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Sequences that omit a box (modulo 1)

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Abstract

Let $\mathcal{S} = (a_j)_{j=1}^{\infty}$ be a strictly increasing sequence of real numbers satisfying

$$a_{j+1} - a_j \geqslant \sigma > 0. \tag{0.1}$$

For an open box I in $[0, 1)^d$, we write

$$E_I^{(d)}(\mathcal{S}) = \{\boldsymbol{x} \in \mathbb{R}^d\colon\ a_j\boldsymbol{x} \notin I \pmod 1 \text{ for } j \geqslant 1\}.$$

It is shown that the Hausdorff dimension of $E_I^{(d)}(\mathcal{S})$ is $d-1$ whenever

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = 1.$$

The case $d = 1$ is due to Boshernitzan. The proof builds on his approach.

Now let $\mathcal{S}_1, \ldots, \mathcal{S}_d$ be strictly increasing in $\mathbb{N}$. Define $E_1' = E_I'(\mathcal{S}_1, \ldots, \mathcal{S}_d)$ to be the set of x in $[0, 1)$ for which

$$x(n_1, \ldots, n_r) \notin I \pmod 1 \quad \text{for } n_j \in \mathcal{S}_j,\ n_1 < \cdots < n_d.$$

A sequence $\mathcal{S}$ is said to fulfill condition $D(C)$ if it contains

$$B_r = [u_r, v_r] \cap \mathcal{S}$$

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for which $v_r - u_r \to \infty$ and

$$1 + v_r - u_r \leqslant C\#(B_r).$$

Kaufman has shown that E'_I is countable whenever $\mathcal{S}_1, \ldots, \mathcal{S}_d$ fulfill condition $D(C)$. Here it is shown that E'_I is finite under this hypothesis. An upper bound for $\#(E'_I)$ is provided.

Keywords: Distribution modulo one; Hausdorff dimension; Hausdorff metric; Granular set; Exponential sums

1. Introduction

Let $\mathcal{S} = (a_j)_{j=1}^{\infty}$ be a strictly increasing sequence of real numbers satisfying

$$a_{j+1} - a_j \geqslant \sigma > 0 \quad (j = 1, 2, \ldots). \tag{1.1}$$

For an open interval I in $[0, 1)$ of length $|I|$, we write

$$E_I(\mathcal{S}) = \{x \in \mathbb{R}: \; a_j x \notin I \pmod 1 \;\text{ for } j \geqslant 1\}.$$

If $\mathcal{S}$ is a sequence in the natural numbers $\mathbb{N}$, then $E_I(\mathcal{S})$ is periodic, and we write

$$E'_I(\mathcal{S}) = E_I(\mathcal{S}) \cap [0, 1).$$

It is a weak consequence of Weyl's work [17] on uniform distribution (mod 1) that $E_I(\mathcal{S})$ has zero Lebesgue measure. It is natural to ask for conditions on $\mathcal{S}$ that will force $E_I(\mathcal{S})$ to be 'smaller' than this, in some sense. The strongest conclusion is obtained when $\mathcal{S} \subseteq \mathbb{N}$ and

$$a_j \leqslant Cj \tag{1.2}$$

for infinitely many j, for some constant C. Both Kahane [9] and Amice [1] found that $E'_I(\mathcal{S})$ is finite in this case. An explicit estimate is given by Baker, Coatney and Harman [2]:

$$\#E'_I(\mathcal{S}) \leqslant \min\left(\frac{288C}{|I|^3}, \frac{144(C \log(2e/|I|))^2}{|I|^2}\right).$$

Here $\#S$ denotes the number of elements in a finite set S. This is close to a sharp bound for $|I|$ tending to 0, as explained in [2].

If we make the hypothesis

$$a_j = O(j^p) \tag{1.3}$$

for some $p > 1$, then the Hausdorff dimension of $E_I(\mathcal{S})$ satisfies

$$\dim E_I(\mathcal{S}) \leqslant 1 - \frac{1}{p},$$

where ‘dim’ denotes Hausdorff dimension. Salem [16] has shown that $(a_j x)_{j=1}^{\infty}$ is uniformly distributed (mod 1) except for a set of x having dimension $\leqslant 1 - 1/p$; see [2] for further results of this kind. I conjecture that $\dim E_I(\mathcal{S}) = 0$ when (1.3) holds, and that there are sequences $\mathcal{S}$ in $\mathbb{N}$ for every $p > 1$ that satisfy (1.3), for which $E_I(\mathcal{S})$ is uncountable for some I.

One reason for believing the first part of the conjecture is that Boshernitzan [5] has proved such a result for real sequences that may grow much more rapidly. He shows that

$$\dim E_I(\mathcal{S}) = 0$$

whenever

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = 1. \tag{1.4}$$

This contrasts neatly with results from de Mathan [6,7] and Pollington [15]: if

$$\liminf_{j\to\infty} \frac{a_{j+1}}{a_j} > 1,$$

then $\dim E_I(\mathcal{S}) = 1$ for a suitably chosen interval I.

How are we to extend the Kahane–Amice and Boshernitzan results to higher dimensions? Take $I = I_1 \times \cdots \times I_d$ to be an open box in $[0, 1)^d$. We write

$$E_I^{(d)}(\mathcal{S}) = \{\boldsymbol{x} \in \mathbb{R}^d \colon a_j \boldsymbol{x} \notin I \pmod 1 \ \text{ for } j = 1, 2, \dots\}.$$

We extend Boshernitzan's result as follows.

Theorem 1. *Under the hypothesis* (1.4), *we have*

$$\dim E_I^{(d)}(\mathcal{S}) = d - 1.$$

Of course the lower bound

$$\dim E_I^{(d)}(\mathcal{S}) \geqslant d - 1$$

is immediate, since

$$(0, x_2, \dots, x_d) \in E_I^{(d)}(\mathcal{S})$$

for every $x_2, \dots, x_d$. The corresponding upper bound will be proved in Section 3.

Kaufman [11] gave an alternative way to obtain a result in higher dimensions as follows. Let $\mathcal{S}_1, \dots, \mathcal{S}_d$ be sequences in $\mathbb{N}$ satisfying (1.1) and again let I be an open box in $[0, 1)^d$. We define $E'_I(\mathcal{S}_1, \dots, \mathcal{S}_d)$ to be the set of x in $[0, 1)$ for which

$$x(n_1, \dots, n_d) \notin I \pmod 1 \quad \text{for } n_j \in \mathcal{S}_j,\ n_1 < \cdots < n_d.$$

Kaufman proves an analog of the Kahane–Amice result. A sequence $\mathcal{S}$ in $\mathbb{N}$ is said to **fulfill condition** $D(C)$ if it contains a sequence of blocks

$$B_r = [u_r, v_r] \cap \mathcal{S}, \quad 1 \leqslant u_r < v_r \tag{1.5}$$

for which $v_r - u_r \to \infty$ and

$$1 + v_r - u_r \leqslant C\#(B_r).$$

His result is that whenever $\mathcal{S}_1, \ldots, \mathcal{S}_d$ all satisfy condition $D(C)$, $E'_I(\mathcal{S}_1, \ldots, \mathcal{S}_d)$ is countable.

We strengthen this as follows:

Theorem 2. *Suppose that $\mathcal{S}_j$ is a sequence in $\mathbb{N}$ that satisfies condition $D(C_j)$ $(j = 1, \ldots, d)$ Then $E'_I(\mathcal{S}_1, \ldots, \mathcal{S}_d)$ is finite. In fact,*

$$\#E'_I(\mathcal{S}_1, \ldots, \mathcal{S}_d) \leqslant \frac{18d(5d)^{2d}}{(|I_1| \ldots |I_d|)^2} \max_j \frac{C_j}{|I_j|},$$

where $I = I_1 \times \cdots \times I_r$.

2. Some lemmas

We write $\mathcal{B}_p(K_1, \ldots, K_p)$ for the set of lattice points $\boldsymbol{\ell}$ in $\mathbb{Z}^p$ with $|\ell_i| \leqslant K_i$ $(1 \leqslant i \leqslant p)$. Let

$$\mathcal{B}_p(K) = \mathcal{B}_p(K, \ldots, K), \qquad \mathcal{B}_p^*(K) = \mathcal{B}_p(K) \backslash \{\mathbf{0}\},$$

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \cdots + x_d y_d, \quad |\boldsymbol{x}| = (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2},$$

$$e(\theta) = e^{2\pi i \theta}, \quad \|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n|.$$

Lemma 1. *Let $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_M \in \mathbb{R}^p$, $\boldsymbol{\xi}_m = (\xi_{m1}, \ldots, \xi_{md})$. Let $\epsilon_i > 0$ $(i = 1, \ldots, p)$. Suppose that*

$$\max_{1 \leqslant i \leqslant p} \frac{\|\xi_{mi}\|}{\epsilon_i} \geqslant 1 \quad (m = 1, \ldots, M).$$

Then

$$M \leqslant 3 \sum_{\substack{\boldsymbol{\ell} \in \mathcal{B}_p(p\epsilon_1^{-1}, \ldots, p\epsilon_p^{-1}) \\ \boldsymbol{\ell} \neq \mathbf{0}}} \left| \sum_{m=1}^{M} e(\boldsymbol{\ell} \cdot \boldsymbol{\xi}_m) \right|.$$

Proof. This is Corollary 2 in Barton, Montgomery and Vaaler [3]. □

Lemma 2. *Let $x_1, \ldots, x_u$ be distinct points of* $[0, 1)$. *Then*

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{v=k}^{2N+k} \left| \sum_{s=1}^{u} b_s\, e(v x_s) \right|^2 = \sum_{s=1}^{u} |b_s|^2$$

uniformly in k.

Proof. By a variant of a theorem of Wiener given by Katznelson [10, p. 47], we have

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{v=k}^{2N+k}\left|\hat{\mu}(v)\right|^2=\sum_{\tau}\left|\mu\{\tau\}\right|^2$$

uniformly in k, for any complex measure μ on $[0, 1)$. Here

$$\hat{\mu}(v)=\int\limits_{[0,1)} e^{-ivt}\,d\mu(t).$$

We obtain the lemma by taking μ to be the measure

$$\mu(E)=\sum_{x_s\in E}\bar{b}_s. \qquad \square$$

For the next two lemmas, we recall some notations from the theory of Hausdorff measures. More details can be found in Falconer [8].

The **diameter** of a nonempty set W in $\mathbb{R}^d$ is

$$|W|=\sup\left\{|\boldsymbol{x}-\boldsymbol{y}|:\ \boldsymbol{x},\boldsymbol{y}\in W\right\}.$$

(This is consistent with our use of $|I|$ as the length of an interval I.)

Let E be a subset of $\mathbb{R}^d$ and $s>0$. For $\delta>0$, define

$$\mathcal{H}^s_\delta(E)=\inf\sum_i|W_i|^s$$

where the infimum is over all sequences of sets (W_i) of diameter $\leqslant\delta$ that cover E. Now

$$\mathcal{H}^s(E)=\lim_{\delta\to\infty}\mathcal{H}^s_\delta(E)$$

is the **Hausdorff s-dimensional outer measure** of E. The restriction of $\mathcal{H}^s$ to a certain σ-field containing the Borel sets is a positive measure on $\mathbb{R}^d$, **Hausdorff s-dimensional measure**. For any E, there is a unique value, $\dim E$, called the **Hausdorff dimension** of E, such that

$$\mathcal{H}^s(E)=\infty\quad\text{if } 0\leqslant s<\dim E,\qquad \mathcal{H}^s(E)=0\quad\text{if } \dim E<s<\infty.$$

For any subset W of $\mathbb{R}^d$ and $\boldsymbol{x}\in\mathbb{R}^d$, we write $W_{\boldsymbol{x}}$ for the translate

$$W_{\boldsymbol{x}}=\{\boldsymbol{w}+\boldsymbol{x}:\ \boldsymbol{w}\in W\}.$$

For a subspace V of $\mathbb{R}^d$, we write $V^\perp$ for the orthogonal complement of V.

Lemma 3. *Let E be a closed subset of $\mathbb{R}^d$ with $\mathcal{H}^s(E)=\infty$. For every $c>0$, there is a compact subset F of E such that $\mathcal{H}^s(F)=c$.*

Proof. See [8, Theorem 5.4]. □

For the next lemma, we need to specify a measure $\gamma_{d,m}$ on the space $G(d,m)$ of all m-dimensional linear subspaces of $\mathbb{R}^d$. For more details, see Mattila [13]. Let $O(d)$ be the orthogonal group of $\mathbb{R}^d$ and let θ_d be the unique Haar measure on $O(d)$ such that

$$\theta_d\big(O(d)\big) = 1.$$

Fix $V \in G(d,m)$; we define the measure $\gamma_{d,m}$ on $G(d,m)$ as follows.

$$\gamma_{d,m}(B) = \theta_d\big(\big\{g \in O(d)\colon\ g(V) \in B\big\}\big).$$

This measure is independent of the choice of V.

Lemma 4. *Let f be a natural number and s a real number such that $f < s < d$. Let A be a Borel set in $\mathbb{R}^d$ with $0 < \mathcal{H}^s(A) < \infty$. Then for almost all $(d-f)$-dimensional subspaces V with respect to $\gamma_{d,d-f}$,*

$$\mathcal{H}^f\big(\big\{\boldsymbol{a} \in V^\perp\colon\ \dim(A \cap V_{\boldsymbol{a}}) = \dim A - f\big\}\big) > 0.$$

This was proved by Marstrand [12] in the planar case. The general case of Lemma 4 is due to Mattila [13].

Let Z be a compact metric space and $d_Z(\cdot,\cdot)$ the associated metric. For nonempty $A \subseteq Z$, we write

$$d_Z(x, A) = \inf\big\{d_Z(x,a)\colon\ a \in A\big\},$$
$$V(A,\epsilon) = \big\{x \in Z\colon\ d_Z(x,A) < \epsilon\big\}.$$

Let $\mathcal{K}(Z)$ denote the family of closed subsets of Z and for $A, B \in \mathcal{K}(Z)$, let

$$D(A,B) = \inf\big\{\epsilon > 0\colon\ A \subseteq V(B,\epsilon) \text{ and } B \subseteq V(A,\epsilon)\big\}.$$

This function on $\mathcal{K}(Z) \times \mathcal{K}(Z)$ is known as the **Hausdorff metric**.

Lemma 5. *$D(A,B)$ is a metric on $\mathcal{K}(Z)$, and with this metric, $\mathcal{K}(Z)$ is compact.*

Proof. See Munkres [14, pp. 280–281]. □

For $\boldsymbol{a}$, $\boldsymbol{u}$ in $\mathbb{R}^d$ with $|\boldsymbol{u}| = 1$, we define the line

$$L(\boldsymbol{a},\boldsymbol{u}) = \{\boldsymbol{a} + t\boldsymbol{u}\colon\ t \in \mathbb{R}\}$$

and the line segment

$$U(\boldsymbol{u}) = \{t\boldsymbol{u}\colon\ 0 \leqslant t \leqslant 1\}.$$

A **closed interval** J of $L(\boldsymbol{a}, \boldsymbol{u})$ is a set of the form

$$J = \boldsymbol{a} + [b, c]\boldsymbol{u}.$$

Given a closed subset X of $L(\boldsymbol{a}, \boldsymbol{u})$, the image of $X \cap J$ under the mapping

$$\boldsymbol{y} \to \frac{\boldsymbol{y} - \boldsymbol{a} - b\boldsymbol{u}}{c - b}$$

is a subset of $U(\boldsymbol{u})$, which we denote by $\Lambda(X, J)$. The **family of limit sets of** X, which we write $FLS(X)$, is the family of sets Y of the form

$$Y = \lim_{i\to\infty} \Lambda(X, J_i),$$

where the diameter $|J_i|$ tends to zero. Here and subsequently we intend the Hausdorff metric on $\mathcal{K}(U(\boldsymbol{u}))$ when referring to the limit of a sequence of sets.

A closed subset X of $L(\boldsymbol{a}, \boldsymbol{u})$ is said to be $\boldsymbol{k}$**-granular** if every set in $FLS(X)$ has cardinality $\leqslant k$.

Lemma 6. *Let X be k-granular. Then* $\dim X = 0$.

Proof. In the case $d = 1$, $L(\boldsymbol{a}, \boldsymbol{u}) = \mathbb{R}$, $U(\boldsymbol{u}) = [0, 1]$, this is due to Boshernitzan [5]. It is simple to extend the result to the general case, but we give the proof for completeness.

Define $f : L(\boldsymbol{a}, \boldsymbol{u}) \to \mathbb{R}$,

$$f(\boldsymbol{a} + t\boldsymbol{u}) = t.$$

Since this is an isometry, we need only to show that $\dim f(X) = 0$, and appealing to Boshernitzan's result, it suffices to show that $f(X)$ is k-granular.

Let $Y \in FLS(f(X))$, then

$$Y = \lim_{i\to\infty} \Lambda\big(f(X), I_i\big) = \lim_{i\to\infty} \frac{f(X) \cap I_i - b_i}{c_i - b_i}$$

for a sequence of intervals $I_i = [b_i, c_i]$ in $\mathbb{R}$ with $c_i - b_i \to 0$.

We observe that

$$f^{-1}(I_i) = \boldsymbol{a} + [b_i, c_i]\boldsymbol{u}$$

is an interval of $L(\boldsymbol{a}, \boldsymbol{u})$ of diameter $c_i - b_i$, and that

$$\Lambda\big(X, f^{-1}(I_i)\big) = \frac{X \cap f^{-1}(I_i) - \boldsymbol{a} - b_i\boldsymbol{u}}{c_i - b_i}.$$

It is easy to see that

$$\boldsymbol{u}\Lambda\big(f(X), I_i\big) = \Lambda\big(X, f^{-1}(I_i)\big).$$

If $\#Y > k$, then there is a set $\boldsymbol{u}Y = \lim_{i\to\infty} \Lambda(X, f^{-1}(I_i))$ in $FLS(X)$ with more than k points, which is absurd. This completes the proof of the lemma. □

Lemma 7. *Let $\boldsymbol{b} \in \mathbb{Z}^d$, $\boldsymbol{u} \in \mathbb{R}^d$ and suppose that $\boldsymbol{b}\cdot\boldsymbol{u} \neq 0$. The relation*

$$a + \boldsymbol{b}\cdot\boldsymbol{y} \equiv 0 \pmod 1 \tag{2.1}$$

holds for at most $|\boldsymbol{b}\cdot\boldsymbol{u}| + 1$ vectors $\boldsymbol{y}$ in $U(\boldsymbol{u})$.

Proof. Let $\boldsymbol{y} = t\boldsymbol{u}$, $0 \leqslant t \leqslant 1$. Then (2.1) yields the equation

$$a + t\boldsymbol{b}\cdot\boldsymbol{u} = n$$

for an integer n, which lies in the closed interval with endpoints a, $a + \boldsymbol{b}\cdot\boldsymbol{u}$. There are at most $|\boldsymbol{b}\cdot\boldsymbol{u}| + 1$ possible n, and each n gives rise to one value of t. □

3. Proof of Theorem 1

A subset S of $[0,1)^d$ is said to be ϵ**-dense** (mod 1) if for every cube C in $\mathbb{R}^d$ of side ϵ,

$$\boldsymbol{s} \in C \pmod 1 \quad \text{for some } \boldsymbol{s} \in S.$$

A theorem of Berend and Peres [4] for the case $d = 1$ states that for every $\epsilon > 0$, there is a $k = k(\epsilon)$ with the following property: Let $Y \subseteq [0,1)$, $\#(Y) > k$. Some dilation mY ($m \in \mathbb{N}$) is ϵ-dense (mod 1). Our first task is to produce a workable substitute for this theorem in dimension d, using Lemma 7 as our jumping off point.

Lemma 8. *Let K, L be natural numbers. Let $\boldsymbol{u} \in \mathbb{R}^d$, $|\boldsymbol{u}| = 1$. Suppose that*

$$\boldsymbol{u}\cdot\boldsymbol{b} \neq 0 \quad \textit{for each } \boldsymbol{b} \in \mathcal{B}_d^*(K).$$

Let $Y \subseteq U(\boldsymbol{u})$,

$$\#Y \geqslant (2K+1)^{dL}(dK+1).$$

Then there is a sequence of distinct elements $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(L)}$ of Y such that

$$\boldsymbol{b}_1\cdot\boldsymbol{y}^{(1)} + \cdots + \boldsymbol{b}_L\cdot\boldsymbol{y}^{(L)} \not\equiv 0 \pmod 1 \tag{3.1}$$

whenever $\boldsymbol{b}_1, \dots, \boldsymbol{b}_L$ are elements of $\mathcal{B}_d(K)$, not all zero.

Proof. We may write $Y = \boldsymbol{u}S$ where $S \subseteq [0,1]$. We select $\boldsymbol{y}^{(j)} = t_j\boldsymbol{u}$ recursively so that

$$\boldsymbol{b}_1\cdot\boldsymbol{y}^{(1)} + \cdots + \boldsymbol{b}_k\cdot\boldsymbol{y}^{(k)} \not\equiv 0 \pmod 1$$

whenever $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$ are in $\mathcal{B}_d(K)$ with $\boldsymbol{b}_k \neq \boldsymbol{0}$. Notice that this condition implies $\boldsymbol{y}^{(k)} \notin \{\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(k-1)}\}$. Evidently this gives a sequence $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(L)}$ with the desired properties.

We apply Lemma 7 repeatedly. The choice of $\boldsymbol{y}^{(1)}$ is possible because the relation

$$\boldsymbol{b}_1 \cdot t_1 \boldsymbol{u} \not\equiv 0 \pmod 1 \quad \bigl(\boldsymbol{b}_1 \in \mathcal{B}_d^*(K)\bigr)$$

holds for any choice of t_1 in S apart from at most $(dK+1)\#\mathcal{B}_d^*(K)$ values, and

$$\#S = \#Y \geqslant (dK+1)(2K+1)^d > (dK+1)\#\mathcal{B}_d^*(K).$$

Once $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(k-1)}$ are chosen, where $2 \leqslant k \leqslant L$, the relation

$$\boldsymbol{b}_1 \cdot \boldsymbol{y}^{(1)} + \dots + \boldsymbol{b}_{k-1} \cdot \boldsymbol{y}^{(k-1)} + \boldsymbol{b}_k \cdot t_k \boldsymbol{u} \not\equiv 0 \pmod 1$$

holds for $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k \in \mathcal{B}_d(K)$, $\boldsymbol{b}_k \neq \boldsymbol{0}$, for any choice of t_k in S apart from at most $(dK+1)(\#\mathcal{B}_d(K))^{k-1}\,\#\mathcal{B}_d^*(K)$ values, and

$$\#S = \#Y \geqslant (dK+1)(2K+1)^{dL} > (dK+1)\bigl(\#\mathcal{B}_d(K)\bigr)^{k-1}\#\mathcal{B}_d^*(K). \qquad \square$$

Lemma 9. *Let $\boldsymbol{\xi} \in \mathbb{R}^p$. Let I be a cube in $[0,1)^d$ of side $1/N$ and suppose that $m\boldsymbol{\xi} \notin I \pmod 1$ $(m = 1, \dots, M)$. Suppose further that $\boldsymbol{\ell} \cdot \boldsymbol{\xi} \not\equiv 0 \pmod 1$ $(\boldsymbol{\ell} \in \mathcal{B}_p^*(2pN))$. Then*

$$M \leqslant \frac{3}{2} \sum_{\boldsymbol{\ell} \in \mathcal{B}_p^*(2pN)} \frac{1}{\|\boldsymbol{\ell} \cdot \xi\|}. \tag{3.2}$$

Proof. We apply Lemma 1 with $\boldsymbol{\xi}_m = m\boldsymbol{\xi} - \boldsymbol{\lambda}$ $(m = 1, \dots, M)$, where $\boldsymbol{\lambda}$ is the center of I. We conclude that

$$M \leqslant 3 \sum_{\boldsymbol{\ell} \in \mathcal{B}_p^*(2pN)} \left| \sum_{m=1}^{M} e(\boldsymbol{\ell} \cdot \boldsymbol{\xi}_m) \right|. \tag{3.3}$$

The lemma follows on inserting a standard estimate for the inner sum in (3.3). $\square$

Lemma 10. *Let $\boldsymbol{u} \in \mathbb{R}^d$, $|\boldsymbol{u}| = 1$ and suppose that*

$$\boldsymbol{u} \cdot \boldsymbol{b} \neq 0 \quad \bigl(\boldsymbol{b} \in \mathcal{B}_d^*\bigl(2dN^{d+1}\bigr)\bigr). \tag{3.4}$$

Let $\epsilon > 0$ and $N = [2/\epsilon] + 1$. Let $Y \subseteq U(\boldsymbol{u})$,

$$\#Y \geqslant \bigl(4dN^{d+1} + 1\bigr)^{dN^d}\bigl(2d^2N^{d+1} + 1\bigr). \tag{3.5}$$

Then there is a natural number m such that mY is ϵ-dense $\pmod 1$ *in* $[0,1)^d$.

Proof. Let $\{C^{(1)}, \dots, C^{(N^d)}\}$ be a partition of $[0,1)^d$ into pairwise disjoint cubes of side $1/N$,

$$C^{(j)} = I_{j1} \times \dots \times I_{jd}.$$

It suffices to show that there is a natural number m such that there is an element of mY in each $C^{(j)}$.

Let $p = dN^d$, $L = N^d$. By (3.4), (3.5), we may apply Lemma 8 with

$$K = 2pN = 2dN^{d+1}$$

to obtain distinct elements $\boldsymbol{y}^{(j)} = (y_1^{(j)}, \ldots, y_d^{(j)})$, $j = 1, \ldots, N^d$, of Y, satisfying (3.1) whenever $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_L$ are points of $\mathcal{B}_d(2pN)$, not all zero. Writing

$$\boldsymbol{\xi} = \left(y_1^{(1)}, \ldots, y_d^{(1)}, \ldots, y_1^{(N^d)}, \ldots, y_d^{(N^d)}\right),$$

this yields

$$\boldsymbol{\ell} \cdot \boldsymbol{\xi} \not\equiv 0 \pmod 1 \quad \left(\boldsymbol{\ell} \in \mathcal{B}_p^*(2pN)\right).$$

In view of Lemma 9, there is a natural number m with

$$m\boldsymbol{\xi} \in I_{11} \times \cdots \times I_{1d} \times \cdots \times I_{N^d 1} \times \cdots \times I_{N^d d},$$

that is,

$$m\boldsymbol{y}^{(j)} \in C^{(j)} \quad \left(j = 1, \ldots, N^d\right)$$

as required. □

The following lemma and its proof are adapted from material in [5].

Lemma 11. *Suppose that $\mathcal{S}$ satisfies* (1.4). *Let $\epsilon > 0$, $N = [2/\epsilon] + 1$. Let $\boldsymbol{a}$, $\boldsymbol{u} \in \mathbb{R}^d$, $|\boldsymbol{u}| = 1$ and suppose that* (3.4) *holds. Let I be an open box in* $[0, 1)^d$, *$I = I_1 \times \cdots \times I_d$, $\min_j |I_j| = 2\epsilon$.*

Let X be a compact subset of $L(\boldsymbol{a}, \boldsymbol{u})$ and suppose that

$$a_j \boldsymbol{x} \notin I \pmod 1 \quad (\boldsymbol{x} \in X,\ j \geqslant 1).$$

Then X is k-granular, where

$$k = \left(4dN^{d+1} + 1\right)^{dN^d} \left(2d^2 N^{d+1} + 1\right).$$

Proof. Suppose the contrary, and let Y be a set in $FLS(X)$ with $\#(Y) > k$. By Lemma 10, there is a natural number m such that mY is ϵ-dense (mod 1).

There exists a sequence of intervals $\{J_i\}_{i \geqslant 1}$ in $L(\boldsymbol{a}, \boldsymbol{u})$ with $|J_i|$ tending to 0 and

$$\lim_{i \to \infty} \Lambda(X, J_i) = Y.$$

Put $J_i = \boldsymbol{a} + [b_i, c_i]\boldsymbol{u}$, $d_i = c_i - b_i$, $X_i = J_i \cap X$, so that

$$\lim_{i \to \infty} d_i = 0. \tag{3.6}$$

In view of (3.6) and (1.4), we can choose s_i in $\{a_j\colon\ j \geqslant 1\}$ such that

$$\lim_{i\to\infty} s_i \frac{d_i}{m} = 1.$$

Now

$$\Lambda(X, J_i) = \frac{X_i - \boldsymbol{a} - b_i \boldsymbol{u}}{d_i} \to Y.$$

This immediately yields

$$\lim_{i\to\infty} \frac{mX_i - m\boldsymbol{a} - mb_i\boldsymbol{u}}{d_i} = mY. \tag{3.7}$$

We claim that

$$\lim_{i\to\infty} D\left(\frac{mX_i - m\boldsymbol{a} - mb_i\boldsymbol{u}}{d_i}, s_i X_i - \frac{m\boldsymbol{a}}{d_i} - s_i b_i \boldsymbol{u}\right) = 0. \tag{3.8}$$

To see this, let $\boldsymbol{x} \in X_i$. Then

$$\begin{aligned}\left|\frac{m\boldsymbol{x} - m\boldsymbol{a} - mb_i\boldsymbol{u}}{d_i} - \left(s_i\boldsymbol{x} - \frac{m\boldsymbol{a}}{d_i} - s_i b_i \boldsymbol{u}\right)\right| &= \left|\left(\frac{m}{d_i} - s_i\right)(\boldsymbol{x} - b_i\boldsymbol{u})\right| \\ &\leqslant (c_i - b_i)\left|\frac{m}{d_i} - s_i\right| = |m - s_i d_i| \to 0.\end{aligned}$$

It follows from (3.7) and (3.8) that

$$\lim_{i\to\infty}\left(s_i X_i - \frac{m\boldsymbol{a}}{d_i} - s_i b_i \boldsymbol{u}\right) = mY.$$

Since mY in ϵ-dense (mod 1), we see that

$$s_i X_i - \frac{m\boldsymbol{a}}{d_i} - s_i b_i \boldsymbol{u}$$

is 2ϵ-dense (mod 1) for sufficiently large i. This implies that the set $s_i X_i$ is 2ϵ-dense (mod 1) for large i, and has a point in common with I. Since $s_i \in \{a_j\colon\ j \geqslant 1\}$ and $X_i \subseteq X$, this gives the desired contradiction. □

Proof of Theorem 1. By combining Lemmas 11 and 6, we see that for any compact subset W of $E_I(\mathcal{S})$, any $\boldsymbol{a} \in \mathbb{R}^d$, and any unit vector $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \cdot \boldsymbol{b} \neq 0 \quad \left(\boldsymbol{b} \in \mathcal{B}_d^*\left(2dN^{d+1}\right)\right)$$

we have

$$\dim\left(W \cap L(\boldsymbol{a}, \boldsymbol{u})\right) = 0. \tag{3.9}$$

Here $N = [2/\epsilon] + 1$; 2ϵ is defined as in Lemma 11.

Suppose now that

$$\dim E_I(\mathcal{S}) > d - 1.$$

Select s, $d-1 < s < \dim E_I(\mathcal{S})$, so that $\mathcal{H}^s(E_I(\mathcal{S})) = \infty$. By Lemma 3, there is a compact subset W of $E_I(\mathcal{S})$ such that

$$0 < \mathcal{H}^s(W) < \infty.$$

We now apply Lemma 4 with $A = W$, $f = d-1$, $d - f = 1$. For almost all lines $V(\boldsymbol{u}) = \{t\boldsymbol{u}\colon t \in \mathbb{R}\}$ with respect to the measure $\gamma_{d,1}$, there exists $\boldsymbol{a} \in V^{\perp}$ such that

$$\dim\bigl(W \cap V(\boldsymbol{u})_{\boldsymbol{a}}\bigr) = s - (d-1). \tag{3.10}$$

We may rewrite $V(\boldsymbol{u})_{\boldsymbol{a}}$ in the form $L(\boldsymbol{a}, \boldsymbol{u})$. Now apart from the set of measure 0 already excluded, say E_1, there is a further set E_2 of measure 0 consisting of lines $V(\boldsymbol{u})$ for which

$$\boldsymbol{u} \cdot \boldsymbol{b} = 0$$

for some $\boldsymbol{b} \in \mathcal{B}_d^*(2dN^{d+1})$. Pick any $\boldsymbol{u}$ such that $V(\boldsymbol{u}) \notin E_1 \cup E_2$. Then (3.10) is in contradiction to (3.9). We conclude that $\dim E_I(\mathcal{S}) \leqslant d - 1$. □

4. Proof of Theorem 2

Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be given with the respective properties $D(C_j)$ $(j = 1, \dots, r)$. Choose C_j' arbitrarily with $C_j' > C_j$. By replacing C_j by C_j', we can suppose that the blocks B_r in (1.5) (with $\mathcal{S} = \mathcal{S}_j$) have the additional property

$$u_r \to \infty.$$

To see this, let $0 < \epsilon < 1$ and

$$B_r' = \bigl[u_r + \epsilon(v_r - u_r), v_r\bigr] \cap \mathcal{S}_j = \bigl[u_r', v_r\bigr] \cap \mathcal{S}_j,$$

say. Then $v_r - u_r' \to \infty$ and $u_r' \to \infty$; moreover,

$$\begin{aligned} C_j\#\bigl(B_r'\bigr) &\geqslant C_j\#(B_r) - C_j\epsilon(1 + v_r - u_r) \\ &\geqslant (1 + v_r - u_r)(1 - C_j\epsilon). \end{aligned}$$

Choosing ϵ so that $C_j' = C_j/(1 - C_j\epsilon)$, we establish the assertion

We may suppose that $E_I'(\mathcal{S}_1, \dots, \mathcal{S}_d)$ is nonempty. Let $x_1, \dots, x_u$ be distinct points of $E_I'(\mathcal{S}_1, \dots, \mathcal{S}_d)$. Let $I = I_1 \times \dots \times I_d$. By hypothesis,

$$x_s(n_1, \dots, n_d) \notin I_1 \times I_2 \times \dots \times I_d \pmod 1 \tag{4.1}$$

for

$$1 \leqslant s \leqslant u, \quad n_j \in \mathcal{S}_j,\ n_1 < \dots < n_d.$$

We select blocks $B^{(1)}, \dots, B^{(d)}$,

$$B^{(t)} = \left[u^{(t)}, v^{(t)}\right] \cap \mathcal{S}_t, \quad 1 \leqslant u^{(t)} < v^{(t)}, \tag{4.2}$$

$$1 + v^{(t)} - u^{(t)} \leqslant C_t' \#\left(B^t\right) \quad (t = 1, \dots, d) \tag{4.3}$$

and moreover

$$v^{(t)} < u^{(t+1)} \quad (t = 1, \dots, d-1). \tag{4.4}$$

Blocks of this kind exist with each $v^{(t)} - u^{(t)}$ arbitrarily large.

By (4.2),

$$x_s(n_1, \dots, n_d) \notin I_1 \times \dots \times I_d \pmod 1 \quad \left(1 \leqslant s \leqslant u,\ n_t \in B^{(t)}\right).$$

We apply Lemma 1 with $\xi_m = x_\ell(n_1, \dots, n_d) - \boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is the center of I, and $\epsilon_j = |I_j|/2$. We obtain

$$u\#\left(B^{(1)}\right) \dots \#\left(B^{(d)}\right) \leqslant 3 \sum_{\substack{\boldsymbol{\ell} \in \mathcal{B}(\frac{2d}{|I_1|}, \dots, \frac{2d}{|I_d|}) \\ \boldsymbol{\ell} \neq \mathbf{0}}} \left| \sum_{s=1}^{u} \sum_{\substack{n_1, \dots, n_d \\ n_t \in B^{(t)}}} e\left(x_s(\ell_1 n_1 + \dots + \ell_d n_d)\right) \right|.$$

For brevity, define K by

$$K^{-1} = 3\left(\frac{4d}{|I_1|} + 1\right) \cdots \left(\frac{4d}{|I_d|} + 1\right).$$

We select $\boldsymbol{\ell} \in \mathcal{B}(\frac{2d}{|I_1|}, \dots, \frac{2d}{|I_d|})$, $\boldsymbol{\ell} \neq \mathbf{0}$, with

$$Ku\#\left(B^{(1)}\right) \dots \#\left(B^{(d)}\right) \leqslant \left| \sum_{s=1}^{u} \sum_{\substack{n_1, \dots, n_d \\ n_t \in B^{(t)}}} e\left(x_s(\ell_1 n_1 + \dots + \ell_d n_d)\right) \right|.$$

Let k be the largest integer with $\ell_k \neq 0$; then

$$Ku\#\left(B^{(1)}\right) \dots \#\left(B^{(k)}\right) \leqslant \left| \sum_{s=1}^{u} \sum_{\substack{n_1, \dots, n_k \\ n_t \in B^{(t)}}} e\left(x_s(\ell_1 n_1 + \dots + \ell_k n_k)\right) \right|.$$

Changing the sign of $(\ell_1, \dots, \ell_k)$ if necessary, we may suppose that $\ell_k > 0$. Now, for some $n_1 \in B^{(1)}, \dots, n_{k-1} \in B^{(k-1)}$ (if $k > 0$), we have

$$Ku\#\left(B^{(k)}\right) \leqslant \sum_{n_k \in B^{(k)}} \left| \sum_{s=1}^{u} e\left(x_s(\ell + \ell_k n_k)\right) \right|$$

with $\ell = \ell_1 n_1 + \cdots + \ell_{k-1} n_{k-1}$. Of course $\ell = 0$ if $k = 1$. By Cauchy's inequality,

$$K^2 u^2 \{\#(B^{(k)})\}^2 \leqslant \#(B_k) \sum_{n_k \in B^{(k)}} \left| \sum_{s=1}^{u} e\big(x_s(\ell + \ell_k n_k)\big) \right|^2.$$

Simplifying,

$$K^2 u^2 \#(B^{(k)}) \leqslant \sum_{n_k \in B^{(k)}} \left| \sum_{s=1}^{u} e\big(x_s(\ell + \ell_k n_k)\big) \right|^2. \tag{4.5}$$

To bound the last expression from above, we write

$$\ell + \ell_k n_k = h + \ell_k\big(n_k - u^{(k)} + 1\big), \quad h = \ell + \ell_k\big(u^{(k)} - 1\big).$$

For $n_k \in B^{(k)}$, the product $v = \ell_k(n_k - u^{(k)} + 1)$ is a natural number between 1 and

$$H = \ell_k\big(v^{(k)} - u^{(k)} + 1\big).$$

Hence

$$\sum_{n_k \in B^{(k)}} \left| \sum_{s=1}^{u} e\big(x_s(\ell + \ell_k n_k)\big) \right|^2 \leqslant \sum_{v=1}^{H} \left| \sum_{s=1}^{u} e\big(x_s(h + v)\big) \right|^2. \tag{4.6}$$

Let $\epsilon > 0$. Taking $v^{(k)} - u^{(k)}$ sufficiently large, the last expression in (4.6) is

$$\leqslant u(1+\epsilon)H$$

by Lemma 2. Recalling (4.5),

$$K^2 u^2 \#(B^{(k)}) \leqslant u(1+\epsilon)H.$$

Since

$$\#(B^{(k)}) \geqslant \frac{1}{C_k'}\big(v^{(k)} - u^{(k)} + 1\big) = \frac{H}{C_k' \ell_k},$$

it follows that

$$\frac{K^2 u}{C_k' \ell_k} \leqslant 1 + \epsilon.$$

Since ϵ is arbitrary, and C_k' can be taken arbitrarily close to C_k, we obtain

$$u \leqslant \frac{C_k \ell_k}{K^2} = 9 C_k \ell_k \left(\frac{4d}{|I_1|} + 1\right)^2 \cdots \left(\frac{4d}{|I_d|} + 1\right)^2.$$

We complete the proof of Theorem 2 on inserting the bounds

$$\ell_k \leqslant \frac{2d}{|I_k|} \quad \text{and} \quad \frac{4d}{|I_j|} + 1 \leqslant \frac{5d}{|I_j|}.$$

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