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The Euler class group of a polynomial algebra II

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1. Introduction

In this paper we continue our study on the theory of the Euler class group of a polynomial algebra $A[T]$, where A is a commutative Noetherian ring (containing $\mathbb{Q}$) of dimension n. For such a ring A, in [9], we defined the notion of the nth Euler class group $E^n(A[T])$ of $A[T]$. For simplicity let us call it $E(A[T])$. In [9] we also studied the relations between $E(A[T])$ and $E(A)$, where $E(A)$ is the nth Euler class group of A. For example, there is canonical map $\Phi : E(A) \to E(A[T])$ which is an injective group homomorphism and it is an isomorphism when A is a smooth affine domain [9, Proposition 5.7]. In general, these two groups are not isomorphic (see discussion preceding [9, Proposition 5.7]). In this context, the following question is natural.

Question. Does there exist a group homomorphism, say, $\Psi : E(A[T]) \to E(A)$ such that the composition $\Psi\Phi$ is the identity map on $E(A)$?

In this paper we give an affirmative answer to the above question.

A few words about the proof are in order. Let R denote A or $A[T]$. We may recall that $E(R)$ is a free abelian group modulo some relation (see Section 2 for definition) and elements of $E(R)$ are classes of pairs (I, ω_I) where I is an ideal of R of height n and $\omega_I : (R/I)^n \twoheadrightarrow I/I^2$ is a surjection. One attempt to define a map from $E(A[T])$ to $E(A)$ could be by restriction at $T = 0$, meaning, given $(I, \omega_I) \in E(A[T])$ we may try to associate it to something like $(I(0), \omega_{I(0)})$ in $E(A)$ where $I(0) = \{f(0) \mid f(T) \in I\}$ and $\omega_{I(0)}$:

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$(A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ is the surjection induced by ω_I. But $I(0)$ may not have height n and therefore $(I(0), \omega_{I(0)})$ may not be a legitimate element of $E(A)$. On the other hand, since A contains $\mathbb{Q}$, there exists $\lambda \in \mathbb{Q}$ such that $I(\lambda)$ is an ideal of height $\geqslant n$. In this case $(I(\lambda), \omega_{I(\lambda)})$ is an element of $E(A)$ but since the element λ depends on I and may vary for different ideals, we may not find a single λ for the whole group $E(A[T])$ so that we can apply the restriction $T = \lambda$.

To tackle this problem we note that, however, the ideal $I(0)/I(0)^2$ is generated by n elements and applying some "moving lemma" (which is an application of Eisenbud–Evans theorem) we can find an ideal K of A of height $\geqslant n$, residual to $I(0)$, and a surjection $\omega_K : (A/K)^n \twoheadrightarrow K/K^2$. We define $\Psi(I, \omega_I) = -(K, \omega_K)$ and prove the following

Theorem 1.1. *The map $\Psi : E(A[T]) \to E(A)$, described above, is a homomorphism of groups such that if $(I, \omega_I) \in E(A[T])$ has the property that $I(0)$ is an ideal of A of height n, then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$.*

To prove that Ψ is well defined and is a group homomorphism we require the so-called "addition" and "subtraction" principles in a little more generality which we prove to fit our needs.

Indeed Ψ is surjective and the composition $\Psi\Phi : E(A) \to E(A[T])$ is the identity map. Furthermore, Ψ is an isomorphism when A is a smooth affine domain.

Let us recall one important result from [9] which is very much relevant to this context. In [9], given an ideal $I \subset A[T]$ of height n and a surjection $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$, we associated an element $(I, \omega_I) \in E(A[T])$. One of the prime objectives was to show that this element $(I, \omega_I) \in E(A[T])$ is the precise obstruction for the surjection ω_I to lift to a surjection $\theta : A[T]^n \twoheadrightarrow I$ [9, Theorem 4.7]. For proving this theorem we first showed that we can assume $\operatorname{ht} I(0) = n$ and then argued "since $(I, \omega_I) = 0$ in $E(A[T])$, we have $(I(0), \omega_I(0)) = 0$ in $E(A)$". This argument essentially assumes the existence of the group homomorphism $\Psi : E(A[T]) \to E(A)$ with the property mentioned in Theorem 1.1 above. However, the question of existence of such a group homomorphism has not been addressed there. In this sense, we now get a complete proof of [9, Theorem 4.7] (in Theorem 3.4 in this paper).

As discussed above, while working on the group homomorphism Ψ, our prime concern was the fact that given an ideal I of $A[T]$ of height n, $I(0)$ may not have height n. But the form of Theorem 1.1 led us to believe that while working on $E(A[T])$, we may restrict ourselves to the "nice" ideals I of $A[T]$ for which $I(0)$ has height n or $I(0) = A$. In this context, we define a "restricted" Euler class group $E'(A[T])$ of $A[T]$ which concerns only those "nice" ideals and prove that $E(A[T])$ is isomorphic to $E'(A[T])$ (Proposition 3.7).

Let A be an affine algebra of dimension n over an algebraically closed field k of characteristic zero. Then $E(A)$ is isomorphic to $E_0(A)$ (can be easily deduced from [4, Lemma 3.4]). In Section 4 we investigate the Euler class group $E(A[T])$ and the weak Euler class group $E_0(A[T])$ when A is an affine algebra over an algebraically closed field and prove that $E(A[T])$ and $E_0(A[T])$ are canonically isomorphic (Corollary 5.4).

In Section 4 we also address the following question.

Question. Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geqslant 3$. Let $(I, \omega_I) \in E(A[T])$ be an arbitrary element. Does there exist a projective module P of rank n (with trivial determinant) together with an isomorphism $\chi : A[T] \simeq \bigwedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$?

In general, the answer to this question is negative as one can take A to be the coordinate ring of an even-dimensional real sphere, any real maximal ideal J of A and set $I = J[T]$. We show, using Corollary 5.4 and the following theorem of Bhatwadekar–Raja Sridharan that the above question has an affirmative answer if A is an affine domain over an algebraically closed field of characteristic zero.

Theorem 1.2. [7, Theorem 2.7] *Let A be an affine domain of dimension $n \geqslant 3$ over an algebraically closed field k of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that I/I^2 is generated by n elements. Then there exists a projective $A[T]$-module P of rank n with trivial determinant and a surjection $\Phi : P \twoheadrightarrow I$.*

We also give an alternative proof of Theorem 1.2 using Euler class computations. Our proof appears simpler with the use of Euler class techniques. We may note that when Bhatwadekar–Raja Sridharan proved this result, the Euler class group of a polynomial algebra was not defined.

2. Preliminaries

In this section we define some of the terms used in the paper and record some results which are used in later sections.

All rings considered in this paper are commutative and Noetherian and all modules considered are assumed to be finitely generated. The projective modules are assumed to have constant rank.

We start with an easy lemma.

Lemma 2.1. *Let B be a Noetherian ring of dimension n and $J \subset B$ be an ideal which is contained in the Jacobson radical of B. Suppose that $K \subset B[T]$ is an ideal such that $K + JB[T] = B[T]$. Then any maximal ideal of $B[T]$ containing K has height $\leqslant n$.*

Now we state a useful lemma. The proof of this lemma can be found in [3, 3.3].

Lemma 2.2. *Let A be a Noetherian ring containing an infinite field k and let $I \subset A[T]$ be an ideal of height n. Then there exists $\lambda \in k$ such that either $I(\lambda) = A$ or $I(\lambda) \subset A$ is an ideal of height n, where $I(\lambda) = \{f(\lambda) \mid f(T) \in I\}$.*

Definition 2.3. Let A be a commutative Noetherian ring and P be a projective A-module of rank $n \leqslant \dim A$. By a *generic surjection* of P we mean a surjection $\alpha : P \twoheadrightarrow J$ where J is an ideal of A of height n. It follows from a theorem of Eisenbud–Evans [11,17] that generic surjections exist.

Definition 2.4. Let A be a commutative Noetherian ring, P a projective $A[T]$-module. Let $J(A,P) \subset A$ consist of all those $a \in A$ such that P_a is extended from A_a. It follows from [18, Theorem 1], that $J(A,P)$ is an ideal and $J(A,P) = \sqrt{J(A,P)}$. This is called the *Quillen ideal* of P in A.

Remark 2.5. Let A, P, $J(A,P)$ be as in the above definition. Then it is easy to deduce from Quillen–Suslin theorem [18,22] that height of $J(A,P)$ is at least one. If determinant of P is extended from A, then by [17, Corollary 2], $\operatorname{ht} J(A,P) \geqslant 2$.

Definition 2.6. Let A be a ring and $A[T]$ be its polynomial extension. We denote by $A(T)$, the ring obtained from $A[T]$ by inverting all the monic polynomials in $A[T]$. It can be proved easily that dimension of $A(T)$ is same as dimension of A.

The proof of the following lemma can be found in [3, Remark 3.9].

Lemma 2.7. *Let A be a ring, $I \subset A[T]$ be an ideal such that $I = (f_1, \ldots, f_n) + I^2$. Assume further that either $I(0) = A$ or $I(0) = (a_1, \ldots, a_n)$ such that $f_i(0) \equiv a_i \bmod I(0)^2$. Then we can find $g_1, \ldots, g_n \in I$ such that $I = (g_1, \ldots, g_n) + (I^2T)$ with the properties*: (1) $g_i \equiv f_i \bmod I^2$, (2) $g_i(0) = a_i$.

We now quote a theorem of Mandal. The following version is implicit in [12, Theorem 1.2].

Theorem 2.8. *Let A be a Noetherian ring and $I \subset A[T]$ be an ideal containing a monic polynomial. Suppose that $I = (f_1, \ldots, f_r) + I^2$, where $r \geqslant \dim(A[T]/I) + 2$. Then, there exist $g_1, \ldots, g_r \in I$ such that $I = (g_1, \ldots, g_r)$ and $f_i \equiv g_i \bmod I^2$.*

The following theorem is also due to Mandal [13, Theorem 2.1].

Theorem 2.9. *Let A be a Noetherian ring and $I \subset A[T]$ be an ideal containing a monic polynomial. Suppose that $I = (f_1, \ldots, f_r) + (I^2T)$, where $r \geqslant \dim(A[T]/I) + 2$. Then, there exist $g_1, \ldots, g_r \in I$ such that $I = (g_1, \ldots, g_r)$ and $f_i \equiv g_i \bmod (I^2T)$.*

The following result is a special case of [14, Theorem 2.3].

Theorem 2.10. *Let A be a Noetherian ring. Suppose $K_3 = K_1 \cap K_2$ be the intersection of two comaximal ideals K_1, K_2 of $A[T]$ such that*:

(1) *K_1 contains a monic polynomial in T.*
(2) *K_2 is an extended ideal.*
(3) *$K_1 = (f_1(T), \ldots, f_n(T))$ with $n \geqslant \dim A[T]/K_1 + 2$.*
(4) *$K_3(0) = (c_1 \ldots, c_n)$ with $c_i - f_i(0) \in K_1(0)^2$.*

Then $K_3 = (h_1(T), \ldots, h_n(T))$ with $h_i(0) = c_i$.

We will refer to the following lemma as "moving lemma". This lemma can easily be proved adapting the proof of [5, Corollary 2.14].

Lemma 2.11. *Let A be a Noetherian ring of dimension $n \geqslant 2$. Let J be an ideal of A of height $\geqslant 1$ such that $J = (a_1, \dots, a_n) + J^2$. Let K be any ideal of A of height at least one. Then there exists an ideal $J' \subset A$ such that*:

(1) *J' is comaximal with $J \cap K$ and* $\operatorname{ht} J' \geqslant n$.
(2) *$J \cap J' = (c_1, \dots, c_n)$ where $c_i \equiv a_i \bmod J^2$.*

In the rest of this section we briefly sketch the definitions of the *Euler class groups* $E(A[T])$ and the *weak Euler class groups* $E_0(A[T])$ (where A is a commutative Noetherian ring containing $\mathbb{Q}$ of dimension $n \geqslant 2$) and quote some results that are relevant to this paper. The notions of $E(A[T])$ and $E_0(A[T])$ have been defined and studied in [9]. We refer the reader to [9] for a detailed account of these topics.

Definitions of $E(A[T])$ and $E_0(A[T])$

Let A be a Noetherian ring of dimension $n \geqslant 2$ containing $\mathbb{Q}$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections α and β from $(A[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(A[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of I*.

It was shown in [9, Proposition 4.4], that if $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of I* if the surjection $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset A[T]$ is an ideal of height n such that $\operatorname{Spec}(A[T]/I)$ is connected, I/I^2 is generated by n elements and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I.

Let $I \subset A[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \dots \cap I_r$, where the I_k's are ideals of $A[T]$ of height n, pairwise comaximal and $\operatorname{Spec}(A[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \leqslant k \leqslant r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G.

Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $A[T]$ of height n generated by n elements and ω_I is a global orientation of I given by the set of generators of I. We define the Euler class group of $A[T]$, denoted by $E(A[T])$, to be G/H.

The weak Euler class group $E_0(A[T])$ is defined in a similar way, just dropping the orientations, as follows.

Let F be the free abelian group on the set of ideals $\mathcal{I}$ where $\operatorname{ht} \mathcal{I} = n$, $\mathcal{I}/\mathcal{I}^2$ is generated by n elements and $\operatorname{Spec}(A[T]/\mathcal{I})$ is connected. For an ideal I of $A[T]$ of height n such that I/I^2 is generated by n elements, we take its decomposition into connected components (as above), say, $I = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_r$, and associate to I the element $(I) := \Sigma \mathcal{I}_k$ of F. Let K

be the subgroup of F generated by elements of the type (I), where $I \subset A[T]$ is an ideal of height n and I is generated by n elements. We define $E_0(A[T])$ to be F/K.

Let P be a projective $A[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : A[T] \simeq \bigwedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a generic surjection (i.e., I is an ideal of height n). Note that, since P has trivial determinant and $\dim A[T]/I \leqslant 1$, P/IP is a free $A[T]/I$-module. Composing $\alpha \otimes A[T]/I$ with an isomorphism $\gamma : (A[T]/I)^n \simeq P/IP$ with the property $\bigwedge^n(\gamma) = \chi \otimes A[T]/I$ we get a local orientation, say ω_I, of I. Let $e(P,\chi)$ be the image in $E(A[T])$ of the element (I,ω_I) of G. (We say that (I,ω_I) is obtained from the pair (α,χ).) It can be proved that the assignment sending the pair (P,χ) to $e(P,\chi)$ is well defined (see [9]). We define the *Euler class* of P to be $e(P,\chi)$.

3. Main results

We begin this section with the following addition and subtraction principles. Here we have only relaxed the condition on height of the ideals concerned. The methods of proof are similar to the usual addition and subtraction principles (one can look at [8, Propositions 3.1, 3.2]). However we include the proofs for the sake of completeness.

Proposition 3.1 *(Addition Principle). Let A be a Noetherian ring of dimension $n \geqslant 3$ and I, J be two comaximal ideals of A, each of height $\geqslant n-1$. Assume further that $I = (a_1,\dots,a_n)$ and $J = (b_1,\dots,b_n)$. Then, $I \cap J = (c_1,\dots,c_n)$ such that $c_i \equiv a_i \bmod I^2$ and $c_i \equiv b_i \bmod J^2$.*

Proof. Note that we can always perform elementary transformations on $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ and no generality is lost doing so. To see this, let us assume that $(a_1,\dots,a_n)$ is elementarliy transformed to $(\widetilde{a_1},\dots,\widetilde{a_n})$ and $(b_1,\dots,b_n)$ is elementarily transformed to $(\widetilde{b_1},\dots,\widetilde{b_n})$. Suppose we can find a set of generators $\widetilde{c_1},\dots,\widetilde{c_n}$ of $I \cap J$ satisfying $\widetilde{c_i} \equiv \widetilde{a_i} \bmod I^2$ and $\widetilde{c_i} \equiv \widetilde{b_i} \bmod J^2$. Then we can use the surjectivity of the canonical map $E_n(A/I \cap J) \to E_n(A/I) \times E_n(A/J)$ to transform $(\widetilde{c_1},\dots,\widetilde{c_n})$ to $(c_1,\dots,c_n)$, so that $I \cap J = (c_1,\dots,c_n)$ with $c_i \equiv a_i \bmod I^2$ and $c_i \equiv b_i \bmod J^2$.

Let $B = A/(b_1,\dots,b_n)$ and bar denote reduction mod $(b_1,\dots,b_n)$. Since $I + J = A$, $(\overline{a_1},\dots,\overline{a_n}) \in Um_n(B)$. Now $\dim B \leqslant 1$ and $n \geqslant 3$. Therefore, we can elementary transform $(\overline{a_1},\dots,\overline{a_n})$ to $(\bar{1},\dots,\bar{0})$. Applying [20, Lemma 2] we can apply an elementary transformation and assume that $\operatorname{ht}(a_1,\dots,a_{n-1}) = n-1$. Note that this transformation preserves the fact that $a_1 \equiv 1$ modulo J. Therefore, $(a_1,\dots,a_{n-1}) + J = A$.

Now let $C = A/(a_1,\dots,a_{n-1})$ and bar denote reduction mod $(a_1,\dots,a_{n-1})$. Consider the unimodular row $(\overline{b_1},\dots,\overline{b_n}) \in Um_n(C)$. Using similar arguments as in the above paragraph we finally obtain:

(1) $(a_1,\dots,a_{n-1}) + (b_1,\dots,b_{n-1}) = A$.
(2) $\operatorname{ht}(a_1,\dots,a_{n-1}) = \operatorname{ht}(b_1,\dots,b_{n-1}) = n-1$.

In $A[T]$ we consider the ideals

$$I_1 = (a_1, \ldots, a_{n-1}, T + a_n), \qquad I_2 = (b_1, \ldots, b_{n-1}, T + b_n)$$

and let $K = I_1 \cap I_2$. Note that $I_1 + I_2 = A[T]$. Therefore, using the Chinese remainder theorem we can choose $g_1(T), \ldots, g_n(T) \in K$ such that

$$K = \big(g_1(T), \ldots, g_n(T)\big) + K^2$$

satisfying $g_i(T) \equiv a_i \bmod I_1^2$, $g_i(T) \equiv b_i \bmod I_2^2$, $1 \leqslant i \leqslant n-1$; $g_n(T) \equiv T + a_n \bmod I_1^2$, $g_n(T) \equiv T + b_n \bmod I_2^2$.

Now $\mathrm{ht}(a_1, \ldots, a_{n-1}) = \mathrm{ht}(b_1, \ldots, b_{n-1}) = n-1$. Also note that $\dim A[T]/I_1 = \dim A/(a_1, \ldots, a_{n-1})$ and $\dim A[T]/I_2 = \dim A/(b_1, \ldots, b_{n-1})$. Therefore, it follows that $\dim A[T]/K \leqslant 1$. Since $n \geqslant 3$, the conditions of Theorem 2.8 are satisfied for K. Applying Theorem 2.8 we obtain $K = (h_1(T), \ldots, h_n(T))$ such that $h_i(T) \equiv g_i(T) \bmod K^2$. Let $h_i(0) = c_i$. Then $I \cap J = (c_1, \ldots, c_n)$ with $c_i \equiv a_i \bmod I^2$ and $c_i \equiv b_i \bmod J^2$. □

Proposition 3.2 *(Subtraction Principle). Let A be a Noetherian ring of dimension $n \geqslant 3$ and I, J be two comaximal ideals of A, each of height $\geqslant n-1$. Assume further that $I = (a_1, \ldots, a_n)$ and $I \cap J = (c_1, \ldots, c_n)$ such that $c_i \equiv a_i \bmod I^2$. Then $J = (b_1, \ldots, b_n)$ such that $c_i \equiv b_i \bmod J^2$.*

Proof. First note that we can perform elementary transformations on $(a_1, \ldots, a_n)$ because we can apply the same elementary transformations on $(c_1, \ldots, c_n)$ to retain the relation that $c_i \equiv a_i \bmod I^2$. Let $B = A/J^2$ and bar denote reduction modulo J^2. Since $\mathrm{ht}(J) = n-1$, $\dim B \leqslant 1$. Therefore, performing elementary transformations as in the proof of the above proposition we may assume that: (1) $\mathrm{ht}(a_1, \ldots, a_{n-1}) = n-1$, (2) $a_n \equiv 1 \bmod J^2$.

Consider the following ideals in $A[T]$:

$$I_1 = (a_1, \ldots, a_{n-1}, T + a_n), \qquad I_2 = JA[T], \qquad K = I_1 \cap I_2.$$

Applying Theorem 2.10 we obtain a set of generators $(h_1(T), \ldots, h_n(T))$ of K such that $h_i(0) = c_i$. Let $b_i = h_i(1 - a_n)$. Then $J = (b_1, \ldots, b_n)$. Since $a_n \equiv 1 \bmod J^2$, $b_i - c_i = h_i(1 - a_n) - h_i(0) \equiv 0 \bmod J^2$. This proves the proposition. □

We are now ready to show that there is a group homomorphism from $E(A[T])$ to $E(A)$ such that if $(I, \omega_I) \in E(A[T])$ has the property that $I(0)$ is an ideal of A of height n, then this group homomorphism takes (I, ω_I) to $(I(0), \omega_{I(0)})$ in $E(A)$, where $\omega_{I(0)}$ is the local orientation of $I(0)$ induced by ω_I. This is done in Theorem 3.3 below. To prove this theorem we mainly need addition and subtraction principles proved above and Lemma 2.11.

Theorem 3.3. *Let A be a Noetherian ring containing $\mathbb{Q}$ of dimension $n \geqslant 3$. There is a group homomorphism $\Psi : E(A[T]) \to E(A)$ such that if $(I, \omega_I) \in E(A[T])$ has the property that $I(0)$ is an ideal of A of height n, then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$ in $E(A)$, where $\omega_{I(0)}$ is the local orientation of $I(0)$ induced by ω_I. If $I(0) = A$, $\Psi((I, \omega_I)) = 0$.*

Proof. We give the proof in steps.

Step 1. Recall that $E(A[T])$ is defined as G/H, where G is the free abelian group on the set of pairs (I, ω_I), where $I \subset A[T]$ is an ideal of height n having the property that $\operatorname{Spec}(A[T]/I)$ is connected and I/I^2 is generated by n elements, and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let us pick one such element (I, ω_I). Let the local orientation ω_I be given by $I = (f_1, \ldots, f_n) + I^2$. Now $I(0)$ is an ideal of A (not necessarily proper) with $\operatorname{ht}(I(0)) \geqslant n-1$ and $I(0) = (f_1(0), \ldots, f_n(0)) + I(0)^2$. Let $J = I \cap A$.

Now applying Lemma 2.11 we can find an ideal $K \subset A$ of height $\geqslant n$ such that K is comaximal with J and $K \cap I(0) = (a_1, \ldots, a_n)$, where $a_i \equiv f_i(0) \bmod I(0)^2$. First assume that both $I(0)$ and K are proper ideals of A. Let us call the local orientation of K, induced by $a_1, \ldots, a_n$, to be ω_K. To the element (I, ω_I) of G we associate the element $-(K, \omega_K)$ of $E(A)$. In the case when $I(0) = A$ or $K = A$, (I, ω_I) is associated to the zero element of $E(A)$.

We need to show that this association does not depend on the choice of K. To prove this, let K' be another ideal of A of height n such that K' is comaximal with J and $K' \cap I(0) = (b_1, \ldots, b_n)$, where $b_i \equiv f_i(0) \bmod I(0)^2$. Let $\omega_{K'}$ be the local orientation of K' induced by $b_1, \ldots, b_n$. We claim that $(K, \omega_K) = (K', \omega_{K'})$ in $E(A)$. In the next paragraph we prove this claim.

First note that, by repeated use of addition and subtraction principles and moving lemma (Lemma 2.11), we may assume that K' is comaximal with K. Now we can find an ideal $L \subset A$ of height n and a local orientation ω_L of L such that L is comaximal with each of J, K and K' and $(L, \omega_L) + (K, \omega_K) = 0$ in $E(A)$. Therefore, it is enough to prove that $(L, \omega_L) + (K', \omega_{K'}) = 0$ in $E(A)$. In order to do so, we first apply the addition principle to the two comaximal ideals $L \cap K$ and $K' \cap I(0)$ to see that $L \cap K \cap K' \cap I(0)$ is generated by n elements (with appropriate set of generators). Next we apply the subtraction principle to the comaximal ideals $K \cap I(0)$ and $L \cap K'$ to conclude that $L \cap K'$ is generated by n elements (with appropriate set of generators), i.e., $(L, \omega_L) + (K', \omega_{K'}) = 0$ in $E(A)$. Thus the claim is proved.

Step 2. Extending to whole of G, we get a group homomorphism $\psi : G \to E(A)$. Note that in the above definition we nowhere used the fact that $\operatorname{Spec}(A[T]/I)$ is connected. So, given any $(I, \omega_I) \in E(A[T])$ (where $\operatorname{Spec}(A[T]/I)$ is not necessarily connected), following the above procedure, we can also associate an element, say $-(K, \omega_K)$ of $E(A)$. We claim that the image of (I, ω_I) under ψ is actually $-(K, \omega_K)$.

Proof of the claim. Consider a decomposition of I into its connected components, say, $I = I_1 \cap \cdots \cap I_r$. Now since I_i's are pairwise comaximal, ω_I induces local orientation ω_{I_i} of I_i, $i = 1, \ldots, r$ and we have, $(I, \omega_I) = \sum_{i=1}^{r} (I_i, \omega_{I_i})$. Suppose that $\psi((I_i, \omega_{I_i})) = -(K_i, \omega_{K_i})$ $(\in E(A))$. In view of Lemma 2.11, we can clearly assume that K_i's are pairwise comaximal. For simplicity we work out the case when $r = 2$.

By definition of ψ, we have $\psi((I, \omega_I)) = -(K_1, \omega_{K_1}) - (K_2, \omega_{K_2})$ in $E(A)$. Since K_1 and K_2 are comaximal, we can write $\psi((I, \omega_I)) = -(K_1 \cap K_2, \omega_{K_1 \cap K_2})$, where $\omega_{K_1 \cap K_2}$ is induced by ω_{K_1} and ω_{K_2}.

Now by the definition of ψ we have $K_1 \cap I_1(0)$ is n-generated with appropriate set of generators. Same is true for $K_2 \cap I_2(0)$. Since the ideals $K_1 \cap I_1(0)$ and $K_2 \cap I_2(0)$ are comaximal and each has height $\geqslant n-1$, applying addition principle we have $K_1 \cap I_1(0) \cap K_2 \cap I_2(0)$ n-generated with appropriate set of generators. In other words, $I(0) \cap (K_1 \cap K_2)$ is n-generated. Now keeping track of the generators and proceeding as in last paragraph of Step 1, we can easily conclude that $(K, \omega_K) = (K_1 \cap K_2, \omega_{K_1 \cap K_2})$. This proves the claim.

Step 3. Recall that $E(A[T]) = G/H$, where H is the subgroup of G generated by pairs $(I, \omega_I) \in G$ such that ω_I is a global orientation. We now show that H is in the kernel of ψ.

First let $(L, \omega_L) \in G$ be such that ω_L is a global orientation. This means that there exist $f_1, \ldots, f_n \in L$ such that $L = (f_1, \ldots, f_n)$ and ω_L is induced by this set of generators of L. But then $L(0) = (f_1(0), \ldots, f_n(0))$ and hence from the definition of ψ it follows that $\psi((L, \omega_L)) = 0$ in $E(A)$. Now an element of H is of the form

$$(I, \omega_I) = \sum_{i=1}^{r} (I_i, \omega_{I_i}) - \sum_{j=r+1}^{s} (I_i, \omega_{I_i}),$$

where $\omega_{I_i}, \omega_{I_j}$ are global orientations. It is now clear that $\psi((I, \omega_I)) = 0$ in $E(A)$ as each of the elements on the right hand side is mapped to zero.

Therefore, we have a group homomorphism $\Psi : E(A[T]) \to E(A)$.

Step 4. Let $(I, \omega_I) \in E(A[T])$ be such that $\mathrm{ht}(I(0)) = n$. In this case ω_I induces a local orientation $\omega_{I(0)}$ of $I(0)$ and $(I(0), \omega_{I(0)}) \in E(A)$. The way we picked up K and ω_K in Step 1 actually means in this case that $(I(0), \omega_{I(0)}) + (K, \omega_K) = 0$ in $E(A)$. Therefore, $\Psi((I, \omega_I)) = -(K, \omega_K) = (I(0), \omega_{I(0)})$. This completes the proof of the theorem. $\square$

We now use the above result to give a complete proof of the following theorem from [9, Theorem 4.7]. As mentioned in the introduction, the question of existence of such a group homomorphism as in Theorem 3.3 has not been addressed in [9] whereas some crucial arguments in the proof of [9, Theorem 4.7] implicitly uses this group homomorphism.

Theorem 3.4. *Let A be a ring of dimension $n \geqslant 3$, $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements and let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, ω_I) is zero in the Euler class group $E(A[T])$ of $A[T]$. Then, I is generated by n elements and ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.*

Proof. Let $\Psi : E(A[T]) \to E(A)$ be the group homomorphism, as defined in the theorem above. Suppose ω_I is given by $I = (f_1, \ldots, f_n) + I^2$. We first assume that $I(0)$ is a proper ideal of A. We have, $I(0) = (f_1(0), \ldots, f_n(0)) + I(0)^2$. Suppose that $\Psi((I, \omega_I)) = -(K, \omega_K)$, where $K \subset A$ is an ideal of height $\geqslant n$ such that $K \cap I(0) = (c_1, \ldots, c_n)$ where $c_i \equiv f_i(0) \bmod I(0)^2$ and ω_K is induced by $c_1, \ldots, c_n$. Since $(I, \omega_I) = 0$ in $E(A[T])$, $\Psi((I, \omega_I)) = 0$ in $E(A)$ and therefore, $(K, \omega_K) = 0$ in $E(A)$. This implies, by [5, Theorem 4.2], that $K = (a_1, \ldots, a_n)$ such that $a_i \equiv c_i \bmod K^2$. Now applying subtraction principle (Proposition 3.2) we see that $I(0) = (b_1, \dots, b_n)$ such that $b_i \equiv c_i \bmod I(0)^2$. Consequently, $b_i \equiv f_i(0) \bmod I(0)^2$. Therefore, using Lemma 2.7, it follows that ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I/(I^2T)$.

On the other hand, if $I(0) = A$, then again, applying Lemma 2.7 we can lift ω_I to a surjection $\theta : A[T]^n \twoheadrightarrow I/(I^2T)$.

In $E(A(T))$ also, the element $(IA(T), \omega_{IA(T)})$ is zero, which, by [5, Theorem 4.2], implies that $\omega_{IA(T)}$ (and hence $\theta \otimes A(T)$) can be lifted to a set of generators of $IA(T)$. Applying [9, Theorem 3.10], we conclude that θ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$. Clearly α lifts ω_I. So ω_I is a global orientation. □

Remark 3.5. Let us review the group homomorphism $\Psi : E(A[T]) \to E(A)$. Recall that we already have a canonical group homomorphism $\Phi : E(A) \to E(A[T])$ and it follows from Theorem 3.4 that Φ is injective. Further, it is easy to see that the composition $\Psi\Phi$ is the identity on $E(A)$. Clearly Ψ is surjective. Of particular interest is the kernel. Let $(I, \omega_I) \in E(A[T])$ be an element of $\mathrm{Ker}(\Psi)$. Then, as shown in the proof of Theorem 3.4 above, ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I/(I^2T)$. $\mathrm{Ker}(\Psi)$ precisely consists of these elements (i.e., roughly, the ideals I of $A[T]$ of height n such that $I/(I^2T)$ is generated by n elements). We may recall that from [3, Theorem 3.8] it follows that if A is a smooth affine domain over an infinite perfect field, $\mathrm{Ker}(\Psi)$ is trivial and hence Ψ becomes an isomorphism. If A is not smooth, there is an example [3, 6.4] of a normal affine domain A for which $\mathrm{Ker}(\Psi)$ is not trivial. We expect $\mathrm{Ker}(\Psi) = 0$ when A is a regular ring containing $\mathbb{Q}$. The "local–global principle" for Euler class groups [9, Theorem 5.4] suggests that it is enough to prove $\mathrm{Ker}(\Psi) = 0$ when A is a regular local ring containing $\mathbb{Q}$.

The main point of Theorem 3.4 is that for an ideal $I \subset A[T]$ of height n, $I(0)$ may not have height n and therefore given $(I, \omega_I) \in E(A[T])$, something like $(I(0), \omega_{I(0)})$ may not make sense. This makes sense only when $I(0)$ has height n or $I(0) = A$. Then we started wondering what happens if we work only with those ideals I for which $I(0)$ has height n or $I(0) = A$. This is reflected in the following definition and the proposition after that.

Definition of a group

We define a group $E'(A[T])$ which may be regarded as the "restricted" Euler class group of $A[T]$. The definition of $E'(A[T])$ is similar to that of $E(A[T])$.

Let G' be the free abelian group on the set of pairs (I, ω_I), where $I \subset A[T]$ is an ideal of height n having the properties: (i) $I(0) \subset A$ is an ideal of height n or $I(0) = A$ (we point out here that this is the "restriction" and only at this point the definition differs from that of $E(A[T])$), (ii) $\mathrm{Spec}(A[T]/I)$ is connected, (iii) I/I^2 is generated by n elements; and ω_I is a local orientation of I.

Let $I \subset A[T]$ be any ideal of height n such that I/I^2 is generated by n elements. Let $I = I_1 \cap \dots \cap I_k$ be the decomposition of I into its connected components. Let ω_I be a local orientation of I. Then ω_I induces local orientations ω_{I_i} of I_i for $i = 1, \dots, k$. By (I, ω_I) we mean the element $\Sigma(I_i, \omega_i)$ of G'.

Let H' be the subgroup of G' generated by the set of pairs (I, ω_I) where ω_I is a global orientation of I.

We define $E'(A[T])$ to be the group G'/H'.

Remark 3.6. Clearly the obvious map $\Delta: E'(A[T]) \to E(A[T])$ which sends $(I, \omega_I) \in E'(A[T])$ to $(I, \omega_I) \in E(A[T])$, is a group homomorphism.

Proposition 3.7. *The map $\Delta: E'(A[T]) \to E(A[T])$, as described above, is an isomorphism of groups.*

Proof. By the very definition of $E'(A[T])$ and by Theorem 3.4, it follows that Δ is injective. To prove the surjectivity, let $(I, \omega_I) \in E(A[T])$ be an arbitrary element. So $I(0)$ may not necessarily be of height n. To prove that Δ is surjective it is enough to find some $(I', \omega_{I'}) \in E(A[T])$ such that $\operatorname{ht}(I'(0)) \geqslant n$ and $(I, \omega_I) = (I', \omega_{I'})$ in $E(A[T])$.

Suppose that ω_I is given by $I = (f_1, \ldots, f_n) + I^2$. Then, $I(0) = (f_1(0), \ldots, f_n(0)) + I(0)^2$. Let $J = I \cap A$. Since $\operatorname{ht} J \geqslant n - 1 \geqslant 2$, we can find an element $s \in J^2$ such that $\operatorname{ht}(s) = 1$. Let bar denote reduction modulo s. Since $\dim \overline{A} \leqslant n - 1$, it follows by a result of Mohan Kumar [15, Corollary 3], that $\overline{I(0)} = (\overline{a_1}, \ldots, \overline{a_n})$, where $a_i \equiv f_i(0)$ modulo $I(0)^2$. By adding suitable multiples of s to $a_1, \ldots, a_n$, we may assume by the Eisenbud–Evans theorem (see [5, Corollary 2.13]) that $(a_1, \ldots, a_n) = I(0) \cap K$, where $K \subset A$ is an ideal of height n and $K + (s) = A$. Note that $K = (a_1, \ldots, a_n) + K^2$. Let us call the local orientation corresponding to this set of generators of K/K^2 by ω_K.

Let $I_1 = I \cap K[T]$. Then I_1 is an ideal of $A[T]$ of height n. Since I and $K[T]$ are comaximal ideals, the local orientations ω_I and $\omega_K \otimes A[T]$, of I and $K[T]$ respectively, induce a local orientation ω_{I_1} of I_1, say, given by $I_1 = (g_1, \ldots, g_n) + I_1^2$. Now

$$I_1(0) = I(0) \cap K = (a_1, \ldots, a_n)$$

and we have $g_i(0) \equiv a_i \bmod I_1(0)^2$. Therefore we can lift $g_1, \ldots, g_n$ to a set of n generators of $I_1/(I_1^2 T)$, which also corresponds to ω_{I_1}. In $E(A[T])$ we have the equation:

$$(I_1, \omega_{I_1}) = (I, \omega_I) + \big(K[T], \omega_K \otimes A[T]\big).$$

Since ω_{I_1} is given by a set of generators of $I_1/(I_1^2 T)$, we can apply [9, Lemma 3.9] to find an ideal I_2 of $A[T]$ of height n and a local orientation ω_{I_2} of I_2 such that (i) $I_2(0) = A$ and (ii) $(I_1, \omega_{I_1}) + (I_2, \omega_{I_2}) = 0$ in $E(A[T])$.

Therefore, we have the equation

$$(I, \omega_I) + \big(K[T], \omega_K \otimes A[T]\big) + (I_2, \omega_{I_2}) = 0$$

in $E(A[T])$. Since $(K[T], \omega_K \otimes A[T])$ and (I_2, ω_{I_2}) both belong to $E'(A[T])$, the result follows. $\square$

4. Some proofs

Let A be a ring (containing $\mathbb{Q}$) of dimension $n \geqslant 3$. Let P be a projective $A[T]$-module of rank n having trivial determinant and χ be a trivialization of $\bigwedge^n P$. To the pair (P, χ) we can associate an element $e(P, \chi)$ in $E(A[T])$, called the Euler class of (P, χ) (see Section 2 for the definition). In [9, 4.11] we proved that $e(P, \chi) = 0$ in $E(A[T])$ if and only if P has a unimodular element. In that proof we crucially used a theorem of Bhatwadekar–Raja Sridharan [6, Theorem 3.4]. Again the proof of [6, Theorem 3.4] depends heavily on a remark from a paper of Bhatwadekar–Lindel–Rao [2, 5.3]. Here we give a straightforward proof of [9, 4.11] which is more in the spirit of Euler class theory. Further, we derive a version of [6, Theorem 3.4] using our theorem.

Theorem 4.1. *Let A be as above. Let P be a projective $A[T]$-module of rank n having trivial determinant and χ be a trivialization of $\bigwedge^n P$. Then, $e(P, \chi) = 0$ in $E(A[T])$ if and only if P has a unimodular element.*

Proof. Let $\alpha : P \twoheadrightarrow I_1$ be a surjection where I_1 is an ideal in $A[T]$ of height n and ω_{I_1} be the local orientation of I_1 induced by (α, χ). Then, $e(P, \chi) = (I_1, \omega_{I_1})$ in $E(A[T])$.

Suppose that P has a unimodular element. We show, under this condition, that $(I_1, \omega_{I_1}) = 0$ in $E(A[T])$. In view of the isomorphism Δ in Proposition 3.7, we can assume that either $I_1(0) = A$ or $\operatorname{ht} I_1(0) = n$. Since P has a unimodular element, it follows that the projective A-module P/TP and the projective $A(T)$-module $P \otimes A(T)$ both have unimodular elements. Consequently, by [5, Corollary 4.4], we have $(I_1(0), \omega_{I_1(0)}) = 0$ in $E(A)$ and $(I_1 A(T), \omega_{I_1} \otimes A(T)) = 0$ in $E(A(T))$. Now following the arguments as in Theorem 3.4, it is easy to see that $(I_1, \omega_{I_1}) = 0$ in $E(A[T])$.

Let us now assume that $e(P, \chi) = 0$ in $E(A[T])$. We prove that then P has a unimodular element. We give the proof in steps.

Step 1. In this step we show that the projective A-module P/TP has a unimodular element.

Recall that we have $\alpha : P \twoheadrightarrow I_1$, a generic surjection of P and ω_{I_1} is the local orientation of I_1 induced by (α, χ). Therefore, $e(P, \chi) = (I_1, \omega_{I_1})$ in $E(A[T])$. As usual, we may assume that either $I_1(0) = A$ or $I_1(0)$ is a proper ideal of height n. If $I_1(0) = A$, then clearly the A-module P/TP has a unimodular element. Now suppose that $I_1(0)$ is a proper ideal of height n. Then, following the definition of the Euler class of a projective module we have, $e(P/TP, \chi \otimes A[T]/(T)) = (I_1(0), \omega_{I_1(0)})$ in $E(A)$. Since $(I_1, \omega_{I_1}) = 0$, in $E(A[T])$, it follows that $(I_1(0), \omega_{I_1(0)}) = 0$ in $E(A)$. Therefore, $e(P/TP, \chi \otimes A[T]/(T)) = 0$ in $E(A)$ and hence by [5, Corollary 4.4], P/TP has a unimodular element.

So in any case P/TP has a unimodular element.

Step 2. Let $J(A, P)$ denote the Quillen ideal of P in A. Write $J = J(A, P)$. In this step we prove, using a theorem of Mandal, that P_{1+J} has a unimodular element.

Since determinant of P is extended (actually free), by Remark 2.5, $\operatorname{ht} J(A, P) \geqslant 2$. Since $\dim A/J \leqslant n - 2$, it follows that the projective $(A/J)[T]$-module $P/J[T]P$ has a unimodular element, i.e., there is a surjection $P/J[T]P \twoheadrightarrow (A/J)[T]$. Using this fact and the Eisenbud–Evans theorem [11,17]) it is easy to see that (since P is projective), there is a generic surjection $\beta : P \twoheadrightarrow I$ such that I is comaximal with $J[T]$.

Let ω_I be the local orientation of I induced by (β, χ). Then $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$.

Consider the ring $B = A_{1+J}$. We want to prove that the projective $B[T]$-module P_{1+J} has a unimodular element. If $IB[T] = B[T]$, we are done. Therefore, suppose that $IB[T]$ is a proper ideal of $B[T]$ of height n and note that it is comaximal with $JB[T]$ and JB is contained in the Jacobson radical of B.

Let us elaborate how ω_I is obtained from (β, χ). Since P has trivial determinant, P/IP is a free $A[T]/I$-module. We choose an isomorphism $\bar{\lambda} : (A[T]/I)^n \simeq P/IP$ such that $\bigwedge^n \bar{\lambda} = \chi \otimes A[T]/I$. ω_I is the surjection $(\beta \otimes A[T]/I)\bar{\lambda}$ from $(A[T]/I)^n$ to I/I^2, say, given by $I = (f_1, \ldots, f_n) + I^2$.

Since $e(P, \chi) = 0$, we have $(I, \omega_I) = 0$ in $E(A[T])$ and hence by Theorem 3.4, $I = (g_1, \ldots, g_n)$ such that $g_i \equiv f_i$ modulo I^2. So we have $IB[T] = (g_1, \ldots, g_n)$ and $IB[T] + JB[T] = B[T]$. Therefore, $(g_1, \ldots, g_n)$ is a unimodular row over $(B/JB)[T]$ and since $\dim(B/JB) \leqslant n-2$, it is elementarily completable. Using this and the fact that elementary matrices can be lifted via surjection of rings, it is easy to see that we can alter the above set of generators of $IB[T]$ by an elementary matrix $\sigma \in E_n(B[T])$ and assume that

(1) $\mathrm{ht}(g_1, \ldots, g_{n-1}) = n - 1$,
(2) $(g_1, \ldots, g_{n-1}) + JB[T] = B[T]$, and hence,
(3) $\dim B[T]/(g_1, \ldots, g_{n-1}) \leqslant 1$.

We set $C = B[T]$, $R = C[Y]$, $K = (g_1, \ldots, g_{n-1}, Y + g_n)$. Let us denote P_{1+J} by P'. Note that

$$C[Y]/K \simeq B[T]/(g_1, \ldots, g_{n-1}),$$

and so we have $\dim C[Y]/K \leqslant 1$. Therefore, it follows that the projective $C[Y]/K$-module $P'[Y]/KP'[Y]$ is a free module of rank n. We choose an isomorphism

$$\tau(Y) : \bigl(C[Y]/K\bigr)^n \xrightarrow{\sim} P'[Y]/KP'[Y]$$

such that $\bigwedge^n \tau(Y) = \chi \otimes C[Y]/K$. Since $\bigwedge^n \bar{\lambda} = \chi \otimes B[T]/IB[T]$, it follows that $\tau(0)$ and $\bar{\lambda}$ differ by an element of $SL_n(B[T]/IB[T])$. Since $IB[T] + JB[T] = B[T]$ and JB is contained in the Jacobson radical of B, by Lemma 2.1, we have $\dim(B[T]/IB[T]) = 0$. Therefore, $SL_n(B[T]/IB[T]) = E_n(B[T]/IB[T])$. Since elementary transformations can be lifted via surjection of rings, we may alter $\tau(Y)$ by an element of $SL_n(C[Y]/K)$ and assume that $\tau(0) = \bar{\lambda}$. Let $\gamma(Y) : (C[Y]/K)^n \twoheadrightarrow K/K^2$ denote the surjection induced by the set of generators $(g_1, \ldots, g_{n-1}, Y + g_n)$ of K.

Thus, we obtain a surjection

$$\delta(Y) = \gamma(Y)\tau(Y)^{-1} : P'[Y]/KP'[Y] \twoheadrightarrow K/K^2.$$

Since $\tau(0) = \bar{\lambda}$, $\beta \otimes B[T]/IB[T] = \omega_I \bar{\lambda}^{-1}$ and $\gamma(0) = \omega_I$, we have $\delta(0) = \beta \otimes B[T]/IB[T]$.

Therefore, applying Mandal's theorem [13, Theorem 2.1], we obtain a surjection $\eta(Y): P'[Y] \twoheadrightarrow K$. Specializing at $Y = 1 - g_n$, we obtain a surjection from P' to $B[T]$.

Step 3. So far we have proved that P/TP has a unimodular element (Step 1) and P_{1+J} has a unimodular element (Step 2), where J is the Quillen ideal of P in A. In this step we combine these two facts and appeal to a patching argument of Plumstead to conclude that P has a unimodular element.

Now P_{1+J} has a unimodular element. Let us call it p_1. We have already seen that P/TP has a unimodular element, say p. We claim that there is an elementary automorphism σ of P_{1+J} such that $\bar{\sigma}\bar{p}_1 = \bar{p}$, where "bar" denotes reduction modulo T. To see this, let us consider the ring $D = B/J(B)$ where $J(B)$ denotes the Jacobson radical of B. Since $\dim D \leqslant n-2$ it follows that there is an elementary automorphism τ of $P_{1+J} \otimes D$ such that $\tau\bar{p}_1 = p$ over D. Since elementary automorphisms can be lifted via a surjection of rings, we have, by repeated use of this argument, a $\sigma \in E(P_{1+J})$ such that $\bar{\sigma}\bar{p}_1 = \bar{p}$. Let q denote the unimodular element σp_1 of P_{1+J}.

Since P_{1+J} has a unimodular element, we can find $s \in J$ such that P_{1+sA} has a unimodular element. We still call it q. Since P_s is extended from A_s, it has a unimodular element, namely p. Since p and q are equal modulo T, i.e., over $A_{s(1+sA)}$, it follows using a patching argument of Plumstead [17] that P has a unimodular element. □

We can now derive the following version of a theorem of Bhatwadekar–Raja Sridharan [6, Theorem 3.4].

Theorem 4.2. *Let A be a Noetherian ring containing $\mathbb{Q}$ of dimension $n \geqslant 3$. Let P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that P_f has a unimodular element for some monic polynomial $f \in A[T]$. Then P has a unimodular element.*

Proof. Since P_f has a unimodular element and f is monic, by [6, Lemma 3.1] it follows that there is an ideal I of $A[T]$ of height at least n and a surjection $\alpha: P \twoheadrightarrow I$ such that I contains a monic polynomial. If $\operatorname{ht} I > n$, it follows that $I = A[T]$ and there is nothing to prove. So we assume $\operatorname{ht} I = n$. Fix an isomorphism $\chi: A[T] \xrightarrow{\sim} \bigwedge^n(P)$. Now (α, χ) induces a local orientation ω_I of I and hence $e(P,\chi) = (I, \omega_I)$ in $E(A[T])$. Since I contains a monic polynomial, it follows from a theorem of Mandal (Theorem 2.8) that ω_I is a global orientation, i.e., $(I, \omega_I) = 0$. Consequently, $e(P,\chi) = 0$ in $E(A[T])$. By the above theorem, P has a unimodular element. This proves the theorem. □

We end this section with another nice application of Proposition 3.7, giving an alternative proof of the main theorem of [10].

Theorem 4.3. *Let A be a commutative Noetherian ring containing the field of rationals with $\dim A = n$ (n even) and let P be a projective $A[T]$-module of rank n such that its determinant is free. Suppose there is a surjection $\alpha: P \twoheadrightarrow I$ where I is an ideal of $A[T]$ of* *height n which is generated by n elements. Assume further that P/TP has a unimodular element. Then P has a unimodular element.*

Proof. Fix a trivialization $\chi : A[T] \simeq \bigwedge^n P$. Then (α, χ) induces $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$, where ω_I is a local orientation of I (induced by α and χ). Now I is generated by n elements, say, $f_1 \ldots, f_n$. Therefore, applying [9, Proposition 6.7] we see that there exists a stably free $A[T]$-module Q' of rank n, a generator χ_1 of $\bigwedge^n(Q')$ such that $e(Q', \chi_1) = (I, \omega_I)$ in $E(A[T])$. Since Q' is stably free of rank n and A contains $\mathbb{Q}$, by a result of Ravi Rao [19], Q' is extended. Therefore, $Q' = Q[T]$ for some stably free A-module Q. So we have $e(Q[T], \chi_1) = (I, \omega_I)$ in $E(A[T])$.

Therefore, in order to prove that P has a unimodular element it is enough to prove that $Q[T]$ has a unimodular element. In what follows we prove that the A-module Q has a unimodular element.

Note that, in view of Proposition 3.7, we may assume that $I(0)$ is an ideal of height n or $I(0) = A$. Since $Q[T]$ maps onto I, it follows that Q maps onto $I(0)$. Therefore, if $I(0) = A$, then Q has a unimodular element and we are done in this case. So assume that $\operatorname{ht} I(0) = n$.

We have a surjection $\alpha \otimes A[T]/(T) : P/TP \twoheadrightarrow I(0)$. Then $(\alpha \otimes A[T]/(T), \chi \otimes A[T]/(T))$ induces the Euler class of P/TP as

$$e\big(P/TP, \chi \otimes A[T]/(T)\big) = \big(I(0), \omega_{I(0)}\big),$$

where $\omega_{I(0)}$ is also the local orientation induced by ω_I.

On the other hand we have, $e(Q[T], \chi_1) = (I, \omega_I)$ in $E(A[T])$. Restricting at $T = 0$ we obtain

$$e\big(Q, \chi_1 \otimes A[T]/(T)\big) = \big(I(0), \omega_{I(0)}\big) = e\big(P/TP, \chi \otimes A[T]/(T)\big).$$

But P/TP has a unimodular element and it implies that $(I(0), \omega_{I(0)}) = 0$. Consequently, $e(Q, \chi_1 \otimes A[T]/(T)) = 0$ and therefore Q has a unimodular element. This proves the theorem. $\square$

5. Polynomial extension of an affine algebra over an algebraically closed field and a theorem of Bhatwadekar–Raja Sridharan

Let A be an affine algebra of dimension n over an algebraically closed field of characteristic zero. It is known that in this case, the canonical map from $E(A)$ to $E_0(A)$ is an isomorphism of groups (can be easily deduced from [4, Lemma 3.4]). In this section we investigate $E(A[T])$ and $E_0(A[T])$ for a ring A as above and prove that $E(A[T])$ and $E_0(A[T])$ are canonically isomorphic. To prove this we first show that if B is an affine algebra of dimension n over a C_1-field of characteristic zero then $E(B) \xrightarrow{\sim} E_0(B)$. Then we use the injectivity of the canonical map from $E(A[T])$ to $E(A(T))$, proved in [9, Proposition 5.8].

Proposition 5.1. *Let R be an affine algebra of dimension $n \geqslant 3$ over a C_1 field k of characteristic zero. Let $J \subset R$ be an ideal of height n such that J is generated by n elements. Then any set of n generators of J/J^2 can be lifted to a set of n generators of J.*

Proof. Suppose $J = (a_1, \dots, a_n)$. Let us take an arbitrary set of generators of J/J^2:

$$J = (b_1, \dots, b_n) + J^2.$$

We want to show that there exists $c_1, \dots, c_n \in J$ such that $J = (c_1, \dots, c_n)$ and $b_i \equiv c_i \bmod J^2$.

Clearly we may assume that $\mathrm{ht}(a_3, \dots, a_n) = n - 2$. Since any two surjections from $(R/J)^n$ to J/J^2 differ by an element of $GL_n(R/J)$, there exists a matrix $\delta \in GL_n(R/J)$ such that $(\overline{a_1}, \dots, \overline{a_n})\delta = (\overline{b_1}, \dots, \overline{b_n})$. Let $u \in R$ be such that $\bar{u} = \det(\delta)^{-1}$. Then $(u, a_1, \dots, a_n) \in Um_{n+1}(R)$. Let $B = R/(a_3, \dots, a_n)$. Then B is an affine algebra over k of dimension $\leqslant 2$. Therefore, all stably free modules over B are free by [23, Theorem 2.4]. So the unimodular row $(u, a_1, a_2) \in Um_3(B)$ is completable. Applying [21, Lemma 2.4] we have a set of generators of J, say $J = (d_1, \dots, d_n)$, and a matrix $\delta' \in SL_n(R/J)$ such that $(\overline{d_1}, \dots, \overline{d_n})\delta' = (\overline{b_1}, \dots, \overline{b_n})$. Since $\dim(R/J) = 0$, we have $SL_n(R/J) = E_n(R/J)$ and therefore we can lift δ' to a matrix $\Delta \in E_n(R)$. Suppose $(d_1, \dots, d_n)\Delta = (c_1, \dots, c_n)$. Then $J = (c_1, \dots, c_n)$ is the desired set of generators. □

Corollary 5.2. *Let R be an affine algebra of dimension $n \geqslant 3$ over a C_1 field k of characteristic zero. Then $E(A) \simeq E_0(A)$.*

Proof. We know that the canonical map from $E(A)$ to $E_0(A)$ is surjective. To prove injectivity, let (J, ω_J) be in the kernel. Then, by [4, Lemma 3.3] we have

$$(J, \omega_J) + \sum_{i=1}^{r} (J_i, \omega_i) = \sum_{k=r+1}^{s} (J_k, \omega_k),$$

where J_i, J_k are ideals of height n such that each of them is generated by n elements. By the above proposition, each of the local orientations $\omega_1, \dots, \omega_s$ is a global one. Consequently $(J, \omega_J) = 0$ in $E(A)$. This proves the corollary. □

Proposition 5.3. *Let A be an affine algebra of dimension $n \geqslant 3$ over an algebraically closed field k of characteristic zero. Let $I \subset A[T]$ be an ideal of height n. Assume that I is generated by n elements. Then any set of n generators of I/I^2 can be lifted to a set of n generators of I. In other words, $(I, \omega_I) = 0$ in $E(A[T])$ for any local orientation ω_I of I.*

Proof. We will use the injectivity of the canonical map from $E(A[T])$ to $E(A(T))$, where $A(T)$ is the ring obtained from $A[T]$ by inverting all monic polynomials. This has been proved in [9, Proposition 5.8].

Note that $A(T)$ is an affine algebra over a C_1 field. Therefore, if we consider the image $(IA(T), \omega_I \otimes A(T))$ of (I, ω_I) in $E(A(T))$, it follows by Proposition 5.1 that

$(IA(T), \omega_I \otimes A(T)) = 0$ as $IA(T)$ is generated by n elements. Therefore, $(I, \omega_I) = 0$ in $E(A[T])$. $\square$

The following corollary is now immediate.

Corollary 5.4. *Let A be an affine algebra of dimension $n \geqslant 3$ over an algebraically closed field k of characteristic zero. Then $E(A[T]) \simeq E_0(A[T])$.*

A theorem of Bhatwadekar–Raja Sridharan

Let A be any commutative Noetherian ring of dimension n containing $\mathbb{Q}$. Now we may ask the following questions.

Question 1. Let (J, ω_J) be any element of $E(A)$. Does there exist a projective A-module of rank n with trivial determinant together with an isomorphism $\chi : A \xrightarrow{\sim} \bigwedge^n(P)$ such that $e(P, \chi) = (J, \omega_J)$?

Question 2. Let (I, ω_I) be any element of $E(A[T])$. Does there exist a projective $A[T]$-module of rank n with trivial determinant together with an isomorphism $\chi : A[T] \xrightarrow{\sim} \bigwedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$?

These questions do not have affirmative answers in general. One can take A to be the coordinate ring of the real three sphere and J be any real maximal ideal. Then it is known that J is not surjective image of a projective A-module of rank n.

If A is an affine algebra over an algebraically closed field, it follows from a theorem of Murthy [16, Theorem 3.3] that Question 1 has an affirmative answer.

In this note we discuss a theorem of Bhatwadekar–Raja Sridharan [7, Theorem 2.7] which essentially says that Question 2 has an affirmative answer when A is an affine algebra over an algebraically closed field of characteristic zero. We may note that when Bhatwadekar–Raja Sridharan proved this theorem, the Euler class group of a polynomial algebra was not defined. Below we give a proof of their theorem using Euler class computations.

The following lemma is an improvement of [6, Lemma 4.1] and is crucial for later discussions.

Lemma 5.5. *Let B be a ring of dimension $n \geqslant 3$ such that height of the Jacobson radical $J(B)$ is at least one. Let $I \subset B[T]$ be an ideal of height n such that*: (1) *$I + J(B)[T] = B[T]$ (so I is zero-dimensional),* (2) *$I = (a_1, \ldots, a_{n-1}, f(T))$ where $a_1, \ldots, a_{n-1} \in B$ and $\mathrm{ht}(a_1, \ldots, a_{n-2}) = n - 2$. Then any set of n generators of I/I^2 can be lifted to a set of n generators of I (i.e., any local orientation of I is a global one).*

Proof. Let ω_I be a local orientation of I, corresponding to a set of generators of I/I^2. We show that $(I, \omega_I) = 0$ in $E(B[T])$. We do this using [6, Lemma 4.1] and the local–global principle for Euler class groups [9, Theorem 5.4].

Let m be any maximal ideal of B of height n. Consider $B_m[T]$. Now by [6, Lemma 4.1], $(I, \omega_I) = 0$ in $E(B_m[T])$. Since it happens for every maximal ideal m of B of height n, we have, by the local–global principle for Euler class groups [9, Theorem 5.4] that (I, ω_I) comes from $E(B)$. But since $\operatorname{ht} J(B) \geqslant 1$, it follows from [15, Corollary 3] that $E(B) = 0$. Therefore, $(I, \omega_I) = 0$ in $E(B[T])$. $\square$

We now quote the following two propositions from [7].

Proposition 5.6. *Let A be a Noetherian ring with $\dim R = d \geqslant 1$. Let $I \subset R[T]$ be an ideal with $\operatorname{ht}(I) \geqslant 2$. Suppose that I/I^2 is generated by n elements where $n \geqslant d + 1$. Then I is generated by n elements.*

Proposition 5.7. *Let A be an affine domain of dimension n over an algebraically closed field of characteristic zero. Let $I \subset A[T]$ be an ideal and let $b \in I \cap A$ be a nonzero element such that A_b is regular. Suppose there exists a projective $A_{1+bA}[T]$-module P' of rank n with trivial determinant and a surjection $\beta : P' \twoheadrightarrow I_{1+bA}$. Assume further that P'_b is free. Then there exists a projective $A[T]$-module P of rank n with trivial determinant and a surjection from P to I.*

We are now ready to prove the following theorem of Bhatwadekar–Raja Sridharan [7, Theorem 2.7].

Theorem 5.8. *Let A be an affine domain of dimension $n \geqslant 3$ over an algebraically closed field k of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that I/I^2 is generated by n elements. Then there exists a projective $A[T]$-module P of rank n with trivial determinant and a surjection $\Phi : P \twoheadrightarrow I$.*

Proof. We will replace Steps 1 and 2 of the proof of [7, Theorem 2.7] by some Euler class computations. We briefly outline the part preceding these steps from their proof.

Let ω_I be a local orientation of I given by $I = (g_1, \ldots, g_n) + I^2$. Let $J = I \cap A$. Let b be a nonzero element of J^2 which also belongs to the singular locus of A. Let $R = A/(b)$. Then $\dim R \leqslant n - 1$. Therefore applying Proposition 5.6, we have $I = (f_1, \ldots, f_n, b)$ where $f_i \equiv g_i$ modulo I^2. Applying Swan's Bertini theorem [4, Theorem 2.11], and adding suitable multiples of b to $f_1, \ldots, f_n$ they obtain an element $(I', \omega_{I'}) \in E(A[T])$ such that:

(1) $(I, \omega_I) + (I', \omega_{I'}) = 0$ in $E(A[T])$.
(2) $I' + (b) = A[T]$ and hence $I' + I = A[T]$.
(3) I' is a prime ideal of height n.

Let $B = A_{1+bA}$. If $I'B[T] = B[T]$, $IB[T]$ is image of a free module and the theorem is proved in this case using Proposition 5.7. Therefore assume that $I'B[T]$ is proper. Since it is prime and is comaximal with the Jacobson radical of B, it is a maximal ideal of height n. To be consistent with their notation, let $I'B[T] = M$.

Using techniques from [1] it follows that there is an ideal $L_1 \subset B[T]$ of height n such that

(1) $M \cap L_1 = (b_1, \dots, b_{n-1}, f(T))$ where $b_i \in B$ and $f(T) \in B[T]$.
(2) $L_1 + M = B[T]$ and $L_1 + bB[T] = B[T]$.

Using a theorem of Murthy they show that there is a projective $B[T]$-module P' with trivial determinant and a surjection $\alpha : P' \twoheadrightarrow L_1$. Some additional arguments imply that P'_b is free.

We now use Euler class computations to show that this P' maps onto $IB[T]$.

We fix an isomorphism $\chi : B[T] \simeq \bigwedge^n(P')$. Then (α, χ) induces a local orientation ω_{L_1} of L_1 and we have $e(P', \chi) = (L_1, \omega_{L_1})$ in $E(B[T])$. We also have $(I, \omega_I) + (M, \omega_M) = 0$ in $E(B[T])$. Since M and L_1 are comaximal, ω_M and ω_{L_1} together induce a local orientation of $M \cap L_1$, say $\omega_{M \cap L_1}$. Note that by Lemma 5.5, the ideal $M \cap L_1 = (b_1, \dots, b_{n-1}, f(T))$ has the property that any local orientation of $M \cap L_1$ is a global one. Therefore, $\omega_{M \cap L_1}$ is a global orientation and hence, $(M, \omega_M) + (L_1, \omega_{L_1}) = 0$ in $E(B[T])$. Consequently, $e(P', \chi) = (I, \omega_I)$ in $E(B[T])$. Now using [9, Corollary 4.10] it follows that there is a surjection $\beta : P' \twoheadrightarrow IB[T]$. Applying Proposition 5.7, the theorem follows. □

Remark 5.9. As remarked earlier, we have replaced Steps 1 and 2 of the proof given in [7] by Euler class computations. One interesting point of our proof is that in this part we have not used the fact that A is an affine domain over an algebraically closed field k of characteristic zero.

In terms of Euler classes the above theorem can be rephrased as:

Theorem 5.10. *Let A, I be as above. Let ω_I be a local orientation of I (so $(I, \omega_I) \in E(A[T])$). Then there exists a projective $A[T]$-module P of rank n with trivial determinant and an isomorphism $\chi : A[T] \simeq \bigwedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$.*

Proof. It follows from Theorem 5.8 that there exists a projective $A[T]$-module of rank n with trivial determinant and a surjection $\alpha : P \twoheadrightarrow I$. Fix an isomorphism $\chi : A[T] \xrightarrow{\sim} \bigwedge^n(P)$. Now (α, χ) induces a local orientation, say $\widetilde{\omega_I}$ of I. Therefore, $e(P, \chi) = (I, \widetilde{\omega_I})$ in $E(A[T])$. Since k is an algebraically closed field of characteristic zero, it follows from Corollary 5.4 that $(I, \omega_I) = (I, \widetilde{\omega_I})$ in $E(A[T])$. Therefore, $e(P, \chi) = (I, \omega_I)$. □

6. The weak Euler class of a projective $A[T]$-module

In [9] we defined the nth weak Euler class group, $E_0(A[T])$, of $A[T]$ and proved results analogous to those on $E_0(A)$ [5]. We further investigated this group in [10]. In [5], there is a notion of the weak Euler class of a projective A-module of top rank. In [9] we did not define the weak Euler class of a projective $A[T]$-module of rank $= \dim A$. The aim of this small section is to give such a definition.

Let A be a Noetherian ring containing $\mathbb{Q}$ of dimension $n \geqslant 2$. Let P be a projective $A[T]$-module of rank n with trivial determinant. Let $\alpha : P \twoheadrightarrow I$ be a generic surjection. We define the weak Euler class of P, denoted $e(P)$, as $e(P) = (I)$ in $E_0(A[T])$.

Proposition 6.1. *The weak Euler class of P, as defined above, is well defined.*

Proof. We prove this using the definition of the Euler class of P. Let us fix an isomorphism $\chi : A[T] \xrightarrow{\sim} \bigwedge^n(P)$. Now (χ, α) induces a local orientation ω_I of I and by the definition of the Euler class of a projective module, $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$.

Next suppose $\beta : P \twoheadrightarrow J$ be another generic surjection. We want to prove that $(I) = (J)$ in $E_0(A[T])$. Now (χ, β) induces a local orientation ω_J of J. Since the Euler class of P is well defined, we have $e(P, \chi) = (I, \omega_I) = (J, \omega_J)$ in $E(A[T])$. Recall that there is a canonical surjective group homomorphism from $E(A[T])$ to $E_0(A[T])$ which sends an element (K, ω_K) of $E(A[T])$ to (K) in $E_0(A[T])$. Therefore, $(I) = (J)$ in $E_0(A[T])$. □

We can now rephrase [9, Proposition 6.6], as:

Proposition 6.2. *Let A be a Noetherian ring of even dimension n. Let P be a projective $A[T]$-module of rank n with trivial determinant. Then $e(P) = 0$ in $E_0(A[T])$ if and only if $[P] = [Q \oplus A[T]]$ in $K_0(A[T])$ for some projective $A[T]$-module Q of rank $n - 1$.*

We can also prove the following analogue of [5, 6.4]. Method of proof of this proposition is similar to [5, 6.4] and hence omitted.

Proposition 6.3. *Let A be a Noetherian ring of even dimension n. Let P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that $e(P) = (I)$ in $E_0(A[T])$, where I is an ideal of $A[T]$ of height n. Then, there exists a projective $A[T]$-module Q of rank n, such that $[P] = [Q]$ in $K_0(A[T])$ and I is a surjective image of Q.*

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