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A Spectral Analysis for Self-Adjoint Operators Generated by a Class of Second Order Difference Equations

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A qualitative spectral analysis for a class of second order difference equations is given. Central to the analysis of equations in this class is the observation that real-valued solutions exhibit a type of stable asymptotic behavior for certain real values of the spectral parameter. This asymptotic behavior leads to the characterization of the limit point and limit circle nature of these equations, and is used to show that a strong nonsubordinacy criterion is satisfied on subintervals of $\mathbb{R}$ for equations of limit point type. These subintervals are part of the absolutely continuous spectrum of the self-adjoint realization of these equations. By other means, the nature of the discrete spectrum for these self-adjoint realizations is also discussed.

1. GENERAL SETTING AND PRINCIPAL RESULTS

Consider the difference equation

$$-a_n y_{n+1} + b_n y_n - a_{n-1} y_{n-1} = z w_n y_n, \tag{1.1}$$

where $z = \lambda + i\sigma$ is complex-valued, and where, for $n = 1, 2, \ldots, w_n > 0$, $a_{n-1} > 0$, and b_n is real-valued.

We shall be interested in this equation when the coefficient sequences have limiting values. In particular, we are interested in sequences which satisfy:

$$\begin{aligned} &\text{(i)} \quad \{w_n\}, \{b_n\}, \{1/a_n\}, \text{ and } \{a_{n-1}/a_n\} \text{ are of bounded variation;} \\ &\text{(ii)} \quad \lim_{n\to\infty} a_{n-1}/a_n > 0. \end{aligned} \tag{H}$$

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A sequence, $\{x_n\}$, is of bounded variation when $\sum_{n=1}^{\infty} |\Delta x_n| < \infty$ and $\Delta x_n = x_{n+1} - x_n$. Such a sequence converges and its limit will be denoted by $\bar{x}$. If a positive term sequence, $\{1/x_n\}$, is of bounded variation, $\{x_n\}$ will either converge to a nonzero limit, or diverge to infinity. Either of these eventualities will be denoted by $\bar{x}$; that is, $0 < \bar{x} \leq \infty$. Thus, when the coefficient sequences satisfy part (i) of (H), $0 \leq \overline{w} < \infty$, $0 \leq |\overline{b}| < \infty$, and $0 < \bar{a} \leq \infty$.

Requiring that $\{1/a_n\}$ be of bounded variation with $\bar{a} < \infty$ is equivalent to requiring that $\{a_n\}$ be of bounded variation with $\bar{a} \neq 0$. When $\{a_n\}$ is of bounded variation, $\bar{a}$ may be zero. And though we are interested in those cases when the coefficient sequences satisfy (H), we shall also consider (1.1) when all the coefficients are of bounded variation and $\bar{a} = 0$.

We shall also be interested in the intervals S_n consisting of those numbers λ which satisfy

$$\frac{b_n - 2a_n}{w_n} < \lambda < \frac{b_n + 2a_n}{w_n}. \tag{1.2}$$

Of particular interest are those cases where the coefficient sequences satisfy

$$-\infty \leq \lim_{n\to\infty} \frac{b_n - 2a_n}{w_n} = \gamma \leq \xi = \lim_{n\to\infty} \frac{b_n + 2a_n}{w_n} \leq \infty. \tag{1.3}$$

When $\gamma < \xi$, let $S = (\gamma, \xi)$, and let $\overline{S}$ denote the closure of S.

If we let $r_n = a_n$ and $q_n = b_n - a_n - a_{n-1}$, (1.1) can be written in Sturm–Liouville form:

$$-\Delta(r_{n-1}\Delta y_{n-1}) + q_n y_n = z w_n y_n.$$

Let $y_n^{[1]} = r_{n-1}\Delta y_{n-1}$. Then for any $\alpha \in [0, 2\pi]$, take $\{u_n(z, \alpha)\}$ and $\{v_n(z, \alpha)\}$ to be the real-valued solutions of (1.1) which satisfy the initial conditions given by

$$\begin{aligned} u_1(z,\alpha) &= -\sin(\alpha), \qquad & u_1^{[1]}(z,\alpha) &= \cos(\alpha), \\ v_1(z,\alpha) &= \cos(\alpha), \qquad & v_1^{[1]}(z,\alpha) &= \sin(\alpha). \end{aligned} \tag{1.4}$$

These sequences form a basis for the linear space of solutions of (1.1). Note that u satisfies the boundary condition given by

$$\cos(\alpha) y_1(z,\alpha) + \sin(\alpha) y_1^{[1]}(z,\alpha) = 0. \tag{1.5}$$

In Section 3 we examine the behavior of the quadratic form

$$Q_n = Q_n(\lambda, \alpha) = u_{n+1}^2(\lambda, \alpha) + u_n^2(\lambda, \alpha), \tag{1.6}$$

where $\{u_n\}$ is defined in (1.4). In this section the following results are proven:

THEOREM 1.1. *Suppose that the coefficient sequences of* (1.1) *satisfy* (H) *and* (1.3); *and suppose that* $\gamma < \xi$. *Let* I *represent a closed, bounded subinterval of* S *and let* $\Lambda = I \times [0, 2\pi]$. *There are constants* $A(\Lambda) > 0$, *and* $B(\Lambda) > 0$ *such that* $A(\Lambda)/a_n < Q_n < B(\Lambda)/a_n$ *for all* $(\lambda, \alpha) \in \Lambda$.

COROLLARY 1.2. *With* $Q_n = y_{n+1}^2(\lambda) + y_n^2(\lambda)$, *where* $\{y_n(\lambda)\}$ *is a nontrivial real-valued solution of* (1.1), *there are constants* $A > 0$ *and* $B > 0$ *such that* $A/a_n < Q_n < B/a_n$ *for all* $\lambda \in I$.

Many of the succeeding results follow from this stability property of the solutions.

The standard Titchmarsh–Weyl theory, long used in the study of Hamiltonian systems of differential equations, is applicable to (1.1) because it is a discrete analog of the differential equation $-(ry')' + qy = zwy$ (see [1, 2]). Equation (1.1) can, in fact, be viewed as part of the larger theory of Volterra–Stieltjes integral equations and generalized differential expressions as seen in [5, 21].

Central to this theory is the ability to classify (1.1) as being either of *limit point*, or of *limit circle* type. For the sequence $\{w_n\}$, with $w_n > 0$, let l_w^2 denote the complex Hilbert space of sequences satisfying $\|y\|_w^2 = \sum_{n=1}^{\infty} w_n|y_n|^2 < \infty$. Equation (1.1) is said to be limit point if, for some value of z, there is a solution $\{y_n(z)\} \in l_w^2$; otherwise, the sequence is said to be limit circle. In [2], Atkinson shows, when $a_n > 0$, that (1.1) is limit circle precisely when, for any complex number z, every solution of (1.1) is in l_w^2. Furthermore, (1.1) is shown to be limit point precisely when, for Im $z \neq 0$, there is exactly one independent solution that is in l_w^2. Thus, when (1.1) is limit point and $z = \lambda \in \mathbb{R}$, at most one independent solution can be in l_w^2. For a particular value of $\lambda \in \mathbb{R}$, it is possible that no solution is in l_w^2.

As a consequence of Theorem 1.1 and its corollary we are able, in Section 4, to show

THEOREM 1.3. *If the coefficient sequences of* (1.1) *Satisfy* (H) *and* (1.3), *and if* $\gamma < \xi$, *then* (1.1) *is limit point if and only if* $\sum_{n=0}^{\infty} w_n/a_n = \infty$.

COROLLARY 1.4. *If* (1.1) *is limit point and the coefficient sequences are all of bounded variation with* $\bar{a} \neq 0$, *then* $\liminf|\Delta a_{n-1}|/w_n = 0$.

A linear operator L defined on l_w^2 by $(Ly)_n = w_n^{-1}\{-a_n y_{n+1} + b_n y_n -$

$a_{n-1}y_{n-1}\}$ is associated with (1.1). Given the definition of $y_n^{[1]}$, y_0 is defined by (1.5), and the domain of L is given by $D(L) = \{y \in l_w^2 : Ly \in l_w^2\}$. When (1.1) is limit point, associated with each $\alpha \in [0, 2\pi]$ is a self-adjoint operator L_α, defined by

$$(L_\alpha y)_n = (Ly)_n, \tag{1.7}$$

whose domain is given by

$$D(L_\alpha) = \{y \in D(L) : y \text{ satisfies } (1.5)\}. \tag{1.8}$$

By imposing a boundary condition at infinity when (1.1) is limit circle, one can, in a similar manner, associate with each $\alpha \in [0, 2\pi]$ a self-adjoint operator. We are concerned here with the spectrum of the self-adjoint operators in these two cases. However, when (1.1) is limit circle, the associated self-adjoint operator has a spectrum which is entirely discrete. For this reason we are most interested in those cases when (1.1) is limit point.

Our principal goal is to obtain a qualitative spectral analysis of the self-adjoint realizations of Eq. (1.1). To this end, the next two results are shown. The first, proven in Section 3, is a direct consequence of the fact that when S is open, Theorem 1.1 implies that *Strong nonsubordinacy* holds on compact subintervals of S—the concept of nonsubordinacy is discussed in Section 2. The second, shown in Section 4, follows as a consequence of a result due to Hinton and Lewis [17].

THEOREM 1.5. *If* (1.1) *is limit point and the coefficient sequences satisfy* (H) *and* (1.3), *and if* $\gamma < \xi$, *then* $S = (\gamma, \xi)$ *is contained in the absolutely continuous spectrum of the self-adjoint operator* L_α *defined in* (1.7) *and* (1.8).

LEMMA 1.6. *If* (1.3) *is satisfied then the following hold*:

(i) *If* $-\infty < \gamma \leq \infty$ *and* $\liminf \Delta a_{n-1}/w_n > -\infty$, *then for all self-adjoint realizations of* (1.1), *the spectrum which lies in the interval* $(-\infty, \mu)$ *is finite for all* $\mu < \gamma + \liminf \Delta a_{n+1}/w_n$.

(ii) *If* $-\infty \leq \xi < \infty$ *and* $\liminf \Delta a_{n-1}/w_n > -\infty$, *then for all self-adjoint realizations of* (1.1), *the spectrum which lies in the interval* (μ, ∞) *is finite for all* $\mu > \xi - \liminf \Delta a_{n-1}/w_n$.

Except for the indeterminate case when $\overline{w} = \overline{b} = \overline{a} = 0$, the next four theorems give a general description of the spectrum for self-adjoint realizations of (1.1) when (1.3) holds and (H) is assumed. The first three theorems concern the case when $\overline{a} < \infty$.

THEOREM 1.7. *If the coefficient sequences of* (1.1) *are of bounded variation*, *and if* $\overline{w} \neq 0$, *then the following hold*:

(i) *Equation* (1.1) *is limit point.*

(ii) *If* $\overline{a} \neq 0$, *then the finite interval S is in the absolutely continuous spectrum of the self-adjoint realization of* (1.1). *In* $\mathbb{R}\backslash\overline{S}$, *the spectrum of the self-adjoint realization is bounded and discrete*, *and the boundary of S contains the only possible limit points of the spectrum.*

(iii) *If* $\overline{a} = 0$, *then the spectrum of the self-adjoint realization of* (1.1) *is discrete and bounded with* $\overline{b}/\overline{w}$ *as the only possible limit point of the discrete spectrum.*

The next two results consider the case $\overline{a} < \infty$ and $\overline{w} = 0$.

THEOREM 1.8. *If the coefficient sequences of* (1.1) *are of bounded variation*, *if* $\overline{w} = 0$, *and if* (1.1) *is limit point*, *then the following hold*:

(i) *If* $0 \le |\overline{b}| < 2\overline{a}$, *then* $\mathbb{R}$ *is the absolutely continuous spectrum of the self-adjoint realization of* (1.1).

(ii) *If* $|\overline{b}| = 2\overline{a} \neq 0$, *if* (1.3) *is satisfied*, *and if* $\gamma < \xi$, *then the half-line*, *S*, *is contained in the absolutely continuous spectrum of the self-adjoint realization of* (1.1). *And if* $\liminf \Delta a_{n-1}/w_n = 0$, *then S is in the absolutely continuous spectrum*, *and the spectrum in* $\mathbb{R}\backslash\overline{S}$ *is discrete and bounded. The boundary of* $\overline{S}$ *has the only possible limit point of the discrete spectrum.*

THEOREM 1.9. *If* (1.3) *is satisfied*, *if* $\overline{w} = 0$, *and if* $\liminf \Delta a_{n-1}/w_n > -\infty$, *then if either* $0 \le 2\overline{a} < |\overline{b}|$ *or* $|\overline{b}| = 2\overline{a} \neq 0$ *and if* $\gamma = \xi$, *it follows that the spectrum of any self-adjoint realization of* (1.1) *is discrete with either* ∞ *or* $-\infty$ *as the only possible limit point of the spectrum.*

And last, we consider the case when $\overline{a} = \infty$.

THEOREM 1.10. *If* (1.1) *is limit point with coefficient sequences satisfying* (H), *and if* $\overline{a} = \infty$, *then* $\mathbb{R}$ *is the absolutely continuous spectrum of the self-adjoint realization of* (1.1).

To prove Theorem 1.7, we begin by noting that $\overline{w} \neq 0$; thus, (1.3) is satisfied. When $\overline{a} \neq 0$ and the coefficient sequences are all of bounded variation, (H) is satisfied and (1.1) is limit point by Theorem 1.3. In this case, S is finite and open and part of the absolutely continuous spectrum by Theorem 1.5. When $\overline{w} \neq 0$, the boundedness of variation of $\{a_n\}$ implies that $\lim_{n\to\infty} \Delta a_{n-1}/w_n = 0$. Consequently, by Lemma 1.6, the spectrum in $R\backslash S$ is bounded and discrete and has limit points only on the boundary of S.

Now, if $\overline{a} = 0$, S is the point $\overline{b}/\overline{w}$. In this case, (1.1) is limit point by [17, Theorem 10]. Caution: The notation used in this paper differs from that used in [17]. By Lemma 1.6 the spectrum in $\mathbb{R}\backslash S$ is bounded and discrete, and the only possible limit point is the point $\overline{b}/\overline{w}$. This completes the proof of Theorem 1.7.

Theorems 1.3 and 1.5 together with Lemma 1.6 can be used to prove Theorems 1.8 and 1.10 in much the same way in which they were used to prove Theorem 1.7. Note that Theorem 1.3 provides a characterization of the limit point assumption made in these two theorems because in both cases we assume that $\overline{a} \neq 0$, and hence in both cases (H) holds. Thus we note, by Corollary 1.4, that it is necessary for $\liminf \Delta a_{n-1}/w_n \leq 0$ for (1.1) to be limit point. However, if in part (ii) of Theorem 1.8, $\liminf \Delta a_{n-1}/w_n$ is assumed to be strictly less than zero rather than equal to zero, we note that S is part of the absolutely continuous spectrum and that a potential gap exists beyond the finite endpoint of $\overline{S}$, which this analysis does not address.

Finally, we note that no assumption of boundedness of variation is made for the coefficient sequences in the statement of Theorem 1.9. In this case, either γ and ξ are both $-\infty$ or both are ∞. Theorem 1.9 follows from Lemma 1.6 alone and addresses both limit point and limit circle cases.

2. The Method of Subordinacy and Absolutely Continuous Spectrum

For $\operatorname{Im} z > 0$, let m be a complex number such that the sequence $y = \{y_n\}$, where

$$y_n = u_n(z, \alpha) + m v_n(z, \alpha), \tag{2.1}$$

represents a solution of (1.1) satisfying the boundary condition given by

$$\cos(\beta) y_N + \sin(\beta) y_N^{[1]} = 0 \tag{2.2}$$

for $N > 1$ and $\beta \in [0, 2\pi]$. The complex numbers $m = m(z, N, \alpha, \beta)$, for fixed values of z, N, and α, form a circle in the complex plane that is parameterized by β. These circles are nested for increasing values of N, and collapse as $N \to \infty$ to either a circle or a point. For this reason, (1.1) is classified as being either limit circle or limit point.

When (1.1) is limit point and $\operatorname{Im} z > 0$, we let $m_\alpha(z)$ be the limit of the complex numbers $m = m(z, n, \alpha, \beta)$ defined in (2.1) and (2.2), i.e., $m_\alpha = \lim_{n\to\infty} m(z, n, \alpha, \beta)$. From the Titchmarsh–Weyl theory the solution $y = \{y_n\}$ of (1.1) defined by

$$y_n = y_n(z, \alpha) = u_n(z, \alpha) + m_\alpha(z) v_n(z, \alpha)$$

is in l_w^2, and its norm satisfies

$$\|y\|_w^2 = \frac{\operatorname{Im} m_\alpha(z)}{\operatorname{Im} z}.$$

Moreover, it is the case that $m_\alpha(z)$ has a unique Pick–Nevanlinna representation given by

$$m_\alpha(z) = A + Bz + \int_{-\infty}^{\infty} \frac{1}{(t-z)} - \frac{t}{(t^2+1)}\, d\rho_\alpha(t),$$

where A is real, $B > 0$, $\rho_\alpha(t)$ is a nondecreasing, left continuous Borel measure satisfying $\rho_\alpha(0) = 0$, and $\int_{-\infty}^{\infty} 1/(1+t^2)\, d\rho_\alpha(t) < \infty$. An inverse relation also exists where at points of continuity for $\rho_\alpha(t)$

$$\rho_\alpha(\delta_2) - \rho_\alpha(\delta_1) = \lim_{\varepsilon \to 0+} \pi^{-1} \int_{\delta_1}^{\delta_2} \operatorname{Im} m_\alpha(t + i\varepsilon)\, dt.$$

The function ρ_α(t) is said to be the spectral density for the self-adjoint operator associated with (1.1) and the singular boundary value problem of limit point type. As the name suggests, ρ_α possesses, in its qualitative characteristics, information about the spectrum of the related self-adjoint operator. The support of ρ_α contains the spectrum. Points of increase for ρ_α correspond to elements of the spectrum; discontinuities correspond to elements of the discrete spectrum; and points of increase at which ρ_α is continuous correspond to elements of the continuous spectrum. A subset of the continuous spectrum on which ρ_α is absolutely continuous with respect to Lebesgue measure is said to be part of the absolutely continuous spectrum.

The theory which underlies the proof of Theorem 1.5 is developed in [14, 15] by Gilbert and Pearson for their study of the equation

$$-y'' + q(x)y = zy, \tag{2.3}$$

for $-\infty \le a < x < b \le \infty$. It has become known as *the method of subordinacy*.

Two lines of thought are pursued in [14], where the one singular endpoint problem is considered. On one hand the limiting behavior of $m_\alpha(\lambda + i\varepsilon)$ as $\varepsilon \to 0^+$ is related to the existence of subordinate solutions: a nontrivial solution $u(x, \lambda)$ of (2.3) being *subordinate* at b when, given that $\|u\|^2 = \int_\alpha^b |u(x)|^2\, dx$, $\|u\|_N^2 = \int_\alpha^N |u(x)|^2\, dx$, and $z = \lambda \in \mathbb{R}$, it is the case that for every independent solution $v(x, \lambda)$,

$$\lim_{N\to b} \frac{\|u(\cdot, \lambda, \alpha)\|_N}{\|v(\cdot, \lambda, \alpha)\|_N} = 0; \tag{2.4}$$

a nontrivial solution being *nonsubordinate* at b when this is not the case. On the other hand, this same limiting behavior of m_α is related to the decomposition of ρ_α into, for example, singular and absolutely continuous parts. Combined, these two lines of thought give a characterization of the decomposition of ρ_α in terms of the existence of subordinate solutions, or more to the point, the asymptotic behavior of certain solutions as expressed in (2.4) (see [14, Theorem 1]). And though developed for (2.3), this theory has been extended by Behncke [3, 4] to Dirac systems of differential equations, by Khan and Pearson [18] to three term recurrence relations, and by Clark and Hinton [9] to Sturm–Liouville systems of sufficient generality to include difference equations like (1.1). Showing the existence of nonsubordinate solutions for values of $\lambda \in \mathbb{R}$ has been particularly useful in showing that certain intervals are part of the absolutely continuous spectrum.

In [9], a solution $\{y_n(\lambda, \alpha)\}$ of (1.1) is said to be subordinate when, with $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} w_n y_n^2(\lambda, \alpha)}{\sum_{n=1}^{N} w_n v_n^2(\lambda, \alpha)} = 0$$

for every independent solution $v = \{v_n(\lambda, \alpha)\}$. In addition to an extension of the method of subordinacy to more general Sturm–Liouville systems, this method is carried a step further in [9] and a criterion is stated for the solution $\{u_n(\lambda, \alpha)\}$ given in (1.4) which, when satisfied on an interval, implies that ρ_α is absolutely continuous on that interval, and that ρ_α' is bounded above and below by a positive constant almost everywhere on that interval. This additional step appears as Theorem 3.1 in [9]. And when stated in terms of (1.1), this theorem becomes

THEOREM 2.1. (Clark and Hinton). *Let $[\lambda_1, \lambda_2]$ be an interval. Suppose there is a number $\beta = \beta(\alpha) > 0$, independent of λ, such that for each $\lambda \in [\lambda_1, \lambda_2]$,*

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} w_n v_n^2(\lambda, \alpha)}{\sum_{n=1}^{N} w_n u_n^2(\lambda, \alpha)} \le \beta^2, \tag{2.5}$$

where $\{u_n(\lambda, \alpha)\}$ and $\{v_n(\lambda, \alpha)\}$ are the real-valued solutions defined by (1.4). *In this case, the spectral density ρ_α satisfies a Lipschitz condition on* $[\lambda_1, \lambda_2]$

with Lipschitz constant $\leq 10\beta/\pi$. *Moreover, if* β *is also independent of* α, *then* $\rho'_\alpha \geq 1/10\beta\pi$ *almost everywhere on* $[\lambda_1, \lambda_2]$.

A *strong nonsubordinacy* criterion is said to be satisfied on $[\lambda_1, \lambda_2]$ when (2.5) holds with β independent of both $\lambda \in [\lambda_1, \lambda_2]$ and $\alpha \in [0, 2\pi]$. In Section 3, Theorem 1.1 is proven, and a consequence of this is

THEOREM 2.2. *If the coefficient sequences satisfy* (H) *and* (1.3), *if* (1.1) *is limit point, and if* $\gamma < \xi$, *then the strong nonsubordinacy criterion is satisfied on each compact subinterval of* $S = (\gamma, \xi)$.

Establishing that strong nonsubordinacy holds on compact subsets of S allows us to conclude, by Theorem 2.1, that the spectral density $\rho_\alpha(\lambda)$, for the self-adjoint operator L_α defined in (1.7) and (1.8), is absolutely continuous on S. On each $[\lambda_1, \lambda_2] \subset S$, $\rho_\alpha(\lambda)$ has a derivative, $\rho'_\alpha(\lambda)$, which is almost everywhere bounded above and below by positive constants. This provides the justification for the claim in Theorem 1.5 that S is contained in the absolutely continuous spectrum of L_α.

We note in passing that the boundedness conditions on ρ'_α have been used for differential equations with two singular endpoints (see Mantlik and Schneider [19], and Castillo [6]) to show that under suitably strong conditions at one endpoint, the absolutely continuous spectrum for the one singular endpoint case is contained in the absolutely continuous spectrum for the two singular endpoint case.

3. STRONG NONSUBORDINACY AND THE LIMIT POINT NATURE OF EQ. (1.1)

Throughout this section we shall assume, in addition to (H), that the coefficient sequences of (1.1) are such that (1.3) is satisfied and $\gamma < \xi$. With S_n defined in (1.2), let I represent a compact subinterval of $S = (\gamma, \xi)$. Let $\Lambda = I \times [0, 2\pi]$. Let $\{u_n(\lambda, \alpha)\}$ be the nontrivial real-valued solution of (1.1) defined in (1.4) for $(\lambda, \alpha) \in \Lambda$. In addition to Q_n defined in (1.6), we define $\tilde{Q}_n$ by

$$\tilde{Q}_n = \tilde{Q}_n(\lambda, \alpha) = a_n u_{n+1}^2 + (\lambda w_n - b_n) u_{n+1} u_n + a_{n-1} u_n^2. \tag{3.1}$$

In the lemmas which follow, explicit representation of the dependence on λ and α will be suppressed. The next three lemmas provide a proof of Theorem 1.1.

LEMMA 3.1. *For all* $(\lambda, \alpha) \in \Lambda$, *and for* n *sufficiently large, there are constants* $C(\Lambda) > 0$ *and* $D(\Lambda) > 0$ *such that*

$$C(\Lambda)a_n < \tilde{Q}_n/Q_n < D(\Lambda)a_n. \tag{3.2}$$

Proof. First, note that

$$\frac{\tilde{Q}_n}{Q_n} = a_n \left\{ \frac{a_n u_{n+1}^2 + a_{n-1}u_n^2}{a_n Q_n} + \frac{\lambda w_n - b_n}{2a_n} \cdot \frac{2u_{n+1}u_n}{Q_n} \right\},$$

and that

$$\frac{a_n u_{n+1}^2 + a_{n-1}u_n^2}{a_n Q_n} = 1 - \left(\frac{u_n^2}{Q_n}\right) \frac{\Delta a_{n-1}}{a_n}.$$

Part (ii) of (H) is equivalent to requiring that $\lim_{n\to\infty} (\Delta a_{n-1}/a_n) < 1$. Thus when $\overline{a} = \infty$, the conclusion follows because $(\lambda w_n - b_n)/2a_n$ tends to zero uniformly for $(\lambda, \alpha) \in \Lambda$. However, when $0 < \overline{a} < \infty$, $\{a_n\}$ is of bounded variation and $\lim_{n\to\infty} (\Delta a_{n-1}/a_n) = 0$. In this case the conclusion follows, given the definition of S_n, because there is a constant $k(\Lambda)$, for n sufficiently large, such that

$$\left| \frac{\lambda w_n - b_n}{2a_n} \right| < k(\Lambda) < 1$$

for all $(\lambda, \alpha) \in \Lambda$.

LEMMA 3.2. *There is an* $N > 0$ *such that, for* $n \geq N$, *the sequence* $\{\Delta\tilde{Q}_n/\tilde{Q}_n\}$ *is absolutely summable for all* $(\lambda, \alpha) \in \Lambda$.

Proof. Using (1.1) to solve for u_{n+2} in terms of u_{n+1} and u_n, we see that

$$\Delta\tilde{Q}_n = \left\{ \lambda\Delta w_n - \Delta b_n + \frac{b_{n+1} - \lambda w_{n+1}}{a_{n+1}} \Delta a_n \right\} u_{n+1}u_n + \left\{ \Delta a_{n-1} - \frac{a_n}{a_{n+1}} \Delta a_n \right\} u_n^2.$$

Now, let p_n and q_n be defined by

$$p_n = \lambda\Delta w_n - \Delta b_n + \frac{b_{n+1} - \lambda w_{n+1}}{a_{n+1}} \Delta a_n,$$

$$q_n = \Delta a_{n+1} - \frac{a_n}{a_{n+1}} \Delta a_n.$$

By Lemma 3.1 there is an $N > 0$ such that, for $n \geq N$, (3.2) holds. Thus for $\lambda \in I \subset S$

$$\begin{aligned}\left|\frac{p_n u_{n+1} u_n}{\tilde{Q}_n}\right| &\leq \Big\{ |\lambda|\, |\Delta w_n| \\ &\qquad + |\Delta b_n| + \left|\frac{b_{n+1} - \lambda w_{n+1}}{a_{n+1}}\right| |\Delta a_n| \Big\} \left|\frac{2u_{n+1}u_n}{Q_n}\right| \frac{Q_n}{2\tilde{Q}_n} \qquad (3.3)\\ &\leq \frac{1}{2C(\Lambda)} \left\{ |\lambda| \frac{|\Delta w_n|}{a_n} + \frac{|\Delta b_n}{a_n} + |b_{n+1} - \lambda w_{n+1}| \frac{|\Delta a_n|}{a_{n+1} q_n} \right\}.\end{aligned}$$

Note that $\Delta a_n / a_{n+1} a_n = \Delta(1/a_n)$. Similarly, we see that

$$\left|\frac{q_n u_n^2}{\tilde{Q}_n}\right| \leq \left|\frac{\Delta a_{n-1}}{a_n} - \frac{\Delta a_n}{a_{n+1}}\right| \frac{1}{C(\Lambda)} = \left|\Delta\left(\frac{a_{n-1}}{a_n}\right)\right| \frac{1}{C(\Lambda)}. \qquad (3.4)$$

Since $\bar{a} > 0$, by (3.3) and (3.4) there are constants, $C_i(\Lambda) \geq 0$, such that for $n \geq N$ and all $(\lambda, \alpha) \in \Lambda$,

$$\begin{aligned}|\Delta \tilde{Q}_n / Q_n| \leq\ & C_1(\Lambda)|\Delta w_n| + C_2(\Lambda)|\Delta b_n| \\ & + C_3(\Lambda)|\Delta(1/a_n)| + C_4(\Lambda)|\Delta(a_{n-1}/a_n)|.\end{aligned} \qquad (3.5)$$

The absolute summability of $\{\Delta \tilde{Q}_n / Q_n\}$ follows from (3.5) and assumption (H).

LEMMA 3.3. *$\tilde{Q}_n$ converges uniformly for all $(\lambda, \alpha) \in \Lambda$ to a positive continuous function.*

Proof. If $F_n = \Delta \tilde{Q}_n / \tilde{Q}_n$ then $\tilde{Q}_{n+1} / \tilde{Q}_n = 1 + F_n$. As noted earlier, $\tilde{Q}_n$ is a continuous function in λ and α for $(\lambda, \alpha) \in \Lambda$. By Lemma 3.1 we may choose N so large that $\tilde{Q}_n > 0$ for all $(\lambda, \alpha) \in \Lambda$ when $n \geq N$. As a result, F_n is continuous on Λ. Furthermore,

$$\prod_{k=N}^{n} (1 + F_k) = \frac{\tilde{Q}_{n+1}}{\tilde{Q}_N}. \qquad (3.6)$$

And when $n > m > N$,

$$\left|\prod_{k=N}^{n}(1+F_k)-\prod_{k=N}^{m}(1+F_k)\right| \le \prod_{k=N}^{m}(1+|F_k|)\left[\prod_{k=m}^{n}(1+|F_k|)-1\right]$$

$$\le \exp\left(\sum_{k=N}^{n}|F_k|\right)-\exp\left(\sum_{k=N}^{m}|F_k|\right) \tag{3.7}$$

$$\le e^{\delta}\sum_{k=m}^{n}|F_k|,$$

where $\sum_{k=N}^{m}|F_k| < \delta < \sum_{k=N}^{n}|F_k|$.

For each $(\lambda, \alpha) \in \Lambda$, $\sum_{k=N}^{\infty}|F_k| < \infty$ by Lemma 3.2. By (3.5), as $n \to \infty$, $\sum_{k=N}^{n}|F_k|$ converges uniformly for $(\lambda, \alpha) \in \Lambda$ to a continuous function. Thus, $\sum_{k=m}^{n}|F_k|$ can be made uniformly small over Λ, and as a result, e^{δ} is uniformly bounded. By (3.6), $\prod_{k=N}^{n}(1+F_k)$ is positive and continuous on Λ; and by (3.7), as $n \to \infty$, this sequence of functions converges uniformly on Λ. The result follows from (3.6).

By Lemmas 3.2 and 3.3 we have

COROLLARY 3.4. *The sequence $\{\tilde{Q}_n\}$ is of bounded variation.*

Theorem 1.1 is a consequence of Lemma 3.3 and inequality (3.2) since the uniform convergence of $\tilde{Q}_n$ to a positive continuous function on Λ implies that $\tilde{Q}_n$ is uniformly bounded and bounded away from zero for n sufficiently large.

We now show that the asymptotic behavior, described in Theorem 1.1 for the solution $\{u_n(\lambda, \alpha)\}$ defined in (1.4), is sufficient to guarantee not only that the limit point nature of Eq. (1.1) can be characterized, but also that the strong nonsubordinacy criterion is satisfied on I.

To prove Theorem 1.3, begin by noting that

$$\begin{aligned} 2\sum_{k=1}^{n} w_k u_k^2 &= w_1 u_1^2 + \sum_{k=1}^{n-1}(w_{k+1}u_{k+1}^2 + w_k u_k^2) + w_n u_n^2 \\ &= w_1 u_1^2 + \sum_{k=1}^{n-1} w_k(u_{k+1}^2 + u_k^2) + \sum_{k=1}^{n-1}\Delta w_k u_{k+1}^2 + w_n u_n^2. \end{aligned} \tag{3.8}$$

As a consequence of Theorem 1.1, there are constants $A(\Lambda) > 0$ and $B(\Lambda) > 0$ such that

$$2\sum_{k=1}^{n} w_k u_k^2 \ge A(\Lambda)\sum_{k=1}^{n-1} w_k/a_k - B(\Lambda)\sum_{k=1}^{n-1}|\Delta w_k|/a_k. \tag{3.9}$$

By Theorem 1.1 it is also true that

$$\sum_{k=1}^{n} w_k u_k^2 \leq B(\Lambda) \sum_{k=1}^{n} w_k/a_k. \tag{3.10}$$

By (H) we see that $\sum_{k=1}^{\infty} |\Delta w_k|/a_k < \infty$. If it is also true that $\sum_{k=1}^{\infty} w_k/a_k = \infty$, then by (3.9), $\{u_n\} \notin l_w^2$. As a consequence, (1.1) is limit point. On the other hand, if (1.1) is limit point, then there is a real-valued solution, $\{y_n\}$, of (1.1) such that $\{y_n\} \notin l_w^2$. By Corollary 1.2, there is an inequality for $\{y_n\}$ like (3.10). By this inequality, we see that $\sum_{k=1}^{\infty} w_k/a_k = \infty$. Theorem 1.1 is thus proven.

To prove Theorem 2.2 we note, given $\{u_n\}$ and $\{v_n\}$ as defined in (1.4), that $u_1(\lambda, \alpha - \pi/2) = v_1(\lambda, \alpha)$ and that $u_1^{[1]}(\lambda, \alpha - \pi/2) = v_1^{[1]}(\lambda, \alpha)$. As a result, both $\{u_n\}$ and $\{v_n\}$ satisfy (3.9) and (3.10); hence

$$\frac{\sum_{k=1}^{n} w_k v_k^2}{\sum_{k=1}^{n} w_k u_k^2} \leq 2 \frac{B_1(\Lambda) \sum_{k=1}^{n-1} w_k/a_k + B_1(\Lambda) w_n/a_n}{B_0(\Lambda) \sum_{k=1}^{n-1} w_k/a_k - B_1(\lambda) \sum_{k=1}^{n-1} |\Delta w_k|/a_k}.$$

By (H) we see that $\overline{w}/\overline{a} < \infty$ and that $\sum_{k=1}^{\infty} |\Delta w_k|/a_k < \infty$. By Theorem 1.1, Eq. (1.1) is limit point precisely when $\sum_{k=1}^{\infty} w_k/a_k = \infty$. It follows that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} w_k v_k^2}{\sum_{k=1}^{n} w_k u_k^2} \leq \frac{2B_1(\Lambda)}{B_0(\Lambda)}.$$

Thus the strong nonsubordinacy criterion holds on each compact interval contained in S and Theorem 2.2 is complete. And as discussed in Section 2, this verifies the claim made in Theorem 1.5 concerning the absolutely continuous spectrum of the self-adjoint operator L_α associated with (1.1) and defined by (1.7) and (1.8).

4. Discrete Spectrum

The proof of Lemma 1.6 follows from a discrete version, due to Hinton and Lewis [17], of an oscillation theorem for second order differential equations due to Glazman [16, p. 34]. This lemma applies to those cases when (1.1) is limit circle as well as to those cases when (1.1) is limit point. Caution: The notation used in [17] differs from that used here.

To prove part (i), we begin by letting D_N be the set of sequences, $x = \{x_n\}$, such that $x_n = 0$ for $n < N$ and only finitely many x_n are nonzero. Define $q(x)$ on D_N by

$$q(x) = \sum_{n=N-1}^{\infty} \{(b_n - a_n - a_{n-1})|x_n|^2 + a_n|\Delta x_n|^2\}.$$

By [17, Theorem 1], if for some $N \geq 1$, $q(x) \geq \mu \sum_{n=N-1}^{\infty} w_n|x_n|^2$ for all $x \in D_N$, then the spectrum of L_α in $(-\infty, \mu)$ is finite. To see that this is the case given the hypotheses of part (i), note that

$$\begin{aligned} q(x) - \mu \sum_{n=N-1}^{\infty} w_n|x_n|^2 &= \sum_{n=N-1}^{\infty} \{(b_n - a_n - a_{n-1} - \mu w_n)|x_n|^2 + a_n|\Delta x_n|^2\} \\ &= \sum_{n=N-1}^{\infty} \left\{ w_n \left(\frac{b_n - 2a_n}{w_n} + \frac{\Delta a_{n-1}}{w_n} - \mu \right) |x_n|^2 + a_n|\Delta x_n|^2 \right\}. \end{aligned}$$

For N sufficiently large, $q(x) - \mu \sum_{n=N-1}^{\infty} w_n|x_n|^2 \geq 0$ for all $x \in D_N$ if $\mu < \gamma + \liminf \Delta a_{n-1}/w_n$.

One can reduce the proof of part (ii) to an argument similar to that of part (i) by means of a simple device introduced to the author by Professor Don Hinton of the University of Tennessee. With $x \in l_w^2$, let $U: l_w^2 \to l_w^2$ be defined by $(Ux)_n = (-1)^{n+1}x_n$. U is a unitary transformation. If the linear operator $L: l_w^2 \to l_w^2$ is defined by

$$(Ly)_n = w_n^{-1}\{-a_n y_{n+1} + b_n y_n - a_{n-1}y_{n-1}\},$$

then the linear operator $T = ULU$ is unitarily equivalent to L. T is given by

$$(Tx)_n = w_n^{-1}\{a_n x_{n+1} + b_n x_n + a_{n-1}x_{n-1}\}.$$

Consider next the equation $-a_n x_{n+1} - b_n x_n - a_{n-1}x_{n-1} = \mu w_n x_n$. With D_N defined above, and $q(x)$ now defined on D_N by

$$q(x) = \sum_{n=N-1}^{\infty} \{(-b_n - a_n - a_{n-1})|x_n|^2 + a_n|\Delta x_n|^2\},$$

we note that

$$\begin{aligned}
q(x) - \mu \sum_{n=N-1}^{\infty} w_n|x_n|^2 &= \sum_{n=N-1}^{\infty} \{(-b_n - a_n - a_{n-1} - \mu w_n)|x_n|^2 + a_n|\Delta x_n|^2\} \\
&= \sum_{n=N-1}^{\infty} \left\{ w_n \left(\frac{-b_n - 2a_n}{w_n} + \frac{\Delta a_{n-1}}{w_n} - \mu \right) |x_n|^2 + a_n|\Delta x_n|^2 \right\}.
\end{aligned}$$

Thus $q(x) - \mu \sum_{n=N-1}^{\infty} w_n|x_n|^2 \geq 0$ for all $x \in D_N$ when $\mu < -\xi + \liminf \Delta a_{n-1}/w_n$ and N is chosen sufficiently large. By [17, Theorem 1] we again see that any self-adjoint realization of $-T$ has finite spectrum in $(-\infty, \mu)$ when $\mu < -\xi + \liminf \Delta a_{n-1}/w_n$; thus, it follows that any self-adjoint realization of T, and hence of L, has finite spectrum in (μ, ∞) when $\mu > \xi - \liminf \Delta a_{n-1}/w_n$.

5. Related Work and Final Observations

This paper presents a qualitative spectral analysis for self-adjoint realizations of Eq. (1.1). The method of subordinacy and a general result concerning the discrete spectrum for self-adjoint realizations of (1.1) are used to locate the absolutely continuous spectrum. Both limit circle and limit point cases are considered.

The results on the absolute continuity of the spectral measure are a direct consequence of the asymptotic behavior observed in Lemma 3.3 for the quadratic form (3.1). This form was derived from a consideration of discrete versions of quadratic forms used by the author while studying linear Hamiltonian systems of ordinary differential equations [7–9]. It was subsequently noticed that a similar quadratic form was used by Professor Nevai and his coauthors in [20, 13] for their studies of orthogonality measures. In this sense, the results presented here are similar to those in the latter two papers mentioned.

General spectral properties of (1.1) were considered in papers by Maté and Nevai [20], Dombrowski [10–12], and more recently in papers by Smith [22] and Stolz [23]. In all of these papers it is assumed that $w_n = 1$. The results presented here represent an extension of the work of Maté and Nevai and that of Dombrowski.

Smith obtains general results relating oscillatory properties of solutions to the essential spectrum of the associated self-adjoint operator. The essential spectrum is observed to be preserved under relatively compact perturbations.

Stolz assumes, additionally, that $a_n = 1$. However, $\{b_n\}$ is a sequence exhibiting a slow oscillatory behavior. Using the method of subordinacy, intervals containing absolutely continuous spectrum are located.

Maté and Nevai view Eq. (1.1) in the context of orthogonal polynomial theory. In their paper the spectral measure is the measure with respect to which the polynomials in z defined by the three term recurrence relation (1.1) are orthogonal. Using standard orthogonal polynomial techniques, they obtain continuous differentiability and positivity of the spectral measure on $(-1, 1)$ when the coefficient sequences satisfy conditions equivalent to that of bounded variation and when in addition to $w_n = 1$, it is assumed that $\lim_{n\to\infty} a_n = \frac{1}{2}$ and that $\lim_{n\to\infty} b_n = 0$.

In [10], Dombrowski considers the spectrum of bounded, cyclic, self-adjoint operators defined on a separable Hilbert space $\mathbb{H}$. In [11, 12] the spectra of unbounded, cyclic, self-adjoint operators defined on dense subsets of $\mathbb{H}$ are considered. In each case, these operators are identified with infinite dimensional tridiagonal matrices. These matrices, which define operators on l^2, are identified with (1.1). Using techniques from orthogonal polynomial theory as well as operator theory, behavior of the coefficient sequences of (1.1) is related to the presence of an absolutely continuous part of the spectral measure for these operators.

In [10], Dombrowski extends the result of Maté and Nevai. The coefficient sequences $\{b_n\}$ and $\{a_n\}$ satisfy conditions equivalent to that of bounded variation and the interval $(\overline{b} - 2\overline{a}, \overline{b} + 2\overline{a})$ is shown to be contained in the absolutely continuous spectrum of the bounded self-adjoint realization of (1.1) when $\overline{a} \neq 0$. Theorem 1.7 of this paper completes this analysis by discussing the discrete spectrum of the operator, by allowing the weight sequence, $\{w_n\}$, to be nonconstant when $\overline{w} \neq 0$, and by characterizing limit pointness in this case.

In [11], $w_n = 1$ and $b_n = 0$. It is also assumed that $\{a_n\}$ monotonically increases to infinity, and that $\sum_{n=1}^{\infty} 1/a_n = \infty$. Monotonicity implies that $\{1/a_n\}$ is of bounded variation. Divergence of the sum is sufficient for limit pointness by Theorem 1.3 of this paper. With the additional assumption that $\{\Delta a_n\}$ is of bounded variation, Dombrowski concludes in Theorem 1 that there are no eigenvalues in the spectrum of the self-adjoint realization of (1.1). Two theorems follow in which additional restrictions on $\{\Delta a_n\}$ are imposed which guarantee that, on $\mathbb{R}$, the spectral measure is absolutely continuous. However, it is not shown that $\mathbb{R}$ is the spectrum.

In [12], it is again assumed that $w_n = 1$, and $b_n = 0$, that $a_n \to \infty$, and that $\sum_{n=1}^{\infty} 1/a_n = \infty$. However, instead of assuming the monotonicity of $\{a_n\}$ and the bounded variation of $\{\Delta a_n\}$, the two principal results assume the bounded variation of $\{|\Delta a_n|\}$. With the additional assumption that $\sum_{n=1}^{\infty}$

$[\Delta a_n]^- < \infty$, the author proves in Theorem 1 that there are no eigenvalues in the spectrum. If instead, the additional assumption is that $\sum_{n=1}^{\infty} [a_{n+1}^2 - a_n^2]^- < \infty$, then it is shown in Theorem 2 that the spectral measure has an absolutely continuous part. The measure is not shown to be absolutely continuous. The author poses the question whether the latter assumption is sufficient to guarantee the absolute continuity of the measure.

Consider the fact that

$$\left|\frac{\Delta a_n}{a_{n+1}} - \frac{\Delta a_{n-1}}{a_n}\right| \leq D_1 + D_2 + D_3 + D_4,$$

where D_1, D_2, D_3, and D_4 are defined by

$$D_1 = \frac{1}{a_{n+1}} |\Delta a_n - |\Delta a_n|| = \frac{2}{a_{n+1}} [\Delta a_n]^- = \frac{2[a_{n+1}^2 - a_n^2]^-}{a_{n+1}(a_{n+1} - a_n)}$$

$$D_2 = \frac{1}{a_n} |\Delta a_{n-1} - |\Delta a_{n-1}|| = \frac{2}{a_n} [\Delta a_{n-1}]^- = \frac{2[a_n^2 - a_{n-1}^2]^-}{a_n(a_n + a_{n-1})}$$

$$D_3 = \frac{1}{a_{n+1}} ||\Delta a_n| - |\Delta a_{n-1}||$$

$$D_4 = |\Delta a_{n-1}| \, |\Delta(1/a_n)|.$$

Observe that the bounded variation of $\{|\Delta a_n|\}$ implies that $\{|\Delta a_n|\}$ has a limit and hence that $\lim_{n\to\infty} (a_n/a_{n-1}) = 1$. If, in addition to either $\sum_{n=1}^{\infty} [\Delta a_n]^- < \infty$ or $\sum_{n=1}^{\infty} [a_{n+1}^2 - a_n^2]^- < \infty$, we assume that $\{1/a_n\}$ is of bounded variation, it follows that $\{\Delta a_n/a_{n+1}\} = \{1 - a_n/a_{n+1}\}$, is of bounded variation and that the hypotheses of (H) are satisfied.

Theorem 1.10 of this paper provides a conditional affirmative in response to the question posed by Dombrowski in [12]: If, in addition to the hypotheses of either theorem in [12], it is assumed that $\{1/a_n\}$ is of bounded variation, then not only is the spectral measure absolutely continuous, but its support is $\mathbb{R}$. If $\{a_n\}$ is monotone increasing to infinity, as is the case in [11], then $\{1/a_n\}$ is of bounded variation, and the conditions posed in Theorem 1 of [12] reduce to those of Theorem 1 of [11]. Thus Thoerem 1.10 informs us that the conditions assumed in Theorem 1 of [11] are sufficient, *alone*, to guarantee that $\mathbb{R}$ is the absolutely continuous spectrum of the self-adjoint realization of (1.1). It should be noted that Theorem 1.10 allows for the possibility of a nonzero sequence, $\{b_n\}$, and a nonconstant weight sequence, $\{w_n\}$, both of which are of bounded variation; in particular for which $\overline{w}$ could be zero. Thus the spectrum remains unchanged when considering this operator with respect to weighted l^2 spaces with a weight sequence that is of bounded variation.

In both [11] and [12], known examples and new constructions of coefficient sequences for (1.1) are presented that produce, as solutions of the three term recursion, systems of orthogonal polynomials for which the measure of orthogonality is some unbounded subset of the real line. One well-known case is that for which $w_n = 1$, $b_n = 0$, and $a_n = \sqrt{n/2}$. This produces Hermite polynomials which can be normalized on $\mathbb{R}$ with respect to the measure $d\mu = \exp(-x^2)\,dx$. The samples and constructions presented in [11, 12] are such that $\{a_n\}$ is eventually monotonically increasing to infinity, and $\{\Delta a_n\}$ is of bounded variation. Thus in each case presented, (H) is satisfied and we see now that the support of the measure of orthogonality is $\mathbb{R}$.

And finally, Theorems 1.8 and 1.9 consider the case when the coefficient sequences are of bounded variation and $\overline{w} = 0$. The resulting operator can be unbounded. In Theorem 1.8, conditions are given such that the spectral measure is absolutely continuous on all of $\mathbb{R}$ or on half-lines. The latter situation is typically for self-adjoint operators associated with singular boundary value problems posed on the half-line for the second order differential equation (2.3) (see [14]). The similar occurrence of absolutely continuous spectrum on a half-line appears to have been unnoticed for (1.1). Theorem 1.9 completes the discussion of the case when $\overline{w} = 0$ by providing minimal conditions which guarantee that the spectrum is discrete in both the limit circle and the limit point case.

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