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A THEORETICAL FRAMEWORK FOR CONVERGENCE AND CONTINUOUS DEPENDENCE OF ESTIMATES IN INVERSE PROBLEMS FOR DISTRIBUTED PARAMETER SYSTEMS

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In this note, we present a general convergence/stability (continuous dependence on observations) framework for methods to treat parameter identification problems (see [BCK]) involving distributed parameter systems. This new parameter estimation convergence framework combines a weak version of the system (in terms of sesquilinear forms [BK]) with the resolvent convergence form of the Trotter-Kato approximation theorem [P]. The very general results depend on three properties of the parameter dependent sesquilinear form describing the system: (A) continuity (with respect to the parameter); (B) uniform (in the parameter) coercivity; and (C) uniform (in the parameter) boundedness. The approach permits one to give convergence and stability arguments in inverse problems under extremely weak compactness assumptions on the admissible parameter spaces Q (equivalent to those in typical variational or weak approaches–see [B], [BCR]) without requiring knowledge of smoothness of solutions usually a part of the variational and general finite element type arguments. Thus this approach combines in a single framework the best features of a semigroup approximation approach (e.e., the Trotter-Kato theorem) with the best features of a variational approach (weak assumptions on Q).

While the approach does involve coercivity of certain sesquilinear forms associated with the system dynamics, its applicability is not restricted to parabolic systems which generate analytic semigroups. As we shall point out below, it can be used to treat problems in which the underlying semigroup is not analytic (i.e., those involving Euler-Bernoulli equations for beams with various types of damping-viscous, Kelvin-Voigt, spatial hysteresis), improving substantially on some of the currently known results for thse problems. With appropriate modifications, we believe this theoretical framework can be generalized to allow an elegant treatment of problems involving functional partial differential equations, i.g., beams with Boltzman damping (i.e., time hysteresis).

The weakening of the compactness criteria on Q is of great computational importance since the constraints associated with these criteria should be implemented in computational procedures to guarantee stability and convergence in inverse problems (see [BI] for further discussion and examples).

We consider first order systems dependent on parameters $q \epsilon Q$ described by an abstract equation

$$\begin{aligned} \dot{u}(t) &= A(q)u(t) + F(t,q) \\ u(0) &= u_0(q) \end{aligned} \tag{1}$$

in a Hilbert space H. The admissible parameter space Q is a metric space with metric d and for $q \epsilon Q$, we assume that $A(q)$ is the infinitesimal generator of a C_0 semigroup $T(t;q)$ on H. We assume that we are given observations $\tilde{u}_i \epsilon H$ for the mild solution values $u(t_i, q)$ of (1); i.e., we solve (1) in the sense

$$u(t;q) = T(t;q)u_0(q) + \int_0^t T(t-s;q)F(s,q)ds \tag{2}$$

in H. We then consider the least squares identification (ID) problem of minimizing over $q \epsilon Q$ the functional

$$J(q) = \sum_i |u(t_i;q) - \tilde{u}_i|^2. \tag{3}$$

Such problems are, in general infinite dimensional in both the state u and the parameter q and thus one must consider a sequence of computationally tractable approximating problems. These can be, for our purposes, best described in terms of parameter dependent sesquilinear forms $\sigma(q)(\bullet,\bullet)$ associated with (1) or (2) (i.e., forms which define the operators $A(q)$ in (1)). For details in the parabolic case, we refer the reader to [BK]. Briefly, let V and H be Hilbert spaces with V continuously and densely imbedded in H. Denote a family of parameter dependent sesquilinear forms by $\sigma(q): V \times V \to \mathbb{C}$, $q \epsilon Q$. We assume that σ possesses the following properties:

(A) Continuity: For $q, \tilde{q} \epsilon Q$, we have for all $\phi, \psi \epsilon V$

$$|\sigma(q)(\phi,\psi) - \sigma(\tilde{q})(\phi,\psi)| \le d(q,\tilde{q})|\phi|_V |\psi|_V.$$

(B) Coercivity: There exist $c_1 > 0$ and some λ such that for $q \epsilon Q$, $\phi \epsilon V$ we have

$$\sigma(q)(\phi,\phi) + \lambda|\phi|_H^2 \ge c_1|\phi|_V^2.$$

(C) Boundedness: There exist $c_2 > 0$ such that for $q \epsilon Q, \phi, \psi \epsilon V$ we have

$$|\sigma(q)(\phi,\psi)| \le c_2|\phi|_V |\psi|_V.$$

Under these assumptions, σ defines in the usual manner (e.g, see [K], [S]) operators $A(q)$ such that $\sigma(q)(\phi,\psi) = < -A(q)\phi, \psi >_H$ for $\phi \epsilon \; dom(A(q))$, $\psi \epsilon V$ with $dom(A(q))$ dense in V. Furthermore, $A(q)$ is the generator of an analytic semigroup $T(t;q)$ on H (indeed $A(q)$ is sectorial with $(\lambda I - A(q) \; dom(A(q)) = H)$. Property (B) guarantees that the resolvent operator $R_\lambda(A(q)) \equiv (\lambda I - A(q))^{-1}$ exists as a bounded operator on H and, moreover, using (B) and (A), one can argue that $q \to R_\lambda(A(q))$ is continuous on Q. It is these ideas that can be modified to give resolvent convergence in the approximation schemes.

We consider Galerkin type approximations in the context of sesquilinear forms (see [BK] for details). Let H^N be a family of finite dimensional subspaces on H satisfying $p^N z \to z$ for $z \epsilon H$ where p^N is the orthogonal projection of H onto H^N. We further assume that $H^N \subset V$ and possess certain V-approximation properties to be specified below. If we now consider the restriction of $\sigma(q)(\bullet,\bullet)$ to $H^N \times H^N$, we obtain operators $A^N(q): H^N \to H^N$ which, because of (B), satisfy a uniform dissipative inequality and can be shown to generate semigroups $T^N(t;q)$ in H^N. These are then used to define approximating systems for (2):

$$u^N(t;q) = T^N(t;q)p^N u_0(q) + \int_0^t T^N(t-s;q)p^N F(s,q)ds. \tag{4}$$

One thus obtains a sequence of approximating ID problems consisting of minimizing over Q.

$$J^N(q) = \sum_i |u^N(t_i;q) - \tilde{u}_i|^2. \tag{5}$$

In problems where Q is infinite dimensional (the usual case in many inverse problems of interest), one must also make approximations Q^M for Q (see [BD] for details). For sake of brevity, we shall not do that here.

To obtain convergence and continuous dependence results for the solutions $\bar{q}^N$ of minimizing J^N in (5), it suffices under the assumption that (Q,d) is a compact space (see [B]) to argue: for arbitrary $\{q^N\} \subset Q$ with $q^N \to q$ we have $u^N(t;q^N) \to u(t;q)$ for each t. Under reasonable assumptions on F and u_0, this can be argued if one first shows that $T^N(t;q^N)p^N \to T(t;q)z$ for arbitrary $q^N \to q$ and $z \epsilon H$. To do this one can use a version of the Trotter-Kato theorem [P] which yields that the convergence to "$T^N(t;q^N)p^N z \to T(t;q)z$ uniformly in t on compact intervals" is equivalent to (i) $|T^N(t;q^N)| \leq Me^{\omega t}$ for M, ω independent of N, and (ii) there exists λ with $Re(\lambda) > \omega$ such that $R_\lambda(A^N(q^N))p^N z \to R_\lambda(A(q))z$ for $z \epsilon H$. Finally, one can use (A), (B), (C) and the V-approximation condition on H^N:

(C1) for each $z \epsilon V$, there exist $\hat{z}^N \epsilon H^N$ such that $|z - \hat{z}^N|_V \to 0$ as $N \to \infty$,

to argue the desired resolvent convergence. We note that one actually obtains all of the convergence statements mentioned above in the V norm which has important consequences for the ID problem when observations are made in a sense that may not be continuous in the H norm (e.g., pointwise in the spatial coordinates as well as time coordinates).

Among the examples that can be treated immediately with the above theory are the usual parabolic systems (see [BK] or [L]) where $V = H_0^1(\Omega)$ and $H = L_2(\Omega)$. This theory also allows an elegant and succinct treatment of domain identification problems arising in thermal tomography and moment estimation problems related to the Fokker-Planck equation for stochastic transition models (details of these two applications will appear elsewhere). In these and many other cases of interest, the continuity assumption (A) on σ is readily verified using a rather weak metric on Q (i.e., the C or L_∞ metric).

The above theory can be modified slightly to treat second order systems of the form (e.g., see [S])

$$\ddot{u}(t) + B(q)\dot{u}(t) + A(q)u(t) = f(t) \tag{6}$$

in a Hilbert space H where the operators A and B are defined via sesquilinear forms σ_1, σ_2 on $V \times V \to \mathbb{C}$. That is, $\sigma_1(q)(\phi,\psi) = < A(q)\phi, \psi >_H, \sigma_2(q)(\phi,\psi) = < B(q)\phi, \psi >_H$. To rewrite (6) in first order form, we let $U = V \times V$ and define a sesquilinear form $\sigma(q): U \times U \to \mathbb{C}$ by

$$\sigma(q)((u,v),(\phi,\psi)) = - < v, \phi >_V + \sigma_1(u,\psi) + \sigma_2(v,\psi). \tag{7}$$

In the usual manner, this gives rise to an operator

$$\mathcal{A}(q) \equiv \begin{bmatrix} 0 & I \\ -A(q) & -B(q) \end{bmatrix}$$

defined densely in $\mathcal{H} = V \times H$. Under assumptions (A), (B), (C) on $\sigma_1(q)$ and assumptions (A), (C) on $\sigma_2(q)$ along with

(B′) H semicoercivity: There exists $b \geq 0$ such that for $q \epsilon Q$ and $\psi \epsilon V$ we have

$$\sigma_2(q)(\psi, \psi) \geq b|\psi|_H^2,$$

one can show that $\mathcal{A}(q)$ generates a C_0 semigroup $\mathcal{T}(t;q)$ on $\mathcal{H}$. If $b > 0$ in (B′), this semigroup is uniformly exponentially stable and if in (B′) we replace $|\psi|_H$ by $|\psi|_V$ and have $b > 0$, the semigroup is analytic. For Euler-Bernoulli beams, the general case handles viscous and spatial hysteresis damping (with uniform exp. stability if the damping coefficient is strictly positive) while Kelvin-Voigt damping is included in the analytic semigroup case.

Identification problems for these second order systems may be formulated in a manner analogous to the first order case outlined above; a convergence/ stability theory under weak compactness assumptions (typically Q can be taken as a subset of $C(\Omega)$ with the supremum metric) can be given using the resolvent form of the Trotter-Kato theorem. This yields results that are a significant improvement over those currently in the research literature. Details will be given in a forthcoming manuscript.

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