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# POISSON OPERATORS FOR BOUNDARY PROBLEMS CONCERNING A CLASS OF DEGENERATE PARABOLIC EQUATIONS

BY

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ABSTRACT. – Using the formal series method in this paper we construct a Poisson operator for classical boundary problems concerning a class of degenerate parabolic equations. © 2001 Éditions scientifiques et médicales Elsevier SAS

## Introduction

Let $T \in ]0, +\infty[$ and let $(t, y, x) \in [0, T] \times R \times R^n$. In [4] we have considered the following degenerate parabolic operator, with real coefficients:

$$L = \partial_t - \partial_y^2 - y^2 \sum_{i,j=1}^{n} a_{ij} \partial_{x_i} \partial_{x_j}$$

and we have constructed a solution for problem:

$$\begin{cases} LU(t, y, x) = F(t, y, x) & (t, y, x) \in ]0, T[\times]0, +\infty[\times R^n, \\ U(t, 0, x) = H(t, x) & (t, x) \in ]0, T[\times R^n, \\ U(0, y, x) = U_0(y, x) & (y, x) \in ]0, +\infty[\times R^n, \end{cases}$$

under the following conditions: the quadratic form

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j$$

has constant coefficients and is definite positive, moreover the data of problem are infinitely differentiable functions and rapidly decreasing respect to $(y, x)$. In particular, if $F$ and $H$ are zero everywhere, the solutions is $U = KU_0$, where $K$ is a Poisson operator of the type

$$(0.1)\quad U_0 \in C_0^\infty\big(]0,+\infty[\times R^n\big) \to (2\pi)^{-n}\int_0^{+\infty} \mathrm{d}y' \int_{R^n} \mathrm{e}^{ix\cdot\xi} k(t, y, y', x, \xi) \underset{x\to\xi}{\mathcal{F}} U_0(y', \xi)\,\mathrm{d}\xi$$

and it can be extended as a linear and continuous operator:

$$E'\big(]0,+\infty[\times R^n\big) \to C^\infty\big([0,+\infty[, D'(]0,+\infty[\times R^n)\big) \cap C^\infty\big(]0,+\infty[\times[0,+\infty[\times R^n\big).$$

If the operator $L$ has variable coefficients and it has pieces of lower order, generally it is no possible obtain an exact solution of type $KU_0$. By pseudodifferential techniques it is possible to construct a Poisson operator $K$ such that, if $U_0$ is a generalized function with compact support in $]0,+\infty[\times R^n$, the distribution $KU_0$ solves the problem for less of infinitely differentiable error, so it is the singular part of the exact solutions (see [5–10]).

In the present paper we talk over a problem of this type. We consider the operator

$$(0.2)\quad L = \partial_t - \partial_y^2 - y^2 \sum_{i,j=1}^n a_{ij}(t,x)\partial_{x_i}\partial_{x_j} + yb(t,x)\partial_y + \sum_{i=1}^n a_i(t,x)\partial_{x_i} + c(t,x)$$

such that the following assumptions hold: $a_{ij}(t,x)$, $a_i(t,x)$, $b(t,x)$, $c(t,x)$ are real valued and infinitely differentiable functions in $[0,T]\times R^n$; the quadratic form:

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j, \quad a_{ij}(t,x) = a_{ji}(t,x),$$

is semi-definite positive, while

$$\omega^2(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(0,x)\xi_i\xi_j \tag{0.3}$$

is definite positive. We have studied the boundary problems:

$$LU = 0, \quad U(t,0,x) = 0, \qquad U(0,y,x) = G(y,x); \tag{0.4}$$

$$LU = 0, \quad \partial_y U(t,0,x) = 0, \quad U(0,y,x) = G(y,x), \tag{0.5}$$

with the following purpose: for every open $A$ with compact closure in $]0,+\infty[\times R^n$, $A \Subset ]0,+\infty[\times R^n$, to construct two Poisson operators, $K_A^{(1)}$ and $K_A^{(2)}$, such that if

$$G(y,x) \in E'\big(]0,+\infty[\times R^n\big),$$

then, for $i = 1, 2$, we have

$$LK_A^{(i)} G \in C^\infty\big([0,T] \times [0,+\infty[\times R^n\big), \tag{0.6}$$

$$\lim_{t\to 0}\big(K_A^{(i)} G - G\big) \in C^\infty\big([0,+\infty[\times R^n\big) \quad \text{if supp } G \subset A, \tag{0.7}$$

$$K_A^{(1)} G(t,0,x) = 0, \quad \partial_y K_A^{(2)} G(t,0,x) = 0, \quad t \in ]0,T]. \tag{0.8}$$

We use the formal series method (see papers mentioned above). For each of problems (0.4) and (0.5), we search a series of pseudo-homogeneous symbols (see [2,9]) with degree negatively diverging:

$$\sum_{j=0}^{+\infty} k_{-j}^{(i)}(t,y,y',x,\xi), \quad i = 1, 2, \tag{0.9}$$

such that, by (0.1), the series (0.9) gives a formal solution of respective problem. Then using classical techniques we construct desired operator.

We obtain the functions $K_{-j}^{(i)}$ by recurrence solving a sequence of differential problems, called transport problems.

Since $L$ is degenerate we use two different processes of homogenization. These processes lead to two different formal series, that act on the distributions $G(y,x)$ such that the support of $\widetilde{G}(\eta,\xi) =$

$\mathcal{F}_{y\to\eta}\mathcal{F}_{x\to\xi}(G(y,x))$ is included in a region of the type $\eta^2 < a|\xi|$, or of the type $\eta^2 > a|\xi|$, $a > 0$, respectively. The final result is contained in Theorem 7.2.

In Sections from 1 to 6 we construct the first series that leads to partial differential equations solved in Section 2. In Sections 3 and 4 we establish estimates for the transport problems solutions. These solutions fit in suitable spaces of symbols of non standard pseudodifferential operators (see Section 5). Section 6 is devoted to the construction of a Poisson operator relative to the formal series found. In Section 7 we construct the second series by classical techniques that lead to transport systems of ordinary differential equations (see [7]). Finally we attain our aim by a suitable connection between the series.

## 1. Pseudo-homogeneous symbols and transport systems

Put $\Omega = R^{n+1} = R_y \times R_x^n$ and $\Omega_T = ]0,T[\times\Omega$ for any $T > 0$, we denote by $\Omega^+$ and $\Omega_T^+$ subsets of $\Omega$ and $\Omega_T$ such that $y > 0$.

Now, let $k(t,y,y',x,\xi) \in C^\infty(\Omega_T \times R_{y'} \times (R_n - \{0\}))$ be a slowly increasing function respect to $\xi$. By $\langle , \rangle$ we denote the duality pairing between $C_0^\infty(\Omega)$ and $D'(\Omega)$. We say that $k$ is a symbol in $\Omega_T$ if for any $\psi \in C_0^\infty(R_{y'})$:

$$\langle k(t,y,y',x,\xi), \psi(y')\rangle = \int_{R_{y'}} k(t,y,y',x,\xi)\psi(y')\,\mathrm{d}y' \tag{1.1}$$

can be extended as a function of class $C^\infty(\bar{\Omega}_T \times (R_n - \{0\}))$. In similar way we define a symbol in $\Omega_T^+$.

If $k$ is a symbol in $\Omega_T$ (respectively in $\Omega_T^+$), infinitely differentiable in $\Omega_T \times R_{y'} \times R_n$ (respectively $\Omega_T^+ \times R_{y'}^+ \times R_n$), we consider the following operator:

$$KG(t,y,x) = \int_{R_n} \mathrm{e}^{ix\cdot\xi}\langle k(t,y,y',x,\xi), \hat{G}(y',\xi)\rangle\,\bar{\mathrm{d}}\xi, \tag{1.2}$$

$$\bar{\mathrm{d}}\xi = (2\pi)^{-n}\,\mathrm{d}\xi,$$

where $G(y,x) \in C_0^\infty(\Omega)$ (respectively $C_0^\infty(\Omega^+)$), and

$$\hat{G}(y',\xi) = \int_{R^n} \mathrm{e}^{-ix\cdot\xi} G(y',x)\,\mathrm{d}x = \mathop{\mathcal{F}}_{x\to\xi}\big(G(y',x)\big).$$

Now let $k = k(t,y,y',x,\xi)$ be a symbol in $\Omega_T$ or in $\Omega_T^+$, and let $m \in R$. We say that $k$ is pseudo-homogeneous of degree $m$ if:

$$k(t\lambda^{-1}, y\lambda^{-1/2}, y'\lambda^{-1/2}, x, \lambda\xi) = \lambda^m k(t,y,y',x,\xi) \quad \forall\lambda \in R^+. \tag{1.3}$$

It is easy to prove that if $k$ is a pseudo-homogeneous symbol of degree $m$, then the symbol:

$$t^p y^h \partial_t^l \partial_y^r \partial_x^\alpha \partial_\xi^\beta k \quad p,h \in R_0^+,\ r,l \in N_0,\ \alpha,\beta \in N_0^n$$

is pseudo-homogeneous of degree $m - p - h/2 + l + r/2 - |\beta|$. This motivates the following definition: if $h \in R$ and $O$ is an operator which does not change the pseudo-homogeneous symbol class, we say that $O$ has pseudo-order $h$ if it sends pseudo-homogeneous symbols of degree $m$ in pseudo-homogeneous symbols of degree $m + h$.

Now we research a symbol $k$ in $\Omega_T$ such that:

$$LKG(t,y,x) = 0 \quad \forall(t,y,x) \in \Omega_T,\ \forall G \in C_0^\infty(\Omega); \tag{1.4}$$

using (1.2), one can prove that (1.4) is equivalent to

$$Mk(t,y,y',x,\xi) = 0, \tag{1.5}$$

where

$$M = M(t,y,x,\xi,\partial_t,\partial_y,\partial_x) = L(t,y,x,\partial_t,\partial_y,\partial_x + i\xi). \tag{1.6}$$

By Mac Laurin series expansion of the coefficients of the operator $L$, with respect to $t$, we have the following decomposition:

$$M = \sum_{h=-1}^{+\infty} M_{-h}, \tag{1.7}$$

where $h$ is an integer, and $M_{-h}$ is an operator of pseudo-order $-h$, for every $h \geqslant -1$; from the definitions

$$M_1 = \partial_t - \partial_y^2 + y^2 \sum_{i,j} a_{ij}(0,x)\xi_i\xi_j + i \sum_j a_j(0,x)\xi_j, \tag{1.8}$$

$$\begin{aligned} M_0 = {} & y^2 \sum_{i,j} t\partial_t a_{ij}(0,x)\xi_i\xi_j - 2iy^2 \sum_{i,j} a_{ij}(0,x)\xi_i\partial_{x_j} \\ & + yb(0,x)\partial_y + it\sum_i \partial_t a_i(0,x)\xi_i \\ & + \sum_i a_i(0,x)\partial_{x_i} + c(0,x) \end{aligned} \tag{1.8$'$}$$

and, for $h > 0$,

$$\begin{aligned} M_{-h} = {} & \frac{t^{h+1}}{(h+1)!}\partial_t^{h+1}\Big[y^2 \sum_{i,j} a_{i,j}(t,x)\xi_i\xi_j \\ & + i\sum_i a_i(t,x)\xi_i\Big](0,x) \\ & + \frac{t^h}{h!}\partial_t^h\Big[-2iy^2 \sum_{i,j} a_{i,j}(t,x)\xi_j\partial_{x_i} + yb(t,x)\partial_y \\ & + \sum_i a_i(t,x)\partial_{x_i} + c(t,x)\Big](0,x) \\ & + \frac{t^{h-1}}{(h-1)!}\partial_t^{h-1}\Big[-y^2 \sum_{i,j} a_{i,j}(t,x)\partial_{x_i}\partial_{x_j}\Big](0,x). \end{aligned} \tag{1.8$''$}$$

We want $k$ as a formal series of pseudo-homogeneous symbols

$$\sum_{s=0}^{\infty} k_{-s}(t,y,y',x,\xi), \tag{1.9}$$

where $k_{-s}$ is pseudo-homogeneous of degree $m-s$, here $s$ is integer and $m$ is a real number to establish. In (1.5) we replace $k$ by (1.9) and we obtain

$$\sum_{h+s=r} M_{-h}k_{-s}(t,y,y',x,\xi) = 0 \quad \forall r \geqslant -1. \tag{1.10}$$

If we consider (1.10) with the initial conditions

$$k_0(0, y, y', x, \xi) = \delta(y' - y);\qquad k_{-s}(0, y, y', x, \xi) = 0, \quad \forall s \in N, \tag{1.11}$$

it is easy to prove that if $k$ is of the type (1.9) and (1.10), (1.11) hold, then the operator $K$ verifies

$$LKG(t, y, x) = 0 \quad \forall (t, y, x) \in \Omega_T,\ \forall G \in C_0^\infty(\Omega), \tag{1.12}$$

$$KG(0, y, x) = G(y, x), \quad \forall G \in C_0^\infty(\Omega). \tag{1.13}$$

Now we suppose that, $\forall s \in N_0, k_{-s}$ keeps the test functions parity. Fixed $G \in C_0^\infty(\Omega^+)$, we denote by $G_d$ and $G_p$ respectively the odd and the even extension of $G$ with respect to $y$. Putting

$$K^{(1)}G = KG_d, \qquad K^{(2)}G = KG_p \tag{1.14}$$

we obtain that $K^{(1)}G$ and $K^{(2)}G$ satisfy (1.12) and (1.13) for $y > 0$. Moreover the functions in (1.14) are solutions of (0.4) and (0.5) respectively, by their symmetry property.

That being stated, we determine the series (1.9) such that (1.10), (1.11) and the condition

$$k_{-s}(t, -y, -y', x, \xi) = k_{-s}(t, y, y', x, \xi) \quad \forall s \in N_0 \tag{1.15}$$

are satisfied.

Fixed $\varphi \in C_0^\infty(R_{y'})$, we put:

$$U_{-s}(t, y, x, \xi) = \langle k_{-s}(t, y, y', x, \xi), \varphi(y')\rangle, \quad s \in N_0. \tag{1.16}$$

So (1.10) and (1.11) entail that we can find the sequence $\{U_{-s}\}_{s\in N_0}$, by recurrence, solving the following transport problems:

$$\begin{cases} \left(\partial_t - \partial_y^2 + \omega^2 y^2 + i\sum_i a_i(0, x)\xi_i\right)U_0 = 0 \\ \quad (t, y, x, \xi) \in \Omega_T \times R_n, \\ U_0(0, y, x, \xi) = \varphi(y) \quad (y, x, \xi) \in \Omega \times R_n, \end{cases} \tag{1.17}$$

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$$(1.18)\quad \begin{cases} M_1 U_{-s} = -(M_0 U_{-s} + M_1 U_{-s+2} + \cdots \\ \qquad\qquad + M_{-s+1} U_0) \quad (t, y, x, \xi) \in \Omega_T \times R_n, \quad s > 1, \\ U_{-s}(0, y, x, \xi) = 0 \quad (y, x, \xi) \in \Omega \times R_n, \end{cases}$$

where $\omega = \omega(x, \xi) \geqslant 0$ is the function in (0.3).

## 2. Resolution of the transport systems

For every $\xi \in R_n - \{0\}$ we set:

$$(2.1)\qquad \tau = t\omega; \qquad z = y\omega^{1/2}; \qquad \dot{\xi} = \xi/\omega;$$

$$(2.2)\qquad g(z, \omega) = \varphi\big(z/\omega^{1/2}\big);$$

$$(2.3)\qquad \mathrm{e}^{i\sum_j a_j(0,x)\dot{\xi}_j\tau} u_{-s}(\tau, z, x, \dot{\xi}, \omega) = \omega^s U_{-s}\big(\tau/\omega, z/\omega^{1/2}, x, \dot{\xi}\omega\big), \quad s \geqslant 0.$$

Then let $m_{-h}^{-s}(\tau, z, x, \dot{\xi}, \omega, \partial_\tau, \partial_z, \partial_{\xi_i})$, $s, h \geqslant 0$ be the operators defined by:

$$(2.4)\qquad \omega^{h+s} M_{-h} U_{-s}\big(\tau/\omega, z/\omega^{1/2}, x, \dot{\xi}\omega\big) = -m_{-h}^{-s} u_{-s}(\tau, z, x, \dot{\xi}, \omega).$$

So the foregoing positions turn the transport systems into the following differential problems in $\bar{R}_\tau^+ \times R_z$, with parameter $(x, \dot{\xi}, \omega) \in R^n \times (R_n - \{0\}) \times R^+$:

$$(2.5)\qquad \begin{cases} \big(\partial_\tau - \partial_z^2 + z^2\big) u_0 = 0, \\ u_0(0, z) = g(z, \omega), \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(2.6)\qquad \begin{cases} \big(\partial_\tau - \partial_z^2 + z^2\big) u_{-s} \\ \quad = m_0^{-s+1} u_{-s+1} + \cdots + m_{-s+1}^0 u_0, \quad s \in N, \\ u_{-s}(0, z) = 0. \end{cases}$$

By imposing to the functions $u_{-s}$, $s \in N_0$, the additional condition of rapidly decreasing on $\bar{R}_\tau^+ \times R_z$, we have that the solutions of the systems (2.5)–(2.6) are unique. So we can obtain their expression using the results in [4].

Let $\{\varphi_k(z)\}_{k\in N_0}$ be the Hermite functions (see [1]). We introduce (see [4]) the fundamental solution of the problem (2.5)

$$\Phi_0(\tau,z,z') = \pi^{-1/2}\sum_{k=0}^{+\infty} \mathrm{e}^{-(2k+1)\tau}\varphi_k(z)\varphi_k(z'), \tag{2.7}$$

it is infinitely differentiable in $R_\tau^+ \times R_z \times R_{z'}$ and belongs to $C^\infty(\bar{R}_\tau^+ \times R_z, D'(R_{z'}))$.

We have proved (see [4, §3]) that the operators

$$T : g \in C_0^\infty(R_z) \to \int_{-\infty}^{+\infty} \Phi_0(\tau,z,z')g(z')\,\mathrm{d}z', \tag{2.8}$$

$$Z : f \in C_0^\infty(R_\tau^+ \times R_z) \to \int_0^\tau \mathrm{d}\tau' \int_{-\infty}^{+\infty} \Phi_0(\tau-\tau',z,z')f(\tau',z')\,\mathrm{d}z'; \tag{2.9}$$

have values in $S(\bar{R}_\tau^+ \times R_z)$ and the function $u = Tg + Zf$ is the unique solution belonging to $S(\bar{R}_\tau^+ \times R_z)$ of the following auxiliary problem

$$\begin{cases} \left(\partial_\tau - \partial_z^2 + z^2\right)u = f(\tau,z), \\ u(0,z) = g(z). \end{cases} \tag{2.10}$$

So we deduce that the sequence defined by recurrence

$$u_0 = Tg; \quad u_{-s} = Z\left(m_0^{-s+1}u_{-s+1} + \cdots + m_{-s+1}^0 u_0\right), \quad s \in N \tag{2.11}$$

solves (2.5) and (2.6).

Now setting

$$B = -\partial_z + z, \qquad \bar{B} = \partial_z + z, \tag{2.12}$$

we consider the following operator

$$D = B^{h_1}\bar{B}^{k_1}B^{h_2}\bar{B}^{k_2}\ldots B^{h_l}\bar{B}^{k_l} \tag{2.13}$$

where $l, h_1, \ldots, h_l, k_1, \ldots, k_l$ are non negative integers.

Put

$$\nu = (k_1 + \cdots + k_l) - (h_1 + \cdots + h_l); \tag{2.14}$$

we call $v$ *index associate to* $D$. In [3] it has been proved (see Proposition 1.2) that

$$DTg = \mathrm{e}^{-2\nu\tau} TDg \quad \forall g \in C_0^\infty(R_z). \tag{2.15}$$

That being stated it is immediate to prove the following composition lemma:

LEMMA 1. – *Let* $a \in R$ *and* $p, q \in N_0$. *Then, if* $D$ *is an operator of the type* (2.13) *with index associate* $\nu$, *it results*:

$$Z\left(\tau^p \mathrm{e}^{a\tau} \partial_\tau^q DTg\right) = \left(\tau^p \mathrm{e}^{a\tau} * \mathrm{e}^{2\nu\tau}\right) \partial_\tau^q DTg \quad \forall g \in C_0^\infty(R_z), \tag{2.16}$$

*where*

$$(f * g)(\tau) = \int_0^\tau f(\tau - \tau')g(\tau')\,\mathrm{d}\tau'. \tag{2.17}$$

The following result holds:

THEOREM 2. – *For every* $s \in N$, *there is a distribution* $\Phi_{-s}(\tau, z, z'; x, \dot{\xi}, \omega)$ *in* $C^\infty(R_\tau^+ \times R_z \times R_{z'} \times R_x^n \times (R_n - \{0\})) \cap C^\infty(\bar{R}_\tau^+ \times R_z, D'(R_{z'}))$ *definable by recurrence from* $\Phi_0(\tau, z, z')$, *such that*

$$u_{-s}(\tau, z, x, \dot{\xi}, \omega) = \int_{-\infty}^{+\infty} \Phi_{-s}(\tau, z, z'; x, \dot{\xi}, \omega) g(z', \omega)\,\mathrm{d}z', \quad \tau > 0. \tag{2.18}$$

*Proof.* – By the structure of the operators $m_{-h}^{-s}$, from Lemma 2.1 and (2.11) it follows that the function $u_{-s}(\tau, z, x, \dot{\xi}, \omega)$, $\forall s \in N$, is finite sum of product of the type $c(x, \dot{\xi})\tau^p \mathrm{e}^{a\tau} \partial_\tau^q DTg$, with $D$ of the type (2.13), $p, q \in N_0$ and $a$ integer. This fact suggests to introduce a family $\mathcal{P}$ of operators:

$$P = P\left(\tau, \mathrm{e}^\tau, \mathrm{e}^{-\tau}, B, \bar{B}; x, \dot{\xi}\right) \tag{2.19}$$

where $P(\zeta_1, \ldots, \zeta_s; x, \dot{\xi})$ is a polynomial in $\zeta$, with $C^\infty$ coefficients depending on $x$ and $\dot{\xi}$. By using the composition lemma we have that for every $P \in \mathcal{P}$ there is a unique operator $P^{(*)} \in \mathcal{P}$ such that

$$ZPTg = P^{(*)}Tg \quad \forall g \in C_0^\infty(R_z). \tag{2.20}$$

Because $m_{-h}^{-s} \in \mathcal{P}$, $\forall s, h \in N_0$, put

$$\Phi_{-s} = \left(m_0^{-s+1}\right)^{(*)}\Phi_{-s+1} + \cdots + \left(m_{-s+1}^{0}\right)^{(*)}\Phi_0 \tag{2.21}$$

from (2.6) we obtain (2.18). □

Now we observe that, thanks to composition lemma, the application $P \to P^{(*)}$ keeps the parity respect to $z$. On the other hand the structure of $L$ implies that the operators $m_{-h}^{-s}$ are even respect to $z$; then, from (2.7) and (2.21) we have

$$\Phi_{-s}(\tau, z, z', x, \dot{\xi}, \omega) \equiv \Phi_{-s}(\tau, -z, -z', x, \dot{\xi}, \omega) \quad \forall s \in N_0. \tag{2.22}$$

That being stated, using Theorem 2.2 and (2.3) we have that the functions

$$\begin{aligned} \omega^s U_{-s}(t, y, x, \xi) &= \omega^{1/2} \mathrm{e}^{it\sum a_j(0,x)\xi_j} \\ &\quad \times \int_{-\infty}^{+\infty} \Phi_{-s}\left(t\omega, y\omega^{1/2}, y'\omega^{1/2}, x, \xi, \omega\right)\psi(y')\,\mathrm{d}y', \quad s \in N_0 \end{aligned} \tag{2.23}$$

are solutions of the transport problems. So, we have proved the following

THEOREM 3. – *For every $s \in N_0$, put*:

$$\begin{aligned} & k_{-s}(t, y, y', x, \xi) \\ &\quad = \omega^{1/2-s} \mathrm{e}^{it\sum a_j(0,x)\xi_j} \Phi_{-s}\left(t\omega, y\omega^{1/2}, y'\omega^{1/2}, x, \xi, \omega\right), \end{aligned} \tag{2.24}$$

*then* (1.10), (1.11) *and* (1.14) *are satisfied. Therefore the series* (1.9), *formed by symbols* (2.24), *gives a formal solution of Eq.* (1.5).

## 3. Estimates for the auxiliary problem

Let $g \in S(R_z)$ and let $f \in S(\bar{R}_\tau^+ \times R_z)$. Let $u \in S(\bar{R}_\tau^+ \times R_z)$ the solution of the problem (2.10) with data $g$ and $f$.

If $p, q$ are seminorms in $S(\bar{R}_\tau^+ \times R_z)$ and if $r$ is a seminorm in $S(R_z)$, the position:

$$p(u) \prec r(g) + g(f)$$

denotes the continuity of the operator $(f, g) \to u$ with respect to the seminorms $r, q, p$.

Now we put:

(3.1) $$[f] = \sup_{\bar{R}_\tau^+ \times R_z} \left|f(\tau, z)\right| \quad \forall f \in S(\bar{R}_\tau^+ \times R_z),$$

(3.2) $$[g] = \sup_{R_z} \left|g(z)\right| \quad \forall g \in S(R_z),$$

(3.3) $$\vartheta(z) = \left(1 + z^2\right)^{1/2},$$

and we prove the following

LEMMA 4. – *For every $h \in R$ it results*:

(3.4) $$\left[\theta^h u\right] \prec \left[\vartheta^h g\right] + \left[\vartheta^{h-2} f\right].$$

*Proof.* – We put

(3.5) $$u = (2 + \cos z)/\left(c^2 + z^2\right)^{h/2} w,$$

where $c$ is a positive number large enough to determine. The function $w \in S(\bar{R}_\tau^+ \times R_z)$ is solution of the problem

(3.6) $$\begin{cases} \partial_\tau w = \partial_z^2 w + b(z)\partial_z w - a(z) w \\ \qquad\qquad + f(\tau, z)(c^2 + z^2)^{h/2}, \quad \tau > 0, \\ w(0, z) = (c^2 + z^2)^{h/2}/(2 + \cos z) g(z), \end{cases}$$

where

(3.7) $$a(z) = z^2 + \cos z/(2 + \cos z) + h(2z \sin z + 1)/\left(c^2 + z^2\right) + h(h-2)z^2/\left(c^2 + z^2\right)^2.$$

Being:

$$z^2 + \cos z/(2 + \cos z) \geqslant \pi^2/9 - 1 \quad \forall z \in R_z$$

fixed $h$, it is possible to take $c$ so large that

(3.8) $$a(z) \geqslant C\left(1 + z^2\right) \quad \forall z \in R_z,$$

where $C > 0$. Then, by classical procedure, one proves that

$$[w] \prec \left[w(0, z)\right] + \left[f(\tau, z)\left(1 + z^2\right)^{h/2-1}\right]$$

so the thesis follows by (3.5) and (3.6). □

Now we introduce the seminorms with two indexes

$$[f]_{h,k} = \sum_{i=0}^{k} \left[\vartheta^{h-i} \partial_z^{k-i} f\right], \quad f \in S(\bar{R}_\tau^+ \times R_z),\ h \in R,\ k \in N_0, \tag{3.9}$$

$$[g]_{h,k} = \sum_{i=0}^{k} \left[\vartheta^{h-i} \partial_z^{k-i} g\right], \quad g \in S(R_z),\ h \in R,\ k \in N_0. \tag{3.10}$$

It is easy to prove that:

$$[f]_{h,k} \leqslant [f]_{h+r,k+r}, \quad [g]_{h,k} \leqslant [g]_{h+r,k+r}, \quad \forall r > 0. \tag{3.11}$$

Reasoning by induction on $k$, from Lemma 3.1 we have:

PROPOSITION 5. – *For every $h \in R$ and $k \in N_0$ it results*:

$$[u]_{h,k} \prec [g]_{h,k} + [f]_{h-2,k}. \tag{3.12}$$

Using the seminorms with three indexes

$$[f]_{h,k,p} = \sum_{p'=0}^{p} \left[\tau^{p'} f\right]_{h-2(p-p'),k} \tag{3.13}$$

one can prove the following:

PROPOSITION 6. – *For every $h \in R$, $k$, $p \in N_0$ it results*:

$$[u]_{h,k,p} \prec [g]_{h-2p,k} + [f]_{h-2,k,p}. \tag{3.14}$$

In order to be able to estimate the generic seminorm $[\tau^p \partial_\tau^q \vartheta^h \partial_z^k u]$ we must define at first the seminorms with four indexes:

$$[f; h, k, p, q] = \sum_{h',k',p',q'} \left[\partial_\tau^{q'} f\right]_{h',k',p'} \quad \forall f \in S(\bar{R}_\tau^+ \times R_z), \tag{3.15}$$

$$[g; h, k, p, q] = \sum_{h',k',p',q'} [g]_{h'-2p',k'+2q} \quad \forall g \in S(R_z), \tag{3.16}$$

where

$$0 \leqslant q' \leqslant q; \quad 0 \leqslant p' \leqslant p; \\ h' - 2p' + k' + 2q \leqslant h - 2p + k + 2q. \tag{3.17}$$

So (3.14) becomes:

$$[u; h, k, p, 0] \prec [g; h, k, p, 0] + [f; h-2, k, p, 0]. \tag{3.18}$$

Now we prove

PROPOSITION 7. – *For every $h \in R$, $k, p \in N_0$ it results*:

$$[u; h, k, p, 1] \prec [g; h, k, p, 1] + [f; h-2, k+2, p, 0], \tag{3.19}$$

*while*, $\forall q \in N$, *we have*:

$$[u; h, k, p, q] \prec [g; h, k, p, q] + [f; h-2, k+2, p, q-1]. \tag{3.20}$$

*Proof.* – We remark that if (3.17) holds it results

$$[f; h', k', p', q'] \prec [f; h, k, p, q], \qquad [g; h', k', p', q'] \prec [g; h, k, p, q].$$

That being stated, from equation in (2.10) we obtain

$$[\partial_\tau u]_{h,k,p} \prec [u]_{h,k+2,p} + [u]_{h+2,k,p} + [f]_{h,k,p}$$

from which, by initial remark:

$$\begin{aligned}[\partial_\tau u]_{h,k,p} &\prec [g; h, k+2, p, 0] + [f; h-2, k+2, p, 0] \\ &\quad + [f; h, k, p, 0] \\ &\prec [g; h, k, p, 1] + [f; h-2, k+2, p, 0]\end{aligned}$$

and so (3.19). Now, differentiating the equation in (2.10)

$$\begin{aligned}\left[\partial_\tau^2 u\right]_{h,k,p} &\prec [u; h, k+2, p, 1] + [u; h+2, k, p, 1] \\ &\quad + [f; h, k, p, 1] \\ &\prec [u; h, k+2, p, 1] + [f; h, k, p, 1],\end{aligned}$$

and using (3.19), we have

$$\begin{aligned}\left[\partial_\tau^2 u\right]_{h,k,p} &\prec [g; h, k+2, p, 1] + [f; h-2, k+4, p, 0] \\ &\quad + [f; h, k, p, 1] \\ &\prec [g; h, k, p, 2] + [f; h, k, p, q-1]\end{aligned}$$

and then (3.20) for $q = 2$. Reasoning by induction the thesis follows. □

Let $f = f(\tau, z; \dot\eta, x, \dot\xi, \omega) \in C^\infty(\bar R^+_\tau, S(R_z))$, infinitely differentiable with respect to the parameters:

$$x \in R^n; \quad (\dot\eta, \dot\xi) \in R \times \big(R_n - \{0\}\big); \quad \omega \in R^+.$$

Fixed $m \in R$ we denote by $I_m$ the space of the functions $f(\tau, z; \dot\eta, x, \dot\xi, \omega)$ such that:

$$\big[\tau^p z^h \partial_\tau^q \partial_z^k f\big] \prec \omega^{n+(h+k)/2+q} \quad \forall p, h, k \in N_0, \tag{3.21}$$

uniformly with respect to $(x, \dot\eta, \dot\xi)$ on the compact subsets of $R^n_x \times R \times (R_n - \{0\})$, and to $\omega$ on the sets of the type $\omega \geqslant a$ with $a > 0$.

The Proposition 3.4 gives

PROPOSITION 8. – *If*

$$\begin{cases} \vartheta^{-2}(\partial_\tau - \partial_z^2 + z^2)u(\tau, z; \dot\eta, \dot\xi, \omega) \in I_m, \\ u(0, z; \dot\eta, x, \dot\xi, \omega) = 0, \end{cases} \tag{3.22}$$

*then we have*

$$u \in I_m. \tag{3.23}$$

## 4. Estimates for transport problems

Let $I^m$ be the space of the functions

$$F(t, y, \eta, x, \xi) = f\big(t\omega, y\omega^{1/2}, \eta\omega^{1/2}, x, \xi/\omega, \omega\big), \quad f \in I_m, \tag{4.1}$$

and let $I = \bigcup_{m\in R} I^m$. It is necessary to point out:

$$F \in I^m \Rightarrow tF \in I^{m-1}, \quad yF \in I^m, \quad \partial_y F \in I^{m+1}. \tag{4.2}$$

Now we introduce the seminorms:

$$\begin{aligned} &\big[t^p y^h \partial_t^q \partial_y^k \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma F\big] \\ &\quad = \sup_{[0,T]\times R_y\times X} \ \sup_{|\eta|\leqslant a|\omega|^{1/2}} \big|t^p y^h \partial_t^q \partial_y^k \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma F(t, y, \eta, x, \xi)\big| \end{aligned} \tag{4.3}$$

where $a$ is a positive number and $X$ is a compact subset of $R^n_x$.

If $p(F)$ is a seminorm of the type (4.3), with

$$p(F) \lesssim |\xi|^\alpha, \quad \alpha \in R, \tag{4.4}$$

we denote that there is a constant $C$, independent of $\xi$, such that

$$p(F) \leqslant C|\xi|^a. \tag{4.5}$$

It is easy to prove that if $F \in I^m$ we have:

$$\left[t^p y^h \partial_t^q \partial_y^k F\right] \lesssim |\xi|^{m-p+2q+k}. \tag{4.6}$$

The following lemma holds

LEMMA 9. – *Let $U(t, y, \eta, x, \xi) \in I$. If*

$$\left(1 + y^2\omega\right)^{-1} M_1 U \in I^m, \qquad U(0, y, \eta, x, \xi) = 0$$

*we have $u \in I^{m-1}$ also.*

That being stated, let $\psi(y) \in C_0^\infty(R_y)$ and let $U_0(t, y, \eta, x, \xi)$ be the solution of the first transport problem with data $\mathrm{e}^{iy\eta}\psi(y)$. We have

PROPOSITION 10. – *The function $U_0(t, y, \eta, x, \xi)$ belongs to $I^0$.*

*Proof.* – By construction we have:

$$\begin{aligned} U_0(t, y, \eta, x, \xi) &= \mathrm{e}^{it\sum_j a_j(0,x)\xi_j} T\left(\mathrm{e}^{i\dot\eta z}\psi\left(z/\omega^{1/2}\right)\right)\left(t\omega, y\omega^{1/2}\right) \\ &= \mathrm{e}^{it\sum_j a_j(0,x)\xi_j} u_0\left(t\omega, y\omega^{1/2}\eta, \omega\right). \end{aligned} \tag{4.7}$$

Put $g(z, \omega) = \mathrm{e}^{i\dot\eta z}\psi(z/\omega^{1/2})$ we get

$$[g; h, k, p, q] \lesssim |\xi|^{\frac{h+k}{2}+q} c_\psi, \tag{4.8}$$

where $c_\psi$ denotes the seminorm on $C_0^\infty(R_y)$. By Proposition 3.4 the thesis follows. □

From (4.7) we have:

$$\partial_\eta^\gamma \partial_x^\alpha \partial_\xi^\beta U_0 \in I \quad \forall \alpha, \beta, \gamma, \tag{4.9}$$

and by Lemma 4.1, with inductive procedure, we obtain that:

$$\partial_\eta^\gamma \partial_x^\alpha \partial_\xi^\beta U_0 \in I^{-|\beta|} \quad \forall \alpha, \beta, \gamma. \tag{4.10}$$

From (4.2) and (1.8), as well as Proposition 4.2, we get that

$$\left(1 + y^2\omega\right)^{-1} M_{-s} U_0 \in I^{-s} \quad \forall s \in N_0. \tag{4.11}$$

Reasoning by induction on transport problems starting by Lemma 4.1, we get

PROPOSITION 11. – *Let* $\{U_{-s}(t, y, \eta, x, \xi)\}_{s \in N_0}$ *be the sequence of solutions of the transport systems, with data* $\mathrm{e}^{iy\eta}\psi(y)$*, such that* $U_{-s} \in I$*,* $\forall s \in N_0$*. Then we have*:

$$\partial_\eta^\gamma \partial_x^\alpha \partial_\xi^\beta U_{-s} \in I^{-|\beta|-s} \quad \forall s \in N_0. \tag{4.12}$$

## 5. A class of symbols

Let $k(t, y, y', x, \xi) \in C^\infty(]0, T] \times R_y \times R_{y'} \times R^n \times R_n)$ and let $m \in R$. We say that $k \in U^m$ if there is a function $\psi_0 \in C_0^\infty(R_y)$, with value 1 in a neighbourhood of manifold $y = 0$, such that putting $\forall r \in N_0$:

$$\begin{aligned} &\psi_r(y) = \psi_0\big(y/(r+1)\big), \\ &U^{(r)}(t, y, \eta, x, \xi) = \big\langle k(t, y, y', x, \xi), \mathrm{e}^{i\eta y'}\psi_r(y')\big\rangle, \end{aligned} \tag{5.1}$$

the functions $U^{(r)}$ extend to $C^\infty(\bar{\Omega}_T \times R_{n+1})$ and the estimates hold:

$$\begin{aligned} &\big[t^p y^h \partial_t^q \partial_y^k \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma U^{(r)}\big] \lesssim \big(1 + |\xi|\big)^{-p+2q+k-|\beta|+m}, \\ &\qquad \forall r \in N_0,\ \forall (p, h, q, k, \alpha, \beta, \gamma) \in N_0^{2n+5}. \end{aligned} \tag{5.2}$$

Now we put $U^\infty = \bigcup_m U^m$; $U^{-\infty} = \bigcap_m U^m$.

If $k \in U^\infty$ and (5.2) holds $\forall \psi_0 \in C_0^\infty(R_y)$, $k$ is a symbol on $\Omega_T$. So we can associate to $k$ the operator $K$ defined in (1.2). For every $\chi \in C^\infty(R_n)$ such that $\chi = 0$ for $|\xi| < \rho$ and $\chi = 1$ for $|\chi| > \rho'$ with $0 < \rho < \rho'$, we have that the functions $\chi(\xi)k_{-s}, k_{-s}$ has been constructed in Section 2, are symbols belonging to $U^{\frac{1}{2}-s}$, as one can deduce from the results of Section 4. However we can associate some local operators to the generic element $k \in U^\infty$.

Let $A \Subset \Omega$, $A$ open, and let $\psi_A \in \{\psi_r\}$ be a function with value 1 on the $y$-projection of $A$. We fix a function $\zeta \in C_0^\infty(R)$, with value 1 in a neighbourhood of zero and put:

$$\zeta(\eta,\xi) = \zeta\bigl(\eta^4/(1+|\xi|^2)\bigr). \tag{5.3}$$

If $k \in U^\infty$, we will say *local operator* associated to $K$, next

$$\begin{aligned} K_A : G \in C_0^\infty(\Omega) \to K_A G \\ = \int\limits_{R_{n+1}} \mathrm{e}^{ix\cdot\xi}\bigl\langle k(t,y,y',x,\xi), \mathrm{e}^{i\eta y'}\psi_A(y')\bigr\rangle \zeta(\eta,\xi)\tilde{G}(\eta,\xi)\,\bar{\mathrm{d}}\eta\,\bar{\mathrm{d}}\xi \end{aligned} \tag{5.4}$$

where $\tilde{G}(\eta,\xi) = \mathcal{F}_{y\to\eta}\mathcal{F}_{x\to\xi}(G(y,x))$.

It is easy to prove that, by (5.2) and by structure of function $\zeta$, we have $K_A G \in C^\infty(\bar{\Omega}_T)$, $\forall G \in C_0^\infty(\Omega)$. If $k$ is also a symbol, we have that $K_A G = KG$ $\forall G \in C_0^\infty(A)$, this is true also if in (5.4) one puts $\zeta = 1$.

Now we prove the following

THEOREM 12. – *If $k \in U^m$, then the local operator $K_A$ associated to $k$ extends as a linear continuous operator*

$$H^\sigma_{\mathrm{comp}}(\Omega) \to C^{q+k}\bigl([0,T]\times R_y, H_{\mathrm{loc}}^{\sigma-2q-k-m-(1/4)}\bigl(R^n\bigr)\bigr)$$

$\forall \sigma \in R$ *and* $\forall q,k \in N_0$.

*Proof.* – Fixed $\sigma, q, k$, we put $\sigma' = \sigma - 2q - k - m - (1/4)$. The thesis is equivalent to

$$\begin{aligned} \sup_{[0,T]\times R_y} \bigl\|\partial_t^q \partial_y^k(\varphi K_A G)\bigr\|_{H^{\sigma'}(R^n)} \leqslant C\|G\|_{H^\sigma(\Omega)} \\ \forall \varphi, G \in C_0^\infty(\Omega), \end{aligned} \tag{5.5}$$

where $C$ is a positive constant independent of $G$.

In order to prove (5.5), we put

$$U_A(t,y,\eta,x,\xi) = \bigl\langle k(t,y,y',x,\xi), \mathrm{e}^{i\eta y'}\psi_A(y')\bigr\rangle$$

and

$$V(t,y,\eta,x,\xi) = \varphi(y,x)\zeta(\eta,\xi)U_A(t,y,\eta,x,\xi) \tag{5.6}$$

such that

$$\varphi K_A G = \int_{R_{n+1}} e^{ix\cdot\xi} V(t, y, \eta, x, \xi)\tilde{G}(\eta, \xi)\, \bar{d}\eta\, \bar{d}\xi.$$

From (5.6), by structure of $\zeta(\eta, \xi)$ we have that the function $V$ satisfies (5.2). Being $V \in C_0^\infty(\Omega, C^\infty([0, T] \times R_{n+1}))$, by a famous theorem about direct product (see [11]), we have

$$V(t, y, \eta, x, \xi) = \sum_{j=0}^{+\infty} \lambda_j \Phi_j(x) V_j(t, y, \eta, \xi), \tag{5.7}$$

where $\sum_j |\lambda_j| < +\infty$, $\{\phi_j(x)\}_{j\in N_0}$ is a bounded sequence of $C_0^\infty(R^n)$, the functions $V_j(t, y, \eta, \xi)$ satisfy (5.2) uniformly respect to $j$ and the $y$-projections of their supports are in the same compact of $R_y$. So, we can suppose $V$ independent of $x$. In that case we have:

$$\mathop{\mathcal{F}}_{x\to\xi}\left(\partial_t^q \partial_y^k \varphi K_A G\right) = \int_R \partial_t^q \partial_y^k V(t, y, \eta, \xi)\tilde{G}(\eta, \xi)\, \bar{d}\eta.$$

Being the support of $V$ included in a region of the type $|\eta| < a|\xi|^{1/2}$, $a > 0$, from (5.2) we deduce:

$$\begin{aligned}&\left(1 + |\xi|\right)^{\sigma'} \Big|\mathop{\mathcal{F}}_{x\to\xi} \partial_t^q \partial_y^k \varphi K_A G\Big| \\ &\quad\lesssim \left(1 + |\xi|\right)^{\sigma'+2q+k+m+(1/4)} \left(\int_R \left|\zeta(\eta, \xi)\right|^2 \left|\tilde{G}(\eta, \xi)\right|^2 \bar{d}\eta\right)^{1/2}.\end{aligned}$$

Since on the support of $\zeta(\eta, \xi)$ results $(1 + |\xi|) \cong (1 + |\xi| + |\eta|)$, from the last inequality the (5.5) follows. □

From this theorem we have

THEOREM 13. – *If $k \in U^\infty$, then the operators $K_A$ are linear and continuous*:

$$E'(\Omega) \to C^\infty\left([0, T] \times R_y, D'\left(R^n\right)\right).$$

*If $k \in U^{-\infty}$, then the operators $K_A$ are regularizing, that is they are linear and continuous from $E'(\Omega)$ to $C^\infty(\bar{\Omega}_T)$.*

Using (5.2) for the function $V$, by well-known procedure, one can prove the next

THEOREM 14. – *If $k \in U^\infty$, then the operators $K_A$ are pseudolocal. This means that if $G \in E'(\Omega) \cap C^\infty(B)$ then $K_A G \in C^\infty([0,T] \times B)$, where $B$ is an open subset of $\Omega$.*

## 6. Construction of a Poisson operator

We consider the formal series (1.7), built by symbols $k_{-s}$ of Section 3. For every $s, r \in N_0$, we put

$$U_{-s}^{(r)}(t,y,\eta,x,\xi) = \langle k_{-s}(t,y,y',x,\xi), \mathrm{e}^{iy'\eta}\psi_r(y')\rangle. \tag{6.1}$$

Let $\{X_s\}_{s\in N_0}$ be a sequence of compact covering $R^n$. However said about $k_{-s}$, from (5.2) we have that, $\forall s \in N_0$, there is a positive constant $C_s$ such that

$$\sup\left|t^p y^h \partial_t^q \partial_y^k \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma U_{-s}^{(r)}\right| \leqslant C_s |\xi|^{-p+2q+k-|\beta|-s+(1/2)} \tag{6.2}$$

respect to

$$t \in [0,T]; \quad y \geqslant 0; \quad x \in X_s; \quad |\eta| \leqslant |\xi|^{1/2}/(s+1), \tag{6.3}$$

and to:

$$r \leqslant s; \quad p+h+q+k+|\alpha|+|\beta|+\gamma \leqslant s. \tag{6.4}$$

On the other hand, by structure of symbols $k_{-s}$, it is easy to prove that is possible to choose the constant $C_s$ such that

$$\sup\left|t^p y^h y'^{h'} \partial_t^q \partial_y^k \partial_{y'}^{k'} \partial_x^\alpha \partial_\xi^\beta k_{-s}(t,y,y',x,\xi)\right| \leqslant C_s |\xi|^{-s} \tag{6.5}$$

respect to:

$$t \in [1/(s+1), T]; \quad y \geqslant 0; \quad y' \geqslant 0; \quad x \in X_s. \tag{6.6}$$

Now we denote by $\chi \in C^\infty$ a function equal to zero in $[-1,1]$ and equal to 1 outside of the interval $(-2,2)$. We put:

$$\rho_s = 2^s C_s. \tag{6.7}$$

In view of the construction, by (6.2), (6.3) and (6.4), we have $\forall s \in N_0$

$$\sup\left|t^p y^h \partial_t^q \partial_y^k \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \chi(|\xi|/\rho_s) U_{-s}^{(r)}\right| \leqslant C 2^{-s} |\xi|^{-p+2q+k-|\beta|+(1/2)}, \tag{6.8}$$

where $C$ is independent of $s$; while by (6.5) and (6.6) we have that there is $\bar{s} \in N_0$ such that, $\forall s \geqslant \bar{s}$:

$$\sup\left|t^p y^h y'^{h'} \partial_t^q \partial_y^k \partial_{y'}^{k'} \partial_x^\alpha \partial_\xi^\beta \chi(|\xi|/\rho_s) k_{-s}\right| \leqslant C 2^{-s}. \tag{6.9}$$

In view of this fact, with the position:

$$k'(t, y, y', x, \xi) = \sum_{s=0}^{+\infty} \chi(|\xi|/\rho_s) k_{-s}(t, y, y', x, \xi) \tag{6.10}$$

we define a function belonging to $U^{1/2}$.

Using the same function $\zeta(\eta, \xi)$ and the open $A \Subset \Omega$ introduced in Section 5, we denote with $K'_A$ the local operators associated to $k'(t, y, y', x, \xi)$ by (5.4). From Theorem 5.1 follows that $K'_A$ is linear and continuous from $E'(\Omega)$ to $C^\infty([0, T] \times R_y, D'(R^n))$.

We prove

THEOREM 15. – *For every $G \in E'(\Omega)$ we have*

$$L K'_A G \in C^\infty(\bar{\Omega}_T).$$

*Proof.* – At first we observe that the operators $L K'_A$ are local operators associated to $Mk'$. By Theorem 5.2 it is necessary to prove that $Mk' \in U^{-\infty}$.

Fixed $h \in N$ we observe that by construction it results

$$M = \sum_{s=1}^{h} M_{-s} + t^h M_h^*, \tag{6.11}$$

where $M_h^*$ is an operator with $C^\infty$ coefficients. So we have

$$Mk' - \sum_{s=-1}^{h} M_{-s} k' \in U^{-h+1/2}. \tag{6.12}$$

That being stated, put $\chi_s(\xi) = \chi(|\xi|/\rho_s)$, we have:

$$\sum_{s=-1}^{h} M_{-s}k' = \sum_{s=0}^{h+1} M_{1-s}k' = \sum_{s=0}^{h+1} M_{1-s} \sum_{s'=0}^{h+1-s} \big(\chi_{s'}(\xi) - 1\big)k_{-s'}$$
$$+ \sum_{s=0}^{h+1} \sum_{s+s' \leqslant h+1} M_{1-s}k_{-s'}$$
$$+ \sum_{s=0}^{h+1} M_{1-s} \sum_{s'=h-s+2}^{\infty} \chi_{s'}(\xi)k_{-s'}$$
$$= h^{(1)} + h^{(2)} + h^{(3)}.$$

It is clear that $h^{(1)} \in U^{-\infty}$, because it is a function equal to zero for $\xi$ large enough. The function $h^{(2)}$ is equal to zero by (1.10), while it is easy to prove that $h^{(3)} \in U^{-h+1/2}$. From (6.12) we have that $Mk' \in U^{-h+1/2}$. The thesis follows because $h$ is arbitrary. □

Fixed $G \in E'(\Omega)$, and put

$$G_1(y,x) = \mathop{\mathcal{F}^{-1}}_{\eta\to y} \mathop{\mathcal{F}^{-1}}_{\xi\to x} \zeta\big(\eta^4/(1+|\xi|^2)\big)\tilde{G}(\eta,\xi) \in S'(R_{n+1}),$$

we have the next

THEOREM 16. – *For every* $G \in E'(\Omega)$ *such that* $\operatorname{supp} G \subset A$, *it results*:

(6.13) $$K'_A G(0,y,x) - G_1(y,x) \in C^\infty(\Omega).$$

*Proof.* – Using the transport systems one can prove that

(6.14) $$K'_A G(0,y,x) - \psi_A(y)G_1(y,x) \in C^\infty(\Omega).$$

Being $\psi_A(y) = 1$ on the $y$-projection of $A$, from (6.14) follows that the left hand in (6.14) belongs to $C^\infty(A)$. On the other hand, if $\operatorname{supp} G \subset A' \subset A$, we have that $G = 0$ in $\Omega - A'$; the thesis follows by the pseudo local theorem. □

## 7. A second process of homogenization

We introduce a new definition of pseudo-homogeneity. Let $a(t,y,\eta,x,\xi)$ a function belongs to $C^\infty(\bar{\Omega}_T \times (R_{n+1} - \{0\}))$ and let $m \in R$. We will say that $a$ is a *pseudo-homogeneous symbol of degree $m$* if, $\forall\lambda > 0$, it results:

$$a\left(t/\lambda^2, y, \eta\lambda, x, \xi\lambda\right) \equiv \lambda^m a(t, y, \eta, x, \xi). \tag{7.1}$$

As in Section 1, let $O$ be an operator that leaves unchanged the class of pseudo-homogeneous symbols. We will say that $O$ has pseudo-order $h$ if, $\forall m \in R$, it transforms pseudo-homogeneous symbols of degree $m$ in orders of degree $m + h$. In particular the operators $t, \partial_t, \partial_\xi, \partial_\eta$ have pseudo-order respectively equal to $-2, 2, -1, -1$.

We now construct $k''(t, y, \eta, x, \xi)$ as a formal series of pseudo-homogeneous symbols such that put:

$$\begin{aligned} K'' : G(y, x) \in C_0^\infty(\Omega) &\to K''G \\ &= \int\limits_{R_{n+1}} \mathrm{e}^{i(x\cdot\xi+y\eta)} k''(t, y, \eta, x, \xi)\tilde{G}(\eta, \xi)\, \bar{\mathrm{d}}\eta\, \bar{\mathrm{d}}\xi \end{aligned} \tag{7.2}$$

the function $K''G$ is a solution of the problem (1.4), $\forall G \in C_0^\infty(\Omega)$.

Putting

$$N(t, y, x, \xi, \eta, \partial_t, \partial_y, \partial_x) = L(t, y, x, \partial_t, i\eta + \partial_y, \cdots i\xi_j + \partial_{x_j} \cdots) \tag{7.3}$$

and reasoning as in Section 1, one can prove that this is obtained if and only if it results

$$Nk''(t, y, \eta, x, \xi) = 0. \tag{7.4}$$

Using the Mac Laurin series expansion with respect to the variable $t$ of the coefficients of $L$, it is possible to exhibit $N$ as follows:

$$N = \sum_{h=-2}^{+\infty} N_{-h} \tag{7.5}$$

where, $\forall h \geqslant -2$, $N_{-h}$ has pseudo-order $-h$. Developing one obtains

$$N_2 = \partial_t + \eta^2 + y^2 \sum_{i,j} a_{ij}(0, x)\xi_i\xi_j, \tag{7.6}$$

$$\begin{aligned} N_1 = -2i\eta\partial_y - 2iy^2 \sum_{i,j} a_{ij}(0, x)\xi_i\partial_{x_j} + iyb(0, x)\eta \\ + i\sum_j a_j(0, x)\xi_j, \end{aligned} \tag{7.7}$$

$$N_0 = -\partial_y^2 + y^2 \sum_{i,j} \bigl( t\partial_t a_{ij}(0,x)\xi_i\xi_j - a_{ij}(0,x)\partial_{x_i}\partial_{x_j} \bigr) + yb(0,x)\partial_y + \sum_j a_j(0,x)\partial_{x_j} + c(0,x) \tag{7.8}$$

and also, for $r \in N_0$

$$N_{-2r-1} = \frac{t^{r+1}}{(r+1)!} \Bigl[ -2iy^2 \sum_{i,j} \partial_t^{r+1} a_{ij}(0,x)\xi_i\partial_{x_j} + iy\partial_t^{r+1} b(0,x)\eta + i\sum_j \partial_t^{r+1} a_j(0,x)\xi_j \Bigr] \tag{7.9}$$

$$N_{-(2r+2)} = \frac{t^{r+2}}{(r+2)!} y^2 \sum_{i,j} \partial_t^{r+2} a_{ij}(0,x)\xi_i\xi_j + \frac{t^{r+1}}{(r+1)!} \times \Bigl( -y^2 \sum_{i,j=1}^{n} \partial_t^{r+1} a_{ij}(0,x)\partial_{x_i}\partial_{x_j} + y\partial_t^{r+1} b(0,x)\partial_y + \sum_{j=1}^{n} \partial_t^{r+1} a_j(0,x)\partial_{x_j} + \partial_t^{r+1} c(0,x) \Bigr). \tag{7.10}$$

We suppose that the symbol $k''$ has the form

$$\sum_{h=0}^{+\infty} k_{-h}(t,y,\eta,x,\xi), \tag{7.11}$$

where $k_{-h}$, $\forall h \in N_0$, is a symbol pseudo-homogeneous of degree $m-h$, where $m$ is a real number to determine. Reasoning as in Section 1 and using (7.2) we arrive to the following transport systems:

$$\begin{cases} N_2 k_0 = 0 & (t,y,\eta,x,\xi) \in \Omega_T \times (R_{n+1} - \{0\}), \\ k_0(0,y,\eta,x,\xi) = 1 & (y,\eta,x,\xi) \in \Omega_T \times (R_{n+1} - \{0\}), \end{cases} \tag{7.12}$$

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$$\begin{cases} N_2 k_{-h} + N_1 k_{-h+1} + \cdots + N_{-h+2} k_0 = 0, \\ k_{-h}(0,y,\eta,x,\xi) = 0. \end{cases} \quad h > 0 \tag{7.13}$$

By virtue of (7.6) and (7.12) we have

$$k_0(t,y,\eta,x,\xi) = \mathrm{e}^{-(\eta^2+\omega^2 y^2)t} \tag{7.14}$$

and so $k_0$ is a pseudo-homogeneous symbol of degree zero. Reasoning by recurrence one can prove that

$$k_{-h}(t,y,\eta,x,\xi) = p_h(t,y,\eta,x,\xi)\mathrm{e}^{-(\eta^2+\omega^2y^2)t}, \quad h \in N, \tag{7.15}$$

where $p_h$ is a polynomial in $(t,\eta,\xi)$, with coefficients $C^\infty(\Omega)$, pseudo-homogeneous of degree $-h$, null for $t = 0$.

Let $X$ be a compact subset of $R^n$, by (7.14) and (7.15) we have

$$\sup_{[0,T]\times X} \left|t^p \partial_t^q \partial_y^k \partial_x^\alpha \partial_\eta^\gamma \partial_\xi^\beta k_{-h}\right| \lesssim \left(\eta^2 + |\xi|^2 y^2\right)^{-p+q-\frac{|\beta|}{2}-\frac{\gamma}{2}-\frac{k}{2}-\frac{h}{2}} |\xi|^k. \tag{7.16}$$

Let $Y$ be a compact subset of $R_y$, we put

$$\left]a(t,y,\eta,x,\xi)\right[ = \sup\left|a(t,y,\eta,x,\xi)\right| \tag{7.17}$$

respect to

$$t \in [0,T], \quad x \in X, \quad y \in Y, \quad |\xi|^{1/2} < c|\eta|, \quad c > 0. \tag{7.18}$$

By (7.16) we have

$$\left]t^p \partial_t^q \partial_y^k \partial_x^\alpha \partial_\eta^\gamma \partial_\xi^\beta k_{-h}\right[ \lesssim \eta^{-2p-|\beta|-\gamma+k-h+4q} \quad \forall (p,k,\alpha,\beta,\gamma) \in N_0^{3+2n}. \tag{7.19}$$

We assume (7.19) as definition of space $V^{-h}$, the meaning of $V^\infty$ and $V^{-\infty}$ is clear.

Let $k \in V^\infty$, let $\zeta(\eta,\xi)$ be the function introduced in Section 5. We put

$$\begin{aligned} G_2(y,x) &= \mathop{\mathcal{F}^{-1}}_{\eta\to y} \mathop{\mathcal{F}^{-1}}_{\xi\to x}\left(\left(1-\zeta(\eta,\xi)\right)\tilde{G}(\eta,\xi)\right) \\ &= G(y,x) - G_1(y,x), \end{aligned}$$

$\forall G \in E'(\Omega)$ and we consider the operator

$$\begin{aligned} G \in C_0^\infty(\Omega) \to KG_2 \\ = \int_{R_{n+1}} \mathrm{e}^{i(x\cdot\xi+y\eta)} k(t,y,\eta,x,\xi)\left(1-\zeta(\eta,\xi)\right)\tilde{G}(\eta,\xi)\,\bar{\mathrm{d}}\eta\,\bar{\mathrm{d}}\xi. \end{aligned} \tag{7.20}$$

By virtue of (7.19) we have that this operator has value in $C^\infty(\bar{\Omega}_T)$. Then, reasoning as in Section 5, one can prove that this operator extends as a linear and continuous operator from $E'(\Omega)$ to $C^\infty([0,T], D'(\Omega))$. If $k \in V^{-\infty}$, this operator is regularizing. By the same technique of Section 6, one proves that there is a diverging sequence $\{\rho_h\}_{h\in N_0}$ of positive number such that the series

$$\sum_{h=0}^{+\infty} \chi\left(|\eta|/\rho_h\right) k_{-h}(t,y,\eta,x,\xi)$$

converges to a symbol $k'' \in V^0$, for which we have

THEOREM 17. – *For every $G \in E'(\Omega)$ results*

$$LK''G_2 \in C^\infty(\bar{\Omega}_T), \tag{7.21}$$

$$K''G_2(0,y,x) - G_2(y,x) \in C^\infty(\Omega). \tag{7.22}$$

Now we are able to construct the Poisson operators for the problems (0.4) and (0.5). Let $A$ be an open of $\Omega^+$, $A \Subset \Omega^+$, and let $K'_A$ be the operator built in Section 6. For (1.14) we define the operators

$$K_A^{(1)} : G \in C_0^\infty(\Omega^+) \to K_A^{(1)}G = K'_{A\cup(-A)}(G_d)_1 + K''(G_s)_2,$$
$$K_A^{(2)} : G \in C_0^\infty(\Omega^+) \to K_A^{(2)}G = K'_{A\cup(-A)}(G_p)_1 + K''(G_p)_2,$$

where $-A = \{(y,x):\ (-y,x) \in A\}$.

It is clear that $K_A^{(i)}G$, $i = 1,2$, belong to $C^\infty(\bar{\Omega}_T^+)$. So, however said we have the following

THEOREM 18. – *For every open $A$, $A \Subset \Omega^+$, the operators $K_A^{(i)}$, $i = 1,2$, extend as linear and contnuous operators*:

$$E'(\Omega^+) \to C^\infty([0,T], D'(\Omega^+)) \cap C^\infty\left(]0,T] \times [0,+\infty[ \times \mathbb{R}^n\right)$$

*and*

$$LK_A^{(i)}G \in C^\infty(\bar{\Omega}_T^+), \quad i = 1,2,$$
$$K_A^{(i)}G(0,y,x) - G(y,x) \in C^\infty(\bar{\Omega}^+), \quad \forall G \textit{ with } \operatorname{supp} G \subset A,$$
$$K_A^{(i)}G(t,0,x) = 0, \quad \partial_y K_A^{(2)}G(t,0,x) = 0, \quad \forall t \in ]0,T].$$

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