ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Topology and its Applications 155 (2008) 1013–1021

Topology and its Applications

www.elsevier.com/locate/topol

An alternative definition of coarse structures

J. Dydak *, C.S. Hoffland

University of Tennessee, Knoxville, TN 37996, USA

Received 2 September 2006; received in revised form 7 January 2008; accepted 7 January 2008

Abstract

Roe [J. Roe, Lectures on Coarse Geometry, University Lecture Series, vol. 31, Amer. Math. Soc., Providence, RI, 2003] introduced coarse structures for arbitrary sets X by considering subsets of $X \times X$. In this paper we introduce large scale structures on X via the notion of uniformly bounded families and we show their equivalence to coarse structures on X. That way all basic concepts of large scale geometry (asymptotic dimension, slowly oscillating functions, Higson compactification) have natural definitions and basic results from metric geometry carry over to coarse geometry.

MSC: primary: 54F45, 54C55; secondary: 54E35, 18B30, 54D35, 54D40, 20H15

Keywords: Asymptotic dimension; Coarse structures; Higson compactification

1. Introduction

Recall that the *star* $\mathrm{St}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. Given two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathrm{St}(\mathcal{B},\mathcal{U})$ is the family $\{\mathrm{St}(B,\mathcal{U})\}$, $B \in \mathcal{B}$, of all stars of elements of $\mathcal{B}$ with respect to $\mathcal{U}$.

Definition 1.1. A *large scale structure* $\mathcal{LSS}_X$ on a set X is a non-empty set of families $\mathcal{B}$ of subsets of X (called *uniformly bounded* or *uniformly* $\mathcal{LSS}_X$*-bounded* once $\mathcal{LSS}_X$ is fixed) satisfying the following conditions:

(1) $\mathcal{B}_1 \in \mathcal{LSS}_X$ implies $\mathcal{B}_2 \in \mathcal{LSS}_X$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.
(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ implies $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}_X$.

We think of (2) above as a generalization of the triangle inequality.

The *trivial extension* $e(\mathcal{B})$ of a family $\mathcal{B}$ is defined as $\mathcal{B} \cup \{\{x\}\}_{x\in X}$. Recall that $\mathcal{B}$ is a *refinement* of $\mathcal{B}'$ if every element of $\mathcal{B}$ is contained in some element of $\mathcal{B}'$. Thus, the meaning of (1) of Definition 1.1 is that if $\mathcal{B} \in \mathcal{LSS}_X$, then all refinements of $e(\mathcal{B})$ also belong to $\mathcal{LSS}_X$.

* Corresponding author.
E-mail addresses: dydak@math.utk.edu (J. Dydak), hoffland@math.utk.edu (C.S. Hoffland).

0166-8641/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2008.01.002

Proposition 1.2. *Any large scale structure $\mathcal{LSS}_X$ on X has the following properties*:

(1) *$\mathcal{B} \in \mathcal{LSS}_X$ if each element of $\mathcal{B}$ consists of at most one point.*
(2) *$\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ implies $\mathcal{B}_1 \cup \mathcal{B}_2 \in \mathcal{LSS}_X$.*

Proof. (1) Pick any $\mathcal{B}_1 \in \mathcal{LSS}_X$ and notice $\mathcal{B}_2 := \mathcal{B}$ satisfies (1) of Definition 1.1.

(2) Let $\mathcal{B}'_i := e(\mathcal{B}_i)$ for $i = 1, 2$. Observe $\mathcal{B}'_i \in \mathcal{LSS}_X$. Therefore $\mathcal{B}_3 = \mathrm{St}(\mathcal{B}'_1, \mathcal{B}'_2) \in \mathcal{LSS}_X$ and notice any element of $\mathcal{B}_1 \cup \mathcal{B}_2$ is contained in an element of $\mathcal{B}_3$. □

We have two basic examples of large scale structures induced by other structures on X. The first one deals with metric spaces, so let us point out there is no need to restrict ourselves to metrics assuming only finite values. To emphasize that, let us call $d : X \times X \to R_+ \cup \infty$ an *∞-metric* if it satisfies all the regular axioms of a metric (with the understanding that $x + \infty = \infty$). Notice that ∞-metrics have the advantage over regular metrics in the fact that one can easily define the *disjoint union* $\bigoplus_{s\in S}(X_s, d_s)$ of any family of ∞-metric spaces (X_s, d_s). Namely, put $d(x, y) = \infty$ if x and y belong to different spaces X_s and X_t (those are assumed to be disjoint). Conversely, any ∞-metric space (X, d) is the disjoint union of its finite components $(C, d|C)$ (two elements belong to the same finite component if $d(x, y) < \infty$).

Proposition 1.3. *Any ∞-metric space (X, d) has a natural large scale structure $\mathcal{LSS}(X, d)$ defined as follows*:

$\mathcal{B} \in \mathcal{LSS}(X, d)$ if and only if there is $M > 0$ such that all elements of $\mathcal{B}$ are of diameter at most M.

Proof. If $\mathcal{B}_1 \in \mathcal{LSS}(X, d)$ and for each $B_\beta \in \mathcal{B}_2$ consisting of more than one point there is a $B_\alpha \in \mathcal{B}_1$ containing B_β, then $\mathrm{diam}(B_\beta) \leqslant \mathrm{diam}(B_\alpha) \leqslant M$ for each $B_\beta \in \mathcal{B}_2$, whence $\mathcal{B}_2 \in \mathcal{LSS}(X, d)$. If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}(X, d)$ then there are $M_1, M_2 > 0$ such that $\mathrm{diam}(B_\alpha) \leqslant M_1$ and $\mathrm{diam}(B_\beta) \leqslant M_2$ for all $B_\alpha \in \mathcal{B}_1$, $B_\beta \in \mathcal{B}_2$. Thus for any $B_\alpha \in \mathcal{B}_1$, $\mathrm{diam}(\mathrm{St}(B_\alpha, \mathcal{B}_2)) \leqslant 2M_2 + M_1$, whence $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}(X, d)$. It follows that $\mathcal{LSS}(X, d)$ is a large scale structure. □

One can generalize Proposition 1.3 as follows: Given certain families $\mathcal{F}$ of positive functions from an ∞-metric space X to reals one can define $\mathcal{LSS}(X, \mathcal{F})$ by declaring $\mathcal{B} \in \mathcal{LSS}(X, \mathcal{F})$ if and only if there is $f \in \mathcal{F}$ such that $\mathcal{B}$ refines the family of balls $\{B(x, f(x))\}_{x\in X}$.

One family of interest is all f such that $\lim_{x\to\infty} \frac{f(x)}{d(x,x_0)} = 0$, where x_0 is a fixed point in a metric space X (if X is an ∞-metric space, one needs to look at each finite component separately). That leads to the *sublinear large scale structure* on X introduced by Dranishnikov and Smith [5] (see also [4]).

Proposition 1.4. *Any group $(X, \cdot)$ has a natural large scale structure $\mathcal{LSS}_l(X, \cdot)$ defined as follows*:

$\mathcal{B} \in \mathcal{LSS}_l(X, \cdot)$ if and only if there is a finite subset F of X such that $\mathcal{B}$ refines the shifts $\{x \cdot F\}_{x\in X}$ of F.

Proof. Notice that if $\mathcal{B} \neq \emptyset$ refines $\{x \cdot F\}_{x\in X}$ for some finite subset F of X, then $e(\mathcal{B})$ also refines $\{x \cdot F\}_{x\in X}$.

Suppose $\mathcal{B}_i$ refines $\{x \cdot F_i\}_{x\in X}$ for $i = 1, 2$, where F_1 and F_2 are finite subsets of X. We may enlarge F_2 and assume it is symmetric ($y \in F_2$ implies $y^{-1} \in F_2$).

Let F be the set of all products $x \cdot y \cdot z$, where $x \in F_1$ and $y, z \in F_2$. Given $B \in \mathcal{B}_1$ pick $a \in X$ such that $B \subset a \cdot F_1$. If $B' \in \mathcal{B}_2$ and $u \in B \cap B'$, choose $y \in X$ so that $B' \subset y \cdot F_2$. Thus $u = a \cdot f_1 = y \cdot f_2$, where $f_1 \in F_1$ and $f_2 \in F_2$. Therefore $y = a \cdot f_1 \cdot f_2^{-1}$ and $B' \subset a \cdot F$ proving that $\mathrm{St}(B, \mathcal{B}_2) \subset a \cdot F$. □

Remark 1.5. Notice that any group $(X, \cdot)$ has another natural large scale structure $\mathcal{LSS}_r(X, \cdot)$ defined as follows:

$\mathcal{B} \in \mathcal{LSS}_r(X, \cdot)$ if and only if there is a finite subset F of X such that $\mathcal{B}$ refines the shifts $\{F \cdot x\}_{x\in X}$ of F.

Clearly, the two structures coincide if X is Abelian. However, they may differ even for finitely presented virtually Abelian groups.

Consider $X = \langle a, t \mid t^2 = 1 \text{ and } tat = a^2 \rangle$. Notice every element of X has unique representation as $t^u a^v$, where $u = 0, 1$. If $\mathcal{LSS}_l(X, \cdot) = \mathcal{LSS}_r(X, \cdot)$, then for $E = \{1, t\}$ there is a finite subset F of X such that for each $x \in X$ there is $y \in X$ satisfying $x \cdot E \subset F \cdot y$. Pick $k \geqslant 1$ such that all elements of F can be represented as $t^u a^v$ so that $u = 0, 1$ and $|v| \leqslant k$. Put $x = ta^{6k}$ and choose $y \in X$ satisfying $x \cdot E \subset F \cdot y$. There is $c = 0, 1$ and i so that $x = t^c a^i y$ and $|i| \leqslant k$. Also, there is $d = 0, 1$ and j so that $x \cdot t = t^d a^j y$ and $|j| \leqslant k$.

Case 1 *($c = 1$).* Now $y = a^{6k-i}$ and $d = 0$, so $y = a^{-j} t a^{6k} t = a^{12k-j}$. That means $6k - i = 12k - j$ and $6k = j - i$ contradicting $|i|, |j| \leqslant k$.

Case 2 *($c = 0$).* Now $y = a^{-i} t a^{6k} = t a^{6k-2i}$ and $d = 1$, so $y = a^{-j} t a^{6k} t = t a^{12k-2j}$. Thus $12k - 2j = 6k - 2i$ and $6k = 2j - 2i$ contradicting $|i|, |j| \leqslant k$.

To create a large scale structure on a set X all one needs is a family $\mathcal{LSS}'_X$ satisfying conditions resembling finite additivity and (2) of Definition 1.1.

Proposition 1.6. *If $\mathcal{LSS}'_X$ is a set of families in X such that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}'_X$ implies existence of $\mathcal{B}_3 \in \mathcal{LSS}'_X$ such that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)$ refines $\mathcal{B}_3$, then the family $\mathcal{LSS}_X$ of all refinements of trivial extensions of elements of $\mathcal{LSS}'_X$ forms a large scale structure on X.*

Proof. It suffices to show that, given $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}'_X$, $\mathrm{St}(e(\mathcal{B}_1), e(\mathcal{B}_2))$ is a refinement of the trivial extension $e(\mathcal{B}_3)$ for some $\mathcal{B}_3 \in \mathcal{LSS}'_X$. Choose $\mathcal{B}_3 \in \mathcal{LSS}'_X$ so that $\mathrm{St}(\mathcal{B}_1 \cup \mathcal{B}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ refines it.

Given $B \in \mathcal{B}_1$ notice $\mathrm{St}(B, e(\mathcal{B}_2)) = B \cup \mathrm{St}(B, \mathcal{B}_2)$ is contained in $\mathrm{St}(B, \mathcal{B}_1 \cup \mathcal{B}_2)$. Also, $\mathrm{St}(x, e(\mathcal{B}_2))$ is either a point or there is $B \in \mathcal{B}_2$ containing x in which case $\mathrm{St}(x, \mathcal{B}_2)$ is contained in $\mathrm{St}(B, \mathcal{B}_1 \cup \mathcal{B}_2)$. □

Remark 1.7. The family $\mathcal{LSS}_X$ in Proposition 1.6 is said to be *generated* by $\mathcal{LSS}'_X$. A good example is the *discrete large scale structure* on any set X generated by all $\mathcal{B}$ such that $\bigcup \mathcal{B}$ is finite.

In [8, Theorem 2.55, p. 34], Roe shows that a course structure is metrizable if and only if it is countably generated. Our analog is the following theorem. Notice the simplicity of our proof.

Theorem 1.8. *Given a large scale structure $\mathcal{LSS}_X$ on a set X the following conditions are equivalent*:

(a) *There is an ∞-metric d_X on X such that $\mathcal{LSS}_X = \mathcal{LSS}(X, d_X)$.*
(b) *$\mathcal{LSS}_X$ is generated by a countable set.*

Proof. (a) $\Rightarrow$ (b) is obvious as any $\mathcal{LSS}(X, d_X)$ is generated by the family of i-balls, $i \geqslant 1$.

(b) $\Rightarrow$ (a). Pick a sequence $\mathcal{B}_i \in \mathcal{LSS}_X$ generating $\mathcal{LSS}_X$. Without loss of generality we may assume $\mathrm{St}(\mathcal{B}_i, \mathcal{B}_i)$ refines $\mathcal{B}_{i+1}$ for all $i \geqslant 1$. Define the ∞-metric d_X on X by setting $d_X(x, y)$ (if $x \neq y$) equal the smallest i such that there is $B \in \mathcal{B}_i$ containing both x and y. If no such i exists, put $d_X(x, y) = \infty$.

To show the triangle inequality notice that $0 < d_X(x, y) \leqslant d_X(y, z) \leqslant i$ implies $d_X(x, z) \leqslant i + 1$ as both x and z belong to $\mathrm{St}(y, \mathcal{B}_i)$ which is contained in some $B \in \mathcal{B}_{i+1}$.

Clearly $\mathcal{LSS}_X \subset \mathcal{LSS}(X, d_X)$ (each $\mathcal{B}_i$ refines the family of $(i + 1)$-balls in (X, d_X)). Also, any family of r-balls in (X, d_X) refines $\mathcal{B}_i$ for all $i > r$. Thus $\mathcal{LSS}_X = \mathcal{LSS}(X, d_X)$. □

2. Coarse structures and their relation to large scale structures

Recall that a *coarse structure* $\mathcal{C}$ on X is a family of subsets E (called *controlled sets*) of $X \times X$ satisfying the following properties:

(1) The diagonal $\Delta = \{(x, x)\}_{x \in X}$ belongs to $\mathcal{C}$.
(2) $E_1 \in \mathcal{C}$ implies $E_2 \in \mathcal{C}$ for every $E_2 \subset E_1$.
(3) $E \in \mathcal{C}$ implies $E^{-1} \in \mathcal{C}$, where $E^{-1} = \{(y, x)\}_{(x,y) \in E}$.

(4) $E_1, E_2 \in \mathcal{C}$ implies $E_1 \cup E_2 \in \mathcal{C}$.
(5) $E, F \in \mathcal{C}$ implies $E \circ F \in \mathcal{C}$, where $E \circ F$ consists of (x, y) such that there is $z \in X$ so that $(x, z) \in E$ and $(z, y) \in F$.

Definition 2.1. Given a family $\mathcal{B}$ of subsets of X define $\Delta(\mathcal{B})$ as $\bigcup_{B\in\mathcal{B}} B \times B$. Given $E \subset X \times X$ define $\mathcal{B}(E)$ as the family of all $B \subset X$ such that $B \times B \subset E$.

Lemma 2.2. *Suppose $\mathcal{B}_1$, $\mathcal{B}_2$ are collections in X. If $\Delta(\mathcal{B}_i) \subset E_i$ for $i = 1, 2$, then $\Delta(\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)) \subset (E_2 \circ E_1) \circ E_2$.*

Proof. Let $(x, y) \in \Delta(\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2))$. Then for some $B \in \mathcal{B}_1$ there are $B_x, B_y \in \mathcal{B}_2$, containing x and y respectively, such that there are $z_x \in B \cap B_x$ and $z_y \in B \cap B_y$. Then

$$(x, z_x) \in B_x \times B_x \subset \Delta(\mathcal{B}_2) \subset E_2,$$
$$(z_y, y) \in B_y \times B_y \subset \Delta(\mathcal{B}_2) \subset E_1,$$
$$(z_x, z_y) \in B \times B \subset \Delta(\mathcal{B}_1) \subset E_1,$$

so there is a $z_x \in X$ such that $(x, z_x) \in E_2$ and $(z_x, z_y) \in E_1$, whence $(x, z_y) \in E_2 \circ E_1$. But then there is also a $z_y \in X$ such that $(z_y, y) \in E_2$, whence $(x, y) \in (E_2 \circ E_1) \circ E_2$ as required. □

Lemma 2.3. *Suppose $\mathcal{B}_1, \mathcal{B}_2$ are collections in X. If $E_i \subset \Delta(\mathcal{B}_i)$ for $i = 1, 2$, then $E_1 \circ E_2 \subset \Delta(\mathrm{St}(\mathcal{B}_2, \mathcal{B}_1 \cup \mathcal{B}_2))$.*

Proof. Suppose $(x, y) \in E_1 \circ E_2$. There is z such that $(x, z) \in E_1$ and $(z, y) \in E_2$. Therefore one has $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ so that $x, z \in B_1$ and $z, y \in B_2$. Put $B_3 = \mathrm{St}(B_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ and notice $B_1 \cup B_2 \subset B_3$. Thus $x, y \in B_3$. □

Proposition 2.4. *Every large scale structure $\mathcal{LSS}_X$ on X induces a coarse structure $\mathcal{C}$ on X as follows*:

A subset E of $X \times X$ is declared controlled if and only if there is $\mathcal{B} \in \mathcal{LSS}_X$ such that $E \subset \bigcup_{B\in\mathcal{B}} B \times B$.

Proof. By the remarks after Definition 1.1, all refinements of $e(\mathcal{B})$, for $\mathcal{B} \in \mathcal{LSS}_X$, themselves belong to $\mathcal{LSS}_X$, meaning that $\{\{x\}\}_{x\in X}$ is a member of $\mathcal{LSS}_X$. Thus

$$\Delta \subset \bigcup_{B\in\{\{x\}\}} B \times B = \bigcup_{x\in X} \{x\} \times \{x\}$$

so $\Delta \in \mathcal{C}$. Let $E_1 \in \mathcal{C}$, so there is a $\mathcal{B} \in \mathcal{LSS}_X$ such that $E_1 \subset \Delta(\mathcal{B})$. $E_2 \subset E_1$ then $E_2 \subset \Delta(\mathcal{B})$ also, whence $E_2 \in \mathcal{C}$. It is clear that if $E \subset \Delta(\mathcal{B})$ then $E^{-1} \subset \Delta(\mathcal{B})$, so $E^{-1} \in \mathcal{C}$. If $E_1, E_2 \in \mathcal{C}$ then there are families $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ such that $E_1 \subset \Delta(\mathcal{B}_1)$ and $E_2 \subset \Delta(\mathcal{B}_2)$. But

$$\begin{aligned} E_1 \cup E_2 \subset \Delta(\mathcal{B}_1) \cup \Delta(\mathcal{B}_2) &= \left(\bigcup_{B\in\mathcal{B}_1} B \times B\right) \cup \left(\bigcup_{B\in\mathcal{B}_2} B \times B\right) \\ &= \bigcup_{B\in\mathcal{B}_1\cup\mathcal{B}_2} B \times B \\ &= \Delta(\mathcal{B}_1 \cup \mathcal{B}_2) \end{aligned}$$

and since, by Proposition 1.2 $\mathcal{B}_1 \cup \mathcal{B}_2 \in \mathcal{LSS}_X$, it follows that $E_1 \cup E_2 \in \mathcal{C}$. Finally, let $E_1, E_2 \in \mathcal{C}$ and $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ again be as above. Then $E_1 \circ E_2 \subset \Delta(\mathrm{St}(\mathcal{B}_2, \mathcal{B}_1 \cup \mathcal{B}_2))$, which, since both $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_2$ are members of $\mathcal{LSS}_X$, is itself a member of $\mathcal{LSS}_X$, completing the proof. □

Proposition 2.5. *Every coarse structure $\mathcal{C}$ on X induces a large scale structure $\mathcal{LSS}_X$ on X as follows*:

$\mathcal{B}$ is declared uniformly bounded if and only if there is a controlled set E such that $\bigcup_{B\in\mathcal{B}} B \times B \subset E$.

Proof. Let $B_1 \in \mathcal{LSS}_X$; then there is a controlled set $E \in \mathcal{C}$ such that $\Delta(\mathcal{B}_1) \subset E$. Suppose $\mathcal{B}_2$ is a family of subsets of X such that for each $B_\beta \in \mathcal{B}_2$ consisting of more than one point there is a $B_\alpha \in \mathcal{B}_1$ containing B_β. Then $\Delta(\mathcal{B}_2) \subset \Delta(\mathcal{B}_1) \cup \Delta \subset E \cup \Delta \in \mathcal{C}$, whence $\mathcal{B}_2 \in \mathcal{LSS}_X$. Now suppose that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$, so there are $E_1, E_2 \in \mathcal{C}$ such that $\Delta(\mathcal{B}_1) \subset E_1$ and $\Delta(\mathcal{B}_2) \subset E_2$. But $(E_2 \circ E_1) \circ E_1$ is controlled, and $\Delta(\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)) \subset (E_2 \circ E_1) \circ E_2$, whence $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}_X$. It follows that $\mathcal{LSS}_X$ is indeed a large scale structure. □

3. Higson functions and Higson compactification

In this section we discuss relation of large scale structures on a topological space X to compactifications of X. Our approach is quite different from that of [8, pp. 26–31] for coarse structures and seems simpler.

Given a large scale structure $\mathcal{LSS}_X$ on X, a subset B of X is *bounded* if $\{B\} \in \mathcal{LSS}_X$.

A bounded continuous function $f : X \to R$ is called *Higson* if for every $\mathcal{B} \in \mathcal{LSS}_X$ and for every $\epsilon > 0$ there is a bounded subset U of X such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in B \setminus U$, $B \in \mathcal{B}$.

If X is a topological space, then using Higson maps one can construct a compact space $h(X, \mathcal{LSS}_X)$ and a natural map $i : X \to h(X, \mathcal{LSS}_X)$. Namely, first we construct $i : X \to \prod_f [\inf(f), \sup(f)]$ by sending x to $\{f(x)\}_f$, and then we declare $h(X, \mathcal{LSS}_X)$ to be the closure of $i(X)$ in $\prod_f [\inf(f), \sup(f)]$.

It is of interest to investigate cases where $h(X, \mathcal{LSS}_X)$ is a compactification of X (called *Higson compactification* of $(X, \mathcal{LSS}_X)$), i.e., $i : X \to i(X)$ is a homeomorphism. Here is the simplest sufficient condition for $h(X, \mathcal{LSS}_X)$ to be a compactification.

Proposition 3.1. *Suppose X is a Tychonoff space. If $\mathcal{LSS}_X$ is a large scale structure such that the family of all open and bounded subsets of X forms a basis of X, then $h(X, \mathcal{LSS}_X)$ is a compactification of X.*

Proof. It suffices to show that the family of Higson maps $f : X \to [0, 1]$ separates points from closed sets. Indeed, given $x_0 \in X \setminus A$, where A is closed, we find U open and bounded such that $x_0 \in U \subset X \setminus A$. Any map $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f|(X \setminus U) \equiv 0$ is a Higson map. □

In case of locally compact Tychonoff spaces X we are interested in the *Higson corona* $\nu(X, \mathcal{LSS}_X) := h(X, \mathcal{LSS}_X) \setminus X$ of X.

Corollary 3.2. *Suppose X is a locally compact Tychonoff space. If $\mathcal{LSS}_X$ is a large scale structure such that all compact subsets of X are bounded, then $h(X, \mathcal{LSS}_X)$ is a compactification of X.*

Proof. Notice all open and relatively compact sets in X are bounded and form a basis of X. □

Given a compactification $c(X)$ of a locally compact Tychonoff space X we are interested in constructing a large scale structure $\mathcal{LSS}(c(X), X)$ on X satisfying the following two conditions:

(a) The bounded subsets of X are precisely relatively compact subsets of X.
(b) The Higson maps of $\mathcal{LSS}(c(X), X)$ include restrictions $f|X$ of all continuous maps $f : c(X) \to R$.

Notice $\mathrm{St}(K, \mathcal{B})$ is bounded for every relatively compact K and every $\mathcal{B} \in \mathcal{LSS}(c(X), X)$. That leads to the following definition.

Definition 3.3. A family $\mathcal{B}$ is *proper* if $\mathrm{St}(K, \mathcal{B})$ is relatively compact for all relatively compact $K \subset X$. Notice that every $B \in \mathcal{B}$ is relatively compact in that case (consider K consisting of a point in B).

Recall $E \subset X \times X$ is *proper* provided both $E[K]$ and $E^{-1}[K]$ are relatively compact for all relatively compact $K \subset X$ (see [8, Definition 2.1, p. 21]).

Lemma 3.4. *If $\mathcal{B}$ is a family of subsets of X, then $\Delta(\mathcal{B})[K] = \mathrm{St}(K, \mathcal{B})$.*

Proof. Recall that $E[K]$ is the set of all x' such that there is $x \in K$ satisfying $(x', x) \in E$. If $E = \Delta(\mathcal{B})$ that means precisely there is $B \in \mathcal{B}$ such that $x', x \in B$ and $x \in K$, i.e., $x' \in \mathrm{St}(K, \mathcal{B})$. □

Corollary 3.5. *$\mathcal{B}$ is proper if and only if $\Delta(\mathcal{B})$ is proper.*

Proposition 3.6. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two proper families, then $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)$ is a proper family.*

Proof. Notice $\mathrm{St}(K, \mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)) \subset \mathrm{St}(\mathrm{St}(K, \mathcal{B}_1), \mathcal{B}_2) \cup \mathrm{St}(\mathrm{St}(K, \mathcal{B}_2), \mathcal{B}_1)$ for every $K \subset X$. If K is relatively compact, so is $\mathrm{St}(\mathrm{St}(K, \mathcal{B}_1), \mathcal{B}_2) \cup \mathrm{St}(\mathrm{St}(K, \mathcal{B}_2), \mathcal{B}_1)$. □

A *Higson family* relative to compactification $c(X)$ is a proper family $\mathcal{B}$ satisfying the following property: For any map $f : c(X) \to R$ and for any $\epsilon > 0$ there is a relatively compact set K in X such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in B \setminus K$, $B \in \mathcal{B}$.

Proposition 3.7. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two Higson families, then $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)$ is a Higson family.*

Proof. Suppose $f : c(X) \to R$ is continuous and $\epsilon > 0$. Find a relatively compact set K such that $|f(x) - f(y)| < \epsilon/4$ for all $x, y \in B \setminus K$, $B \in \mathcal{B}_1$ or $B \in \mathcal{B}_2$. Put $L = \mathrm{St}(\mathrm{St}(K, \mathcal{B}_1), \mathcal{B}_2) \cup K$. Suppose $x, y \in \mathrm{St}(B, \mathcal{B}_2) \setminus L$ for some $B \in \mathcal{B}_1$ and $|f(x) - f(y)| > \epsilon$. Clearly, both x and y cannot belong to B. We will discuss the case of $x, y \in X \setminus B$, the other cases are similar. Thus $x \in B_x \in \mathcal{B}_2$ and $y \in B_y \in \mathcal{B}_2$ so that there exist $a \in B \cap B_x$ and $b \in B \cap B_y$. Notice $a, b \in X \setminus K$ (otherwise $x, y \in L$). Therefore $|f(a) - f(b)| < \epsilon/4$, $|f(a) - f(x)| < \epsilon/4$, and $|f(y) - f(b)| < \epsilon/4$ resulting in $|f(x) - f(y)| < 3 \cdot \epsilon/4$, a contradiction. □

Define $\mathcal{LSS}(c(X), X)$ as consisting of all Higson families $\mathcal{B}$. It is a large scale structure as the trivial extension of a Higson family is a Higson family and refinements of Higson families are Higson as well. Notice every continuous $f : c(X) \to R$ restricts to a Higson map $f|X$ of $\mathcal{LSS}(c(X), X)$.

Using Example 2.34 in [8, p. 28] consider the compactification $c(Z)$ of integers (Z is equipped with the discrete topology) obtained by identifying two different points u and v in the Čech–Stone corona $\beta(Z) \setminus Z$. Notice $\mathcal{LSS}(c(Z), Z)$ is the discrete large scale structure on Z (generated by families $\mathcal{B}$ such that $\bigcup \mathcal{B}$ is finite) whose Higson functions are all bounded functions $f : Z \to R$, a set larger than restrictions $f|Z$ of all continuous maps $f : c(Z) \to R$. Thus, the Higson compactification of $\mathcal{LSS}(c(Z), Z)$ may be larger than $c(X)$.

4. Asymptotic dimension

Large scale structures offer a very simple definition of *asymptotic dimension*. Namely, $\mathrm{asdim}(X, \mathcal{LSS}_X) \leqslant n$ if $\mathcal{LSS}_X$ is generated by families $\mathcal{B}$ such that the *multiplicity* of $\mathcal{B}$ is at most $n + 1$ (that means each point $x \in X$ is contained in at most $n + 1$ elements of $\mathcal{B}$).

It is well known that for metric spaces the condition $\mathrm{asdim}(X) \leqslant n$ can be expressed by one of the following equivalent conditions (see [7]):

(a) For every uniformly bounded family $\mathcal{B}$ in X there is a uniformly bounded family $\mathcal{B}'$ on X of which $\mathcal{B}$ is a refinement such that the multiplicity of $\mathcal{B}'$ is at most $n + 1$.
(b) For every $r > 0$ there is a decomposition of X as $X_0 \cup \cdots \cup X_n$ such that the family of r-components of each X_i is uniformly bounded.

Our first observation is that one can generalize it to ∞-metric spaces without changing the proof.

Proposition 4.1. *Suppose (X, d) is an ∞-metric space. If $n \geqslant 0$, then the following conditions are equivalent*:

(a) *For every uniformly bounded family $\mathcal{B}$ in X there is a uniformly bounded family $\mathcal{B}'$ on X of which $\mathcal{B}$ is a refinement such that the multiplicity of $\mathcal{B}'$ is at most $n + 1$.*

(b) *For every $r > 0$ there is a decomposition of X as $X_0 \cup \cdots \cup X_n$ such that the family of r-components of each X_i is uniformly bounded.*

Let us point out another benefit of ∞-metric spaces. Namely, Bell and Dranishnikov [1] define $\operatorname{asdim}(X_s, d_s) \leqslant n$ *uniformly* for all $s \in S$ if for every $r > 0$ there is $M(r) < \infty$ such that each X_s decomposes as $X_0^s \cup \cdots \cup X_n^s$ and r-components of each X_i^s are of diameter at most $M(r)$. In our language we may simply state $\operatorname{asdim}(\bigoplus_{s\in S} X_s) \leqslant n$.

We would like to generalize Proposition 4.1 to arbitrary large scale structures. For that we need the notion of *$\mathcal{B}$-components*. Those are equivalence classes of the relation $x \sim_{\mathcal{B}} y$ meaning that there is a finite chain of points $x_0 = x, \dots, x_k = y$ such that for every $i \geqslant 0$ (and $i \leqslant k-1$) there is $B_i \in \mathcal{B}$ satisfying $x_i, x_{i+1} \in B_i$.

Our generalization of Proposition 4.1 has the advantage that its proof is by reduction to Proposition 4.1 which shows that the asymptotic dimension of arbitrary large scale structures can be reduced to asymptotic dimension of ∞-metric spaces. Compare our approach to that of [6].

Corollary 4.2. *Suppose $\mathcal{LSS}_X$ is a large scale structure on a set X. If $n \geqslant 0$, then the following conditions are equivalent*:

(a) *For every uniformly bounded family $\mathcal{B}$ in X there is a uniformly bounded family $\mathcal{B}'$ on X of which $\mathcal{B}$ is a refinement such that the multiplicity of $\mathcal{B}'$ is at most $n+1$.*
(b) *For every uniformly bounded family $\mathcal{B}$ in X there is a decomposition of X as $X_0 \cup \cdots \cup X_n$ such that the family of $\mathcal{B}$-components of each X_i is uniformly bounded.*

Proof. (a) $\Rightarrow$ (b). Given $\mathcal{B} \in \mathcal{LSS}_X$ construct inductively a sequence of elements $\mathcal{B}_i \in \mathcal{LSS}_X$ satisfying the following conditions:

(1) $\mathcal{B}_1 = \mathcal{B}$,
(2) $\operatorname{St}(\mathcal{B}_i, \mathcal{B}_i)$ is a refinement of $\mathcal{B}_{i+1}$ for each $i \geqslant 1$,
(3) the multiplicity of $\mathcal{B}_i$ is at most $n+1$ for $i > 1$.

Given two points $x, y \in X$ we define $d(x, y)$ as the smallest integer i such that $x, y \in B \in \mathcal{B}_i$ for some i. If such integer does not exist, we put $d(x, y) = \infty$.

Notice $\operatorname{asdim}(X, d) \leqslant n$. Therefore one can decompose (X, d) as $X_0 \cup \cdots \cup X_n$ such that the family of 2-components of each X_i is uniformly bounded by a fixed integer M. That can be translated into $\mathcal{B}$-components of each X_i being contained in an element of $\mathcal{B}_{M+1}$.

(b) $\Rightarrow$ (a). Given $\mathcal{B}_1$ put $\mathcal{B}_2 = \operatorname{St}(e(\mathcal{B}_1), e(\mathcal{B}_1))$ and find a decomposition of X as $X_0 \cup \cdots \cup X_n$ such that the family of $\mathcal{B}_2$-components of each X_i is uniformly bounded. Consider $\mathcal{B}_3$ consisting of stars $\operatorname{St}(C, \mathcal{B}_1)$, where C is a $\mathcal{B}_2$-component of some X_i. Clearly, $\mathcal{B}_1$ refines $\mathcal{B}_3$, so it remains to show that the multiplicity of $\mathcal{B}_3$ is at most $n+1$. That follows from the observation that $\operatorname{St}(C, \mathcal{B}_1) \cap \operatorname{St}(C', \mathcal{B}_1) = \emptyset$ for every two different $\mathcal{B}_2$-components C and C' of the same X_i (otherwise $\operatorname{St}(x, \mathcal{B}_1)$ would intersect both C and C' for any $x \in \operatorname{St}(C, \mathcal{B}_1) \cap \operatorname{St}(C', \mathcal{B}_1)$, a contradiction). □

Our final task is to generalize the Hurewicz Theorem for asymptotic dimension of [1] and [2].

First let us point out that *large scale uniform* functions (or *bornologous functions* in the terminology of [8]) between metric spaces have a very simple generalization to large scale structures: $f : (X, \mathcal{LSS}_X) \to (Y, \mathcal{LSS}_Y)$ is large scale uniform if $f(\mathcal{B}) \in \mathcal{LSS}_Y$ for all $\mathcal{B} \in \mathcal{LSS}_X$.

Given a function $f : (X, \mathcal{LSS}_X) \to (Y, \mathcal{LSS}_Y)$ we need to define the concept of $\operatorname{asdim}(f) \leqslant n$. Since that has to do with $f^{-1}(\mathcal{B})$ for $\mathcal{B} \in \mathcal{LSS}_Y$, let us define a natural large scale structure on the disjoint union $\bigoplus_{s\in S} A_s$ for any family $\{A_s\}_{s\in S}$ of subsets of X. Since we want the natural projection $\bigoplus_{s\in S} A_s \to X$ to be large scale uniform, the natural choice is to call $\mathcal{B}$ uniformly bounded in $\bigoplus_{s\in S} A_s$ if and only if there is $\mathcal{C} \in \mathcal{LSS}_X$ such that $\mathcal{B}|A_s$ refines $\mathcal{C}$ for all $s \in S$.

Let us adopt the notation of $\bigoplus \mathcal{B}$ for the disjoint union of any family $\mathcal{B}$. Now, $\operatorname{asdim}(f) \leqslant n$ means that $\operatorname{asdim}(\bigoplus f^{-1}(\mathcal{B})) \leqslant n$ for all $\mathcal{B} \in \mathcal{LSS}_Y$.

Theorem 4.3. *If $f:(X,\mathcal{LSS}_X)\to(Y,\mathcal{LSS}_Y)$ is a large scale uniform function, then*

$$\operatorname{asdim}(X,\mathcal{LSS}_X)\leqslant \operatorname{asdim}(f)+\operatorname{asdim}(Y,\mathcal{LSS}_Y).$$

Proof. Let $\operatorname{asdim}(f)=n$ and $\operatorname{asdim}(Y,\mathcal{LSS}_Y)=k$.

Suppose $\mathcal{B}_1\in\mathcal{LSS}_X$ is a cover. Let us construct by induction a sequence of covers $\mathcal{B}_i\in\mathcal{LSS}_X$ and a sequence of covers $\mathcal{C}_i\in\mathcal{LSS}_Y$ satisfying the following conditions:

(1) $\operatorname{St}(\mathcal{B}_i,\mathcal{B}_i)$ refines $\mathcal{B}_{i+1}$ for all $i\geqslant 1$.
(2) $f(\mathcal{B}_i)$ refines $\mathcal{C}_i$.
(3) The multiplicity of $\mathcal{C}_i$ is at most $k+1$.
(4) The cover of $\bigoplus f^{-1}(\mathcal{C}_i)$ induced by $\mathcal{B}_i$ refines a cover of multiplicity at most $n+1$ that is a refinement of the cover of $\bigoplus f^{-1}(\mathcal{C}_i)$ induced by $\mathcal{B}_{i+1}$.
(5) $\operatorname{St}(\mathcal{C}_i,\mathcal{C}_i)$ refines $\mathcal{C}_{i+1}$ for all $i\geqslant 1$.

Define the ∞-metric d_X on X by setting $d_X(x,y)$ equal the smallest i such that there is $B\in\mathcal{B}_i$ containing both x and y. If no such i exists, put $d_X(x,y)=\infty$. Create a ∞-metric d_Y on Y the same way using the sequence $\mathcal{C}_i$. Notice the following properties of $f:(X,d_X)\to(Y,d_Y)$:

(a) $\operatorname{asdim}(Y,d_Y)\leqslant n$.
(b) $\operatorname{asdim}(f)\leqslant n$.
(c) $f:(X,d_X)\to(Y,d_Y)$ is large scale uniform.

Indeed, $\mathcal{LSS}(X,d_X)$ is generated by $\mathcal{B}_i$'s and $\mathcal{LSS}(Y,d_Y)$ is generated by $\mathcal{C}_i$'s (see the proof of Theorem 1.8), so (a) and (c) follow. Similarly, (b) holds.

Since the proof of Hurewicz Theorem in [2] is valid for ∞-metric spaces, one concludes $\operatorname{asdim}(X,d_X)\leqslant n+k$. In particular there is a uniformly bounded family $\mathcal{U}$ in (X,d_X) such that $\mathcal{B}_1$ refines $\mathcal{U}$ and the multiplicity of $\mathcal{U}$ is at most $k+n+1$. Notice $\mathcal{U}$ refines $\mathcal{B}_M$ for some large M. Thus, $\mathcal{U}\in\mathcal{LSS}_X$ which completes the proof. □

5. Švarc–Milnor lemma

Ref. [3] gives a simple proof of Švarc–Milnor lemma. It gives sufficient conditions for an action by isometries of a group G on a metric space X to induce a quasi-isometry between G (equipped with a word metric) and X via the map $g\to g\cdot x_0$:

Theorem 5.1 *(Švarc–Milnor). A group G acting properly and cocompactly via isometries on a length space X is finitely generated and induces a quasi-isometry equivalence $g\to g\cdot x_0$ for any $x_0\in X$.*

Let us use the approach of this paper to offer an explanation of assumptions in the Švarc–Milnor lemma.

Given a function $f:X\to Y$ and given a large scale structure $\mathcal{LSS}_Y$ on Y let us define *the induced large scale structure* $f^*(\mathcal{LSS}_Y)$ on X as that generated by $f^{-1}(\mathcal{B})$, $\mathcal{B}\in\mathcal{LSS}_Y$.

Lemma 5.2. *If a group $(G,\cdot)$ acts on the left by isometries on a metric space (X,d), then $\mathcal{LSS}_l(G,\cdot)\subset f^*(\mathcal{LSS}(X,d))$ for any $x_0\in X$, where $f(g):=g\cdot x_0$ for $g\in G$.*

Proof. Suppose $F\subset G$ is finite. Put $r=\max\{d(x_0,h\cdot x_0)\mid h\in F\}$. Given $g\in G$ let $U=B(g\cdot x_0,r)$. It suffices to show $f(g\cdot F)\subset U$. That is obvious as $d(g\cdot h\cdot x_0,g\cdot x_0)=d(h\cdot x_0,x_0)<r$ if $h\in F$. □

Lemma 5.3. *Suppose a group $(G,\cdot)$ acts on the left by isometries on a metric space (X,d), $x_0\in X$ and $f(g):=g\cdot x_0$ for $g\in G$.*

$$\mathcal{LSS}_l(G,\cdot)=f^*\big(\mathcal{LSS}(X,d)\big)$$

if and only if for any bounded subset U of $G\cdot x_0$ containing x_0 the set $\{g\in G\mid (g\cdot U)\cap U\neq\emptyset\}$ is finite.

Proof. In view of Lemma 5.2, we need to analyze $f^*(\mathcal{LSS}(X,d)) \subset \mathcal{LSS}_l(G,\cdot)$. It holds if and only if, for any $r > 0$, there is a finite subset F_r of G such that for any $x \in G \cdot x_0$ there is $g_x \in X$ so that $f^{-1}(B(x,r)) \subset g_x \cdot F_r$.

Put $U = B(x_0, r)$ and assume $F_r = \{g \in G \mid (g \cdot U) \cap U \neq \emptyset\}$ is finite. If $x = g_x \cdot x_0$, then $f^{-1}(B(x,r)) = \{g \in G \mid d(g \cdot x_0, g_x \cdot x_0) < r\} = \{g \in G \mid g_x^{-1} g \cdot x_0 \in B(x_0, r)\} \subset g_x \cdot F_r$.

Assume that, for any $r > 0$, there is a finite subset F_r of G and $g_0 \in X$ so that $f^{-1}(B(x_0, r)) \subset g_0 \cdot F_r$. Consider $U = B(x_0, r)$ (any bounded subset of $G \cdot x_0$ is contained in such ball). If $h \cdot x_0 \in (g \cdot U) \cap U$, then $h \in f^{-1}(B(x_0, r))$, so $h \in g_0 \cdot F_r$. Also, $g^{-1} \cdot h \in g_0 \cdot F_r$ which means the set $\{g \in G \mid (g \cdot U) \cap U \neq \emptyset\}$ is finite. □

References

[1] G. Bell, A. Dranishnikov, A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory, math.GR/0407431.

[2] N. Brodskiy, J. Dydak, M. Levin, A. Mitra, Hurewicz theorem for Assouad–Nagata dimension, math.MG/0605416, J. London Math. Soc., in press.

[3] N. Brodskiy, J. Dydak, A. Mitra, Švarc–Milnor lemma: A proof by definition, math.GT/0603487, Topology Proc. 31 (2007), in press.

[4] M. Cencelj, J. Dydak, J. Smrekar, A. Vavpetič, Sublinear Higson corona and Lipschitz extensions, math.MG/0608686.

[5] A.N. Dranishnikov, J. Smith, On asymptotic Assouad–Nagata dimension, math.MG/0607143.

[6] B. Grave, Asymptotic dimension of coarse spaces, available at http://www.uni-math.gwdg.de/bgr/veroeffentl/.

[7] M. Gromov, Asymptotic invariants for infinite groups, in: G. Niblo, M. Roller (Eds.), Geometric Group Theory, vol. 2, Cambridge Univ. Press, 1993, pp. 1–295.

[8] J. Roe, Lectures on Coarse Geometry, University Lecture Series, vol. 31, Amer. Math. Soc., Providence, RI, 2003.