{268} B. Pausader / J. Differential Equations 241 (2007) 237–278}

Now, we want to estimate the contribution of the forcing term. For any $0 \leq t_1 \leq t$ we let

$$r_3(t, t_1) = -1_S \int_{t-t_1}^{t} \pi_1 \mathcal{W}(t-s)(0, P_{<N} u^p(s)) \, ds,$$

and

$$r'_3(t, t_1) = - \int_{t-t_1}^{t} \pi_1 \mathcal{W}(t-s)(0, u^p(s)) \, ds,$$

(6.23)

where $S_t$ is as in (6.8). The next step in the proof of Lemma 6.1 states as follows.

**Step 6.2.** There exists $t_2 > 0$, depending only on $E$ and $\epsilon$, such that

$$\|r_3(t, t_1)\|_{L^{p\alpha}} \leq M \epsilon$$

(6.24)

for all $t \geq 0$, where $t_1 = \min(t_2, t)$.

**Proof.** We remark that since $p < 2^*-1$, we have that $\frac{4(p+1)-n(p-1)}{2(p+1)} > 0$. Then by (3.46), and the Sobolev embedding theorem,

$$\|r'_3(t, t_1)\|_{L^{p+1}} \leq \int_{t-t_1}^{t} \|\pi_1 \mathcal{W}(t-t')(0, u^p(t'))\|_{L^{p+1}} \, dt'$$

$$\leq C \int_{t-t_1}^{t} (t-t')^{1-\frac{p-1}{2(p+1)}} \|u^p\|_{L^\infty([t-t_1, t], L^{\frac{p+1}{p}})} \, dt'$$

$$\leq C |t_1| \frac{4(p+1)-n(p-1)}{2(p+1)} \epsilon_0$$

(6.25)

for all $t \geq 0$ and all $t_2 \leq 1$ sufficiently small, depending only on $n$, $p$, $E$, $\epsilon_0$, where $\epsilon_0$ is some small parameter to be chosen later on. Besides, for any $t \geq 0$ and any $t_1 \in [0, t]$,

$$r'_3(t, t_1) = u(t) - u(t-t_1) - \omega(t) + \omega(t-t_1).$$

Hence, by conservation of the energy as in (3.13), $r'_3$ is bounded in $L^2$ uniformly in $t$, $t_1$, so that for any $t \geq 0$, and any $t_1 \in (0, t_2)$, since $P_{<N}$ is bounded on $L^{p+1}$,
\[ \|r_3(t, t_1)\|_{L^{p+1}} \leq \|P_{<N}r'_3(t, t_1)\|_{L^{p+1}} \leq C \|r'_3\|_{L^{p+1}} \leq C \epsilon_0, \quad \text{and} \]
\[ \|r_3(t, t_1)\|_{L^2} \leq \|r'_3(t, t_1)\|_{L^2} \leq 4\sqrt{E}. \]  
(6.26)

By Hölder’s inequality and (6.26), for \( \epsilon_0 \) correctly chosen depending only on \( \epsilon, E, \) and \( n \), we get that (6.24) holds true. This ends the proof of Step 6.2. □

Now, for any \( t' \geq 0 \), we split \( u(t') \) into
\[ u(t') = 1_{S'_t}u(t') + 1_{S_t}u(t') = u_c(t') + u_f(t'), \]  
(6.27)
where \( S'_t \) stands for the complement of \( S_t \). The forcing term also splits as \( u^P = u^P_c + u^P_f \). In what follows we estimate the contribution from \( u_c \). For any time \( t \geq 0 \), we let
\[ r_4(t) = -1_{S_t} \int_{t-t_1}^t \pi_1 \mathcal{W}(t-t')\left(0, P_{<N}u^P_c(t')\right) dt', \]  
(6.28)
where \( t_1 \) is as in Step 6.2, and \( S_t = \{ |x| \geq R(2 + Kt) \} \) is as in (6.8). A third step in the proof of Lemma 6.1 is as follows.

**Step 6.3.** Let \( \epsilon > 0 \). There exists \( R > 0 \) such that
\[ \|r_4(t)\|_{L^{q'}} \leq \epsilon^p \]  
(6.29)
for all \( t \geq 0 \), where \( r_4 \) is as in (6.28).

**Proof.** For any \( t \) and \( t' \), we define the operator \( \mathcal{V}_{t,t'} \) on \( L^1 \) by
\[ \mathcal{V}_{t,t'} h(x) = 1_{S_t}(x) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\varphi(t-t',\xi,x-y)} \tilde{s}(\xi) 1_{S'_t}(y) h(y) dy d\xi, \]  
(6.30)
where \( h \in L^1 \) and \( \varphi \) is as in (6.6). Also, for any \( \xi \in \mathbb{R}^n \), we let
\[ \tilde{s}(\xi) = \psi\left(\frac{2\xi}{N}\right)/\sqrt{1 + |\xi|^4}, \]
where \( \psi \) is as in (2.2). We claim that this operator satisfies that for any \( q \geq 2 \), there exists \( C \) independent of \( K, N, \epsilon, \) and \( R \geq 2 \) such that for any for any \( t \geq t' \geq 0 \), and any function \( h \in L^1 \cap L^{q'}, \)
\[ \|\mathcal{V}_{t,t'} h\|_{L^{q'}} \leq C\left((t-t')(R-1)\right)^{-n(1-\frac{2}{q})} \|h\|_{L^{q'}}. \]  
(6.31)
We prove (6.31) in what follows.

First, we note that when \( t \geq t' \), \( x \in S_t \), and \( y \in S'_t \), then \( |x - y| \geq KR(t-t') \) and under these conditions, if \( e = (x - y)/\|x - y\| \), we get that
\[ |\partial_\xi \varphi(t-t',\xi,x-y)| \geq K(t-t')(R-1). \]  

(6.32)

Then, again, let \( L_{x,y} \) be as in (6.16). When \( x \in S_t \), after \( n \) integrations by parts in the \( \xi \) variable, we get that

\[
|\mathcal{V}_{t,t'} h(x)| \leq \int_{\mathbb{R}^n} |S_t^N(y)| h(y) \left| \int_{\mathbb{R}^n} e^{i\varphi(t-t',\xi,x-y)} (L^*_x y)^n d\xi \right| dy \\
\leq C |\text{supp} \tilde{\xi}| (K(R-1)(t-t'))^{-n} \|h\|_{L^1}.
\]  

(6.33)

where \( C \) does not depend on \( h, N, K, R, t', t \). Furthermore, by Parseval’s theorem,

\[
\|\mathcal{V}_{t,t'} h\|_{L^2} \leq C \|h\|_{L^2},
\]  

(6.34)

where \( C \) is independent of \( h, N, t, t', R, \text{ and } K \). By the Riesz–Thorin theorem, we deduce from (6.33) and (6.34) that for any \( q \geq 2 \),

\[
\|\mathcal{V}_{t,t'} h\|_{L^q} \leq C (K(R-1)(t-t'))^{-n(1-\frac{1}{2}q)} N^{n(1-\frac{1}{2}q)} \|h\|_{L^q}.
\]  

(6.35)

As is easily checked, (6.35) implies (6.31) since by (6.7), \( N/K \leq 1 \).

Now we prove (6.29). We let \( t \geq 0 \) and \( t_1 \) be as in Step 6.2. Using (6.4), (6.28), (6.31), and since \( n(1-2/\alpha') > 1 \), we get that, for \( R \) sufficiently large, depending only on \( p, E, \epsilon, \text{ and } t_2 \), and for any \( t \geq 0 \),

\[
\|r_4(t)\|_{L^{\alpha'}} = \left\| 1_{S_t} \int_0^{t-t_1} \pi_1 W(t-t')(0, P_{<N} u_c(t')) dt' \right\|_{L^{\alpha'}} \\
\leq \frac{1}{(2\pi)^n} \int_0^{t-t_1} \|\mathcal{V}_{t,t'} u_c^p(t')\|_{L^{\alpha'}} dt' \\
\leq C \int_0^{t-t_1} (R(t-t'))^{-n(1-\frac{1}{2}q)} \|u_c^p\|_{L^\infty(\mathbb{R}_+,L^{\alpha'})} dt' \\
\leq CR^{-n(1-\frac{1}{2}p)} \|u\|_{L^\infty(\mathbb{R}_+,H^2)}^p \\
\leq \epsilon^p.
\]

This proves (6.29), and thus Step 6.3. \( \Box \)

With Steps 6.1–6.3 we are now in position to prove Lemma 6.1.

**Proof of Lemma 6.1.** Let \( t \geq 0 \), and \( t_1 \) be as in Step 6.2. Let also \( r_1 \) and \( r_5 \) be given by
\[ r_1(t) = 1_{s_t} P_{\geq N} u(t), \quad \text{and} \]
\[ r_5(t) = -1_{s_t} \int_0^{t-t_1} \pi_1 \mathcal{W}(t-s) (0, P_{<N} u^p_f(s)) \, ds. \]  
(6.36)

By (5.3) and (6.4), we have that
\[ \| r_1 \|_{L^\infty(\mathbb{R}_+, L^p)} \leq M \epsilon, \]  
(6.37)
where \( \alpha \) is as in (5.7). Independently, by Duhamel’s formula (3.47), we can write that
\[ u_f(t) = r_1(t) + r_2(t) + r_3(t, t_1) + r_4(t) + r_5(t), \]  
(6.38)
where \( u_f \) is as in (6.27), and the \( r_i \)'s are as in (6.8), (6.23), (6.28), and (6.36). Besides, for any \( t \geq 0 \),
\[ r_4(t) + r_5(t) = 1_{s_t} P_{<N} \pi_1 \left[ \mathcal{W}(t_1) \left( u(t - t_1), u_1(t - t_1) \right) - \mathcal{W}(t)(u_0, u_1) \right], \]  
(6.39)
and, by boundedness of \( P_{<N} \) on \( L^2 \), conservation of the energy as in (3.13), and since \( \mathcal{W} \) is a unitary operator, we thus get from (6.39) that
\[ \| r_4 + r_5 \|_{L^2} \leq 2 \sqrt{E}. \]  
(6.40)
Now, we remark that for \( t \leq t_2 \), we get that \( r_4, r_5 = 0 \). As a consequence, we have that
\[ \| u_f(t) \|_{L^p} \leq r_1(t) + r_2(t) + r_3(t, t_1) \leq 3M \epsilon \]  
(6.41)
for all \( t \leq t_2 \). We let
\[ t_0 = \sup \{ t \geq 0 : \forall s \in [0, t], \| u_f(s) \|_{L^p} \leq 4M \epsilon \} \]  
(6.42)
and we assume that \( t_0 < +\infty \). We know from (6.41) that \( t_0 > t_2 \). By continuity we then get that
\[ \| u_f(t_0) \|_{L^p} = 4M \epsilon. \]  
(6.43)
However, by the decay estimates (3.46), (5.8) equation (iv), (6.29), (6.40), and since \( P_{<N} \) is bounded on \( L^\alpha \), we can write that
\[
\begin{align*}
\| u_f(t_0) \|_{L^p} & \leq \| r_1(t_0) \|_{L^p} + \| r_2(t_0) \|_{L^p} + \| r_3(t_0, t_1) \|_{L^p} + \| r_4(t_0) \|_{L^p} + \| r_5(t_0) \|_{L^p} \\
& \leq 3M \epsilon + (2\sqrt{E})^{1-\theta} \left( \int_0^{t_0-1} \| \pi_1 \mathcal{W}(t_0 - t') \left( 0, P_{<N} u^p_f(t') \right) \|_{L^\alpha'} \, dt' \right)^\theta
\end{align*}
\]
\[
\leq 3M \varepsilon + C \left( \int_0^{t_0-1} (t_0 - t')^{-\frac{q}{2}} \left\| P_{<N} u_{f}' \right\|_{L^\infty([0,t_0), L^\alpha)} \right)^\theta \\
\leq 3M \varepsilon + \tilde{C} \left\| u_{f} \right\|_{L^\infty([0,t_0), L^\alpha)}^\theta,
\]

where \( \theta \) is such that \( 0 < \theta < 1 < p \theta \) as in (5.8), equation (iv). The first inequality in (6.44) is by (6.38), the second inequality is by (6.8), (6.24), (6.37), and (5.8), equation (iv), the third inequality is by the second equation in (6.36) and (6.40), the fourth inequality is by the first bound in (3.46) and (6.29), and the fifth inequality is by boundedness of \( P_{<N} \) on \( L^\alpha \) and the fact that \( \frac{n}{4} (1 - \frac{2}{\alpha'}) > 1 \) since \( \alpha < \frac{2n}{n + 4} \). Then, by (6.42), we get that

\[
\left\| u_{f}(t_0) \right\|_{L^{p\alpha}} < 4M \varepsilon
\]

for \( \varepsilon \leq \varepsilon' \), where \( \varepsilon' \) is chosen sufficiently small such that \( \varepsilon'^{p\theta-1} \tilde{C} (1 + (4M)^p)^\theta < M \), and \( \tilde{C} \) depends only on \( E, n, \) and \( p \). Clearly, (6.45) is in contradiction with (6.43). Hence \( t_0 = +\infty \), where \( t_0 \) is given by (6.42). This proves (6.1) and Lemma 6.1. \( \square \)

7. Proof of the theorem

We prove our theorem in this section. In addition to the material developed in the preceding sections, a key ingredient we need in the proof is a Morawetz estimate [26] obtained by Levandosky and Strauss [19]. We refer also to Lin [21]. Let \( n \geq 5 \) and \( u \in \mathbb{R}^n \) be a forward global solution of the nonlinear equation (0.1) with \( 1 + \frac{8}{n} \leq p \leq 2^* - 1 \) and \( \lambda < 0 \). Then, as proved in Levandosky and Strauss [19], it holds that

\[
\int_0^\infty \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|} \, dt \, dx \leq C,
\]

where \( C > 0 \) depends on \( n \) and \( u \) only through the energy. We prove our theorem in what follows, using the method developed by Lin and Strauss [22] and Morawetz and Strauss [27] for the Schrödinger and Klein–Gordon equations.

**Proof of the theorem.** As above, we may assume that \( m = 1 \) and \( \lambda = -1 \). Let \( n \geq 5 \) and \( u \) be a solution of (0.1) with \( p \) such that (5.1) holds true. By Corollary 4.1 in Section 4 it suffices to prove that

\[
\left\| u(t) \right\|_{L^{p+1}} \to 0
\]

as \( t \to +\infty \). The surjectivity of \( W_+ \), and the continuity of its inverse, come from the remark after Corollary 4.1. In order to prove (7.2) we claim that for any \( \varepsilon > 0 \), \( t_0 \geq 0 \), and \( t_1 > 0 \), there exists \( t_2 > t_0 \) such that

\[
\sup_{t' \in [t_2-t_1,t_2]} \left\| u(t') \right\|_{L^{p+1}} \leq \varepsilon.
\]
We prove (7.3) in what follows. Applying Corollary 6.1, we get that there exist \( T, R > 0 \) such that

\[
\int_{|x| \geq R(1+t)} |u(t, x)|^{p+1} \, dx \leq \epsilon_1
\]  

(7.4)

for all \( t \geq T \), where \( \epsilon_1 \), depending only on \( n, E, \) and \( p, \) is to be chosen later on. We let \( t'_0 = \max(T, t_0) \). Given \( \epsilon_0 > 0 \) and \( \tau > 0 \), there exists \( \tilde{t} > t'_0 + 2\tau \) such that

\[
\int_{\tilde{t}-2\tau}^{\tilde{t}} \int_{|x| \leq R(1+t')} |u(t', x)|^{p+1} \, dx \, dt' \leq \epsilon_0.
\]  

(7.5)

Indeed, by the Morawetz estimate (7.1), we can write that

\[
\int_{t'_0}^{\infty} \int_{|x| \leq R(1+t')} |u(t', x)|^{p+1} \, dx \, dt' \geq \sum_{k=0}^{\infty} \frac{1}{R(1 + (t'_0 + 2(k+1)\tau))} \int_{t'_0+2k\tau}^{t'_0+2(k+1)\tau} \int_{|x| \leq R(1+t')} |u(t', x)|^{p+1} \, dx \, dt'.
\]

Since

\[
\sum_k \frac{1}{R(1 + (t'_0 + 2(k+1)\tau))} = \infty,
\]

there exists \( k_0 > 0 \) such that

\[
\int_{t'_0+2(k_0+1)\tau}^{\infty} \int_{|x| \leq R(1+t')} |u(t', x)|^{p+1} \, dx \, dt' \leq \epsilon_0.
\]  

(7.6)

Letting \( \tilde{t} = t'_0 + 2(k_0 + 1)\tau \), (7.6) gives that (7.5) holds true. Now that we have (7.5), we write with Duhamel’s formula (3.47) that for any \( t \geq \sigma \),

\[
(u(t), u_t(t)) = \mathcal{W}(t)(u_0, u_1) - \int_0^{t-\sigma} \mathcal{W}(t-t')(0, u^p(t')) \, dt' - \int_{t-\sigma}^{t} \mathcal{W}(t-t')(0, u^p(t')) \, dt'
\]

\[
= (v(t), v_t(t)) + (w(t, \sigma), w_t(t, \sigma)) + (z(t, \sigma), z_t(t, \sigma)),
\]  

(7.7)

where \( \sigma \geq 0 \) is to be chosen later on. We observe that

\[
\| v(t) \|_{L^{p+1}} \to 0
\]  

(7.8)
as \( t \to +\infty \). Indeed, let \( \delta > 0 \) arbitrary, and let \( \phi_0, \phi_1 \in C_c^\infty (\mathbb{R}^n) \) be such that \( E_0(u_0 - \phi_0, u_1 - \phi_1) \leq \delta \). Then we define \((\omega(t), \omega_t(t)) = W(t)(\phi_0, \phi_1)\). By conservation of the energy, the Sobolev embedding theorem, and the decay estimate (3.18),

\[
\|v(t)\|_{L^{p+1}} \leq C \|\omega(t) - v(t)\|_{H^2} + \|\omega(t)\|_{L^{p+1}} \\
\leq C \delta + C \left( t^{-\frac{n}{2}(1-\frac{2}{p+1})} + t^{-\frac{n}{2}(1-\frac{2}{p+1})} \right) \leq 2C \delta
\]

for \( t \geq t_0 \) sufficiently large depending only on \( n, p, \phi_0, \phi_1, \) and \( \delta \). This prove (7.8). As a consequence of (7.8), we get that there exists \( t_0'' \) such that for any time \( t' \geq t_0'' \),

\[
\|v(t')\|_{L^{p+1}} \leq \epsilon \quad (7.9)
\]

Now, we let \( \beta \geq 1 \) be such that

\[
\beta = \begin{cases} 
\frac{2}{p} & \text{if } p \leq 2, \\
1 & \text{otherwise}.
\end{cases} \quad (7.10)
\]

Then, \( 1 - 2/\beta' = \min(1, p-1) \), and \( \beta < 2n/(n+4) \). By the decay estimate (3.46) and the Sobolev embedding theorem, for \( \beta \) as in (7.10), we get that

\[
\|w(t, \sigma)\|_{L^{\beta'}} \leq C \int_0^{t-\sigma} (t-t')^{-\frac{n}{4}(1-2/\beta')} \|u(t')\|_{L^{p\sigma}} dt' \\
\leq C \sigma^{-\frac{4-n\min(1,p-1)}{4}} \sup_{t'} \|u(t')\|_{H^2}, \quad (7.11)
\]

where \( C > 0 \) depends only on \( n \). Independently, we see from (7.7) that

\[
(w(t, \sigma), w_t(t, \sigma)) = W(\sigma)(u(t - \sigma), u_t(t - \sigma)) - W(t)(u_0, u_1). \quad (7.12)
\]

Since \( W \) is unitary on \( \mathcal{E} \), and \( E_0(u, u_t) \) remains bounded, we see from (7.12) that \( w(t, \sigma) \) remains bounded in \( L^2 \). Hence, since \( 2 < p + 1 < \beta' \leq +\infty \), by Hölder’s inequality, (7.11), and the boundedness of \( w(t, \sigma) \) in \( L^2 \), we get that there exist positive constants \( C \) and \( K \), depending only on \( n, p, \) and \( E \), such that for any \( \sigma > 0 \), and any \( t \geq \sigma \),

\[
\|w(t, \sigma)\|_{L^{p+1}} \leq C \|w(t, \sigma)\|_{L^{\beta'}} \leq K \sigma^{-\frac{(n(p-1)-4\max(1,p-1))}{4(p+1)}}. \quad (7.13)
\]

As a consequence, there exists \( \sigma_0 \) such that

\[
\|w(t, \sigma)\|_{L^{p+1}} \leq \frac{\epsilon}{3} \quad (7.14)
\]

for all \( \sigma \geq \sigma_0 \), and all \( t \geq \sigma \).
Finally, we remark that since $p < 2^2 - 1$, we may find $q \in [1, 4(p + 1)/n(p - 1))$ such that $pq' \geq p + 1$, and, again with the decay estimate (3.46) and the fact that $u$ remains bounded in $L^{p+1}$, we get, for $z(t, \sigma)$ as in (7.7), that

$$\|z(t, \sigma)\|_{L^{p+1}} \leq C \int_{t-\sigma}^{t} (t-t')^{-\frac{n(p-1)}{4(p+1)}} \|u(t')\|_{L^{p+1}}^p dt'$$

$$\leq C \left( \int_{t-\sigma}^{t} (t-t')^{-\frac{n(p-1)q}{4(p+1)}} dt' \right)^{\frac{1}{q}} \left( \int_{t-\sigma}^{t} \|u(t')\|_{L^{p+1}}^{pq'} dt' \right)^{\frac{1}{q'}}$$

$$\leq C \sigma^\delta \left( \int_{t-\sigma}^{t} \|u(t')\|_{L^{p+1}}^{p+1} dt' \right)^{\frac{1}{q'}}$$

(7.15)

for some $\delta > 0$, where $C$ depends only on $n$, $p$, $\sigma$, and $E$ but not on $t$. Note that $\frac{n(p-1)q}{4(p+1)} < 1$ thanks to our assumption on $q$. Now, for $\sigma_0$ as in (7.14) and $t''_0$ as in (7.9), we let

$$t_1 = \max(\sigma_0, t''_0),$$

(7.16)

and we choose $t_2 > t'_0 + 2t_1$ such that (7.5) holds true for $\tau = t_1$, $\tilde{t} = t_2$ and $\epsilon_0$ small in a sense to be made precise below. Since $[t - t_1, t] \subset [t_2 - 2t_1, t_2]$ when $t \in [t_2 - t_1, t_2]$, we get with (7.4), (7.5), and (7.15) that

$$\|z(t, t_1)\|_{L^{p+1}} \leq C t_1^\delta \left( \int_{t-t_1}^{t} \int_{|x| \leq R(1+t')} |u(t', x)|^{p+1} dx dt' + t_1 \sup_{t' \in [t, t-t_1]} \|u(t')\|_{L^{p+1}}^{p+1} \right)^{\frac{1}{q'}}$$

$$\leq C t_1^\delta (\epsilon_0 + t_1 \epsilon_1)^{\frac{1}{q'}} \leq \frac{\epsilon}{3}$$

(7.17)

for $\epsilon_0$ and $\epsilon_1$ sufficiently small depending only on $n$, $p$, and $t_1$. Estimates (7.15)–(7.17) can be regarded as the key estimates in this section.

By combining (7.9), (7.14), (7.16), and (7.17) we get that (7.3) holds true. Now that we have (7.3), we prove that (7.2) also holds true. Given $\epsilon > 0$ sufficiently small, we let $\sigma_\epsilon$ large be such that

$$K \sigma_\epsilon^{-\frac{n(p-1)-4 \max(1, p-1)}{4(p+1)}} = \frac{\epsilon}{4},$$

(7.18)

where $K$ is the constant (depending only on $E$, $n$, and $p$) appearing in (7.13). By (7.7), we can write that $u(t) = v(t) + w(t, \sigma) + z(t, \sigma)$ with $\sigma = \sigma_\epsilon$. We let $t''_0$ be such that (7.9) holds true for $t' \geq t''_0$. For $t \geq \max(t''_0, \sigma_\epsilon)$,

$$\|u(t)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + \|z(t, \sigma_\epsilon)\|_{L^{p+1}}.$$  

(7.19)
and by the decay estimate (3.46) we get that there exist $C, C' > 0$, depending only on $p$ and $n$, such that

$$
\|z(t, \sigma \epsilon)\|_{L_p} \leq C \int_{t-\sigma \epsilon}^{t} (t-t')^{-\frac{n(p-1)}{4(p+1)}} \|u(t')\|_{L_p+1}^p dt'.
$$

$$
\leq C' \sigma \epsilon^{1-\frac{n(p-1)}{4(p+1)}} \sup_{[t-\sigma \epsilon, t]} \|u\|_{L_p+1}^p.
$$

(7.20)

There exists $t_2 \geq \max(t_0'', \sigma \epsilon)$ such that (7.3) holds true with $t_1 = \sigma \epsilon$. We let

$$
t_\epsilon = \sup\{ t \geq t_2 : \forall s \in [t_2 - \sigma \epsilon, t], \|u(s)\|_{L_p+1} \leq \epsilon \}.
$$

(7.21)

Assuming that $t_\epsilon \neq \infty$, and since the map $t \mapsto u(t)$ is continuous on $L_p+1$, we get that $\|u(t_\epsilon)\|_{L_p+1} = \epsilon$. From this, (7.19), and (7.20), we see that

$$
\epsilon \leq \epsilon + C' \sigma \epsilon^{1-\frac{n(p-1)}{4(p+1)}} \epsilon^p.
$$

Hence, $\sigma \epsilon^{1-\frac{n(p-1)}{4(p+1)}} \epsilon^{p-1} \geq 1/2C'$, and, by (7.18), we get that

$$
\sigma \epsilon^{\gamma} \geq \frac{1}{2C'(4K)^{p-1}},
$$

(7.22)

where $K$ and $C'$ depend only on $n, p,$ and $E$, and where

$$
\gamma = -\frac{np(p-1) - 4(p+1) + (p-1) \max(1, p-1))}{4(p+1)}.
$$

(7.23)

When $p < 2$, we get with (7.23) that

$$
\gamma = -\frac{np(p-1) - 8p}{4(p+1)} = \frac{2p}{p+1} \left(1 - np + \frac{1}{8}\right)
$$

(7.24)

and $\gamma$ is negative when $p$ satisfies (5.1), while if $p \geq 2$, we get from (7.23) that

$$
\gamma = -\frac{n-4}{4(p+1)} \left(p^2 - p - \frac{8}{n-4}\right).
$$

(7.25)

If we let $h(x) = x^2 - x - 8/(n-4)$, then $h$ is increasing for $x \geq 1$ and, hence, $h(p) > h((n+8)/n) > 0$ when $n \geq 8$. Finally, in the two cases (7.24) and (7.25), we have that $\gamma < 0$. Then, (7.22) together with (7.18) imply that

$$
\epsilon \geq 4K \left(2C'(4K)^{p-1}\right)^{\frac{n(p-1) - 4\max(1, p-1)}{4(p+1)\gamma}},
$$

(7.26)
where the right-hand side depends only on $E$, $p$, and $n$. Letting $\epsilon_0 > 0$ be smaller than the right-hand side in (7.26), we get a contradiction for any $\epsilon \leq \epsilon_0$. This proves that for such $\epsilon$'s, $t_\epsilon = +\infty$. In particular, for any $\epsilon > 0$ sufficiently small, there exists $T > 0$ such that $\|u(t)\|_{L^{p+1}} \leq \epsilon$ for all $t \geq T$. Replacing the $L^{p+1}$-norm by a $L^q$-norm for $q < 2^k - 1$ close to $2^k - 1$, the above argument also gives the result when $5 \leq n \leq 7$. This proves (7.2). As already mentioned, this also proves our theorem. □

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