A Non-oscillation Theorem for the Emden–Fowler Equation: Ground States for Semilinear Elliptic Equations with Critical Exponents

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Let \( \Omega \) be a ball centered at the origin in \( \mathbb{R}^N \) \((N > 2)\), \( \partial \Omega \) its boundary, and \( f: [0, \infty) \rightarrow [0, \infty) \) a given function. This article is concerned with radial solutions of the boundary value problem

\[
Au + f(|x|) u^p = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \tag{1}
\]

Radial solutions of (1) are functions \( u \) of the variable \( r = |x| \) that satisfy the ordinary differential equation

\[
u'' + \frac{N-1}{r} u' + f(r) u^p = 0. \tag{2}\]

It is shown that, if \( p = p^* \), where \( p^* = (N+2)/(N-2) \), then (2) is non-oscillatory at the origin if there exists a \( \sigma > 0 \) such that the function \( r \mapsto f(r)(\log (1/r))^{\sigma} \) is non-decreasing near the origin.

The result is a corollary of a similar result for the Emden-Fowler equation

\[
x'' + t^{-2-1/\gamma} g(t) x^{1+2/\gamma} = 0, \tag{3}
\]

where \( g \) is a given positive-valued function and \( \gamma \) a positive constant. The relation between the solutions of (2) and (3) is established via the transformations \( x(t) = u(r) \), \( g(t) = f(r) \), where \( t = ((N-2)/r)^{\gamma} \). It is shown that (3) is non-oscillatory at infinity if there exists a \( \sigma > 0 \) such that the function \( t \mapsto g(t)(\log t)^{\sigma} \)

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N > 2)$, $\partial \Omega$ its (smooth) boundary. It is well known that the problem

$$ Au + u^p = 0, \quad u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega, \quad (1) $$

has a solution for any star-shaped domain $\Omega$ if $p$ is less than the critical (Sobolev) exponent $p^* = (N + 2)/(N - 2)$, but no solution for any star-shaped domain $\Omega$ if $p \geq p^*$ (cf. [11]).

Recently, several authors have considered problems like (1), where the nonlinear term $u^p$ is perturbed by some other term, as in

$$ Au + u^p + \lambda u^q = 0, \quad u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega, \quad (2) $$

$0 < q < p$. The dichotomy at $p = p^*$ noted above can then be resolved by means of the additional parameters $\lambda$ and $q$. We refer the reader to the investigations by Atkinson and Peletier [2, 3, 4], Brezis [5], Brezis and Nirenberg [6], Budd [7, 8], and Budd and Norbury [9].

In this article we are concerned with another type of perturbation of the non-linear term in (1). We consider the case where $\Omega$ is a ball in $\mathbb{R}^N$ and the problem (1) is modified to

$$ Au + f(|x|) u^p = 0, \quad u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega. \quad (3) $$

Here $f$ is a given positive-valued function, which depends on the radial variable $r = |x|$ only.

It follows from the results of Gidas, Ni, and Nirenberg [10], that any solution $u$ of (3) is radially symmetric, i.e., $u$ depends on the radial variable $r = |x|$ only. Consequently, $u$ satisfies the ordinary differential equation

$$ u'' + \frac{N - 1}{r} u' + f(r) u^p = 0, \quad (4) $$

where $'$ denotes differentiation with respect to $r$. In this article we are interested in finding conditions on $f$ which guarantee that Eq. (4) is non-oscillatory at the origin if $p = p^*$.

We recall that a solution $u$ of (4) is oscillatory at the origin if, for every $r_1 > 0$, there exists an $r_2 \in (0, r_1)$ where $u(r_2) = 0$. A non-trivial solution of
(4) that is not oscillatory at the origin is called non-oscillatory at the origin, and Eq. (4) is called non-oscillatory at the origin if every non-trivial solution is non-oscillatory at the origin.

Clearly, if (4) is non-oscillatory at the origin, it is meaningful to ask for solutions $u$ of (4) that are positive on some interval $[0, R)$ of positive length. Such solutions, if they exist, give rise to ground states for the semilinear elliptic problem (3). Thus, the objective of this investigation bears directly upon the investigation of the existence of ground states for problems described by (3).

The change of variables

$$t = ((N - 2)/r)^{N-2}, \quad x(t) = u(r),$$

transforms the differential equation (4) into an equation of the Emden-Fowler type [11, Chap. XII; 12]. If $u$ satisfies (4), where $p = p^*$, then $x$ satisfies

$$x'' + t^{-2-1/\gamma} g(t) x^{1+2/\gamma} = 0,$$

where $g(t) = f(r)$ and $\gamma = \frac{1}{2}(N - 2)$. Notice that $0 < \gamma < \infty$.

The transformation (5) maps the origin to the point at infinity. Thus we are led to study the oscillatory behavior of non-trivial solutions of (6) for large $t$. In particular, our interest focuses on establishing conditions on the function $g$ which guarantee that Eq. (6) is non-oscillatory at infinity. (Henceforth, we omit the quantifier at infinity when we discuss solutions of (6).)

A study of the oscillatory or non-oscillatory behavior of solutions of the Emden–Fowler equation (6) is of independent interest as well. Such studies have been undertaken before by researchers in oscillation theory, notably Nehari [13], Coffman and Wong [14, 15], and Chiou [16, 17].

In [13], Nehari proved that (6) is non-oscillatory if there exists a $\sigma > 2 + 1/\gamma$ such that $t \mapsto g(t)(\log t)^\sigma$ is non-increasing for all sufficiently large $t$. This result was subsequently improved by Chiou [16], who showed that the conclusion still holds if $\sigma$ satisfies the weaker inequality $\sigma > \frac{1}{2}(3 + 1/\gamma)$. A further improvement was announced by Chiou [17], who claimed that it was sufficient that $\sigma$ satisfy the inequality $\sigma > 0$. But, as was pointed out by Nehari [18], Chiou's proof contained an error; after correction, the result was that (6) is non-oscillatory if $\sigma > (\frac{1}{2} + \gamma)/(\gamma(1 + \gamma))$. The corrected result still amounted to an improvement of Chiou's earlier result [16].

Finally, we mention a result of Erbe and Muldowney [19], which is a refinement of Chiou's. These authors proved that (6) is non-oscillatory if $t \mapsto g(t)(\log t)^\sigma$ is non-increasing and $t \mapsto g(t)(\log t)^\sigma$ is bounded for some pair $(\sigma, \eta) \in D_\gamma$, where $D_\gamma = \{ (\sigma, \eta) \in \mathbb{R}^2 : \sigma > 0, \eta + \sigma/(1 + 2\gamma) > 1/\gamma \}$. This result reduces to Chiou's if $\sigma = \eta$.

In this article we shall prove the following theorem.
THEOREM 1. The differential equation

\[ x'' + t^{-2 - \frac{1}{\gamma}} g(t) x^{1 + \frac{2}{\gamma}} = 0, \]  

(7)

where \( g \) is a positive-valued function and \( \gamma \) a positive constant, is non-oscillatory at infinity if there exists a \( \sigma > 0 \) such that the function \( t \mapsto g(t)(\log t)^{\sigma} \) is non-increasing for all sufficiently large \( t \).

Since the conclusion of the theorem is not true if \( g \) is constant, the result is likely to be best possible.

As a consequence of Theorem 1 we have the following result for the partial differential equation (3).

COROLLARY 1. Let \( \Omega \) be a ball centered at the origin in \( \mathbb{R}^N \). Let \( f \) be a positive-valued function of the radial variable only. Let \( p \) be equal to the critical exponent \( p^* \) (\( p^* = (N + 2)/(N - 2) \)). If there exists a neighborhood \( O \) of the origin and a \( \sigma > 0 \) such that \( r \mapsto f(r)(\log(1/r))^\sigma \) is a non-decreasing function in \( O \), then every non-trivial radial solution of the semilinear elliptic equation

\[ \Delta u + f(|x|) u^p = 0, \quad x \in \Omega, \]  

(8)

is non-oscillatory at the origin.

Because of the result of [17], as corrected by Nehari [18], it suffices to prove the theorem for \( 0 < \sigma \leq \left( \frac{1}{2} + \gamma \right)/(\gamma(1 + \gamma)) \). Our proof, which takes up the entire Section 2, is based on the use of Lyapunov functions. At several places we apply a generalized Sturm Comparison Theorem from the theory of linear ordinary differential equations. This theorem, together with its proof, is given in the Appendix.

2. PROOF OF THEOREM 1

The proof of Theorem 1 is by contradiction, where it is assumed that Eq. (7) admits a non-trivial oscillatory solution. We explore in detail the consequences of this assumption and eventually derive a contradiction. Thus, the existence of non-trivial oscillatory solutions of (7) is ruled out, and the theorem follows.

2.1. Preliminaries

Before embarking on the proof, we establish some notational conventions and state three equivalent forms of the differential equation (7).

To explicitly bring out the logarithmic factor in the function \( g \), we introduce the function \( a \) by the definition

\[ a(t) = g(t)(\log t)^{\sigma}. \]  

(9)
We assume throughout that the function \( a \) is non-increasing. Without loss of generality we may assume that \( 0 < a(t) \leq 1 \) for all \( t \).

With a slight abuse of notation, we will often omit the generic argument of a function. For example, we may write \( x \) when \( x(t) \) is meant.

When considering the value of a function at a specific argument value, where the latter is characterized by an index to the generic argument, we will often omit the generic argument and attach its index immediately to the function symbol. For example, we may use \( a_n \) as a shorthand notation for \( a(t_n) \).

In terms of \( a \), the original equation (7) reads

\[
x'' + \frac{a(t)}{t^{2 + 1/2}(\log t)^\sigma} x^{1 + 2/\gamma} = 0.
\]

Changing variables,

\[
s = \log t, \quad y(s) = t^{-1/2} x(t),
\]

we see that (10) is equivalent with

\[
y'' - \frac{1}{4} y + \frac{a(s)}{s^\sigma} y^{1 + 2/\gamma} = 0.
\]

Here, \( a(s) \equiv a(t(s)) \).

We obtain a third form of the equation upon introduction of the function \( z \),

\[
z(s) = s^{-(1/2) \gamma \sigma} y(s).
\]

The function \( z \) satisfies the differential equation

\[
z'' + \frac{\gamma \sigma}{s} z' + \left[ a(s) z^{2/\gamma} - \left( \frac{1}{4} + \frac{\alpha}{s^2} \right) \right] z = 0,
\]

where the constant \( \alpha \) is defined in terms of \( \gamma \) and \( \sigma \) by the relation

\[
\alpha = \frac{1}{2} \gamma \sigma \left( 1 - \frac{1}{2} \gamma \sigma \right).
\]

Since it suffices to consider values of \( \sigma \) in the interval \((0, (1 + \gamma)/(\gamma(1 + \gamma)))\), we may assume that \( \alpha \) satisfies the condition \( \alpha > 0 \).

The transformations (11) and (13) preserve oscillation. Hence, if \( x \) is an oscillatory solution of (10), as we suppose throughout the proof of the theorem, then there is an infinite sequence of points \( s_0, s_1, \ldots \) with \( s_0 < s_1 < \cdots \), such that \( y(s_0) = 0 \) and \( z(s_i) = 0 \) for \( i = 0, 1, \ldots \). In the following subsections we explore in detail the consequences of this supposition.
2.2. Lyapunov Functions

In this subsection we introduce two Lyapunov functions, one associated with the differential equation (12), the other with the differential equation (14), and establish some of their elementary properties.

The first functional $\psi$ is defined by the expression

$$
\psi(s) \equiv \psi(y(s)) = \frac{1}{2} (y')^2 - \frac{1}{8} y^2 + \frac{a(s)}{(2 + 2/\gamma) s^\sigma} y^2 + 2/2/\gamma.
$$

Since $y$ satisfies (12), we have

$$
\psi'(s) = \frac{1}{2 + 2/\gamma} \left[ \frac{\sigma a(s)}{s^{\sigma+1}} - \frac{a'(s)}{s^\sigma} \right] y^2 + 2/2/\gamma.
$$

Since $a$ is non-increasing, the expression inside the brackets is always positive, so $\psi$ is monotone non-increasing. This is the first assertion of the following lemma.

**Lemma 1.** The functional $\psi$ is monotone non-increasing, the limit $\lim_{s \to \infty} \psi(s)$ exists, and $\psi(s) \geq \psi(\infty) > 0$.

**Proof.** At a zero $s_i$ of $y$, the value of $\psi$ is $\psi(s_i) = \frac{1}{2} (y'_i)^2$, which is positive, so $\psi(s) \geq \psi(s_i) > 0$ for all $s \leq s_i$. As $x$ is oscillatory, $s_i$ can be chosen arbitrarily large. Hence, $s \mapsto \psi(s)$ is non-increasing and bounded below by 0, so $\psi(s)$ must tend to a limit as $s \to \infty$. Clearly, the limit $\psi(\infty)$ satisfies the inequality $\psi(\infty) > 0$.

For future reference, we also give the expressions for $\psi$ and $\psi'$ in terms of $z$.

$$
\psi(s) = \frac{s^{\gamma}}{2} \left\{ \frac{1}{2} (z')^2 + \frac{\gamma s}{2s} z^2 + \left[ \frac{a(s)}{2 + 2/\gamma} z^{2/\gamma} - \frac{1}{8} \left(1 - \frac{(\gamma s/2)^2}{s^2}\right) z^2 \right] \right\}, \tag{18}
$$

$$
\psi'(s) = -\frac{s^{\gamma}}{2 + 2/\gamma} \left[ \frac{\sigma a(s)}{s} - a'(s) \right] z^2 + 2/2/\gamma. \tag{19}
$$

The boundedness of the function $az^{2/\gamma}$ is an immediate consequence of Lemma 1. As this boundedness will play an essential role in the following analysis, we state it in a separate lemma.

**Lemma 2.** The function $az^{2/\gamma}$ is bounded.

**Proof.** It follows from Lemma 1 that there exist constants $M$ and $s_M$ such that $\psi(s) \leq M$ for all $s \geq s_M$. Then also

$$
\frac{1}{2} s^{\gamma} \left[ \frac{a(s)}{1 + 1/\gamma} z^{2/\gamma} - \frac{1}{4} \right] z^2 \leq M.
$$
Let $A = az^{2/\gamma}$. Then

$$A^y(s) \left[ A(s) - \frac{1}{4} (1 + 1/\gamma) \right] \leq \frac{(2 + 2/\gamma) M a^y_{\gamma}}{s^{\gamma_{\sigma}}} \leq \frac{(2 + 2/\gamma) M}{s^{\gamma_{\sigma}}}.$$ 

The upper bound becomes arbitrarily small as $s$ increases, so $A(s)$ tends to 0 or $\frac{1}{4}(1 + 1/\gamma)$. In either case, it is bounded. 

The second functional $\phi$ has the unusual feature of being defined non-locally. Let $s_i$ and $s_{i+1}$ be two successive zeros of $z$. Let $s_{m(i)}$ be the point in $(s_i, s_{i+1})$ where $z$ has its extremum. Then $\phi$ is defined by the expression

$$\phi(s) \equiv \phi(z(s)) = \left[ \frac{1}{2} (z')^2 + \frac{a(s)}{2 + 2/\gamma} z^2 + \frac{a'(s)}{s^{\gamma_{\sigma}}} \right] a^y(s_{m(i)}), \quad s \in (s_i, s_{i+1}).$$

This function is discontinuous at each zero of $z$; in fact, because $a$ is non-increasing, $\phi$ has a downward jump at each zero of $z$. This downward jump is an essential feature in the proof of the theorem.

Since $z$ satisfies (14), we have

$$\phi'(s) = \left[ -\frac{\gamma_{\sigma}}{s} (z')^2 + \frac{a'(s)}{2 + 2/\gamma} z^2 + \frac{a^y_{\gamma}}{s^{3/2}} z^2 \right] a^y(s_{m(i)}), \quad s \in (s_i, s_{i+1}).$$

The derivative may be positive or negative, so $\phi$ may not be monotone. A major thrust of the proof of the theorem is to assess the relative importance of each of the three terms inside the brackets.

Notice that $\phi$ is positive at a zero of $z$. The following lemma shows that $\phi(s)$ converges to 0 as $s \to \infty$.

**Lemma 3.** $\lim_{s \to \infty} \phi(s) = 0$.

**Proof.** (We write $m$, instead of $m(i)$.) At $s_i$ we have $\phi(s_i) = \frac{1}{2} (z_i')^2 a^y_{m(i)}$, which is positive. From (18) we see that $\psi(s_i) = \frac{1}{2} s_i^{\gamma_{\sigma}} (z_i')^2$. so

$$\phi(s_i) = a^y_{m(i)} s_i^{-\gamma_{\sigma}} \psi(s_i) \leq s_i^{-\gamma_{\sigma}} \psi(s_i).$$

As $x$ is oscillatory, $s_i$ grows beyond bounds as $i \to \infty$. Because $\psi$ is bounded, it follows that

$$\lim_{i \to \infty} \phi(s_i) = 0.$$
For any \( s \in (s_i, s_{i+1}) \) we have
\[
\phi(s) = \phi(s_i) + \int_{s_i}^{s} \phi'(t) \, dt.
\]
From (21) it follows that
\[
\phi'(s) \leq \frac{2}{s^2} z^2(s) a_m^n.
\]
so
\[
\phi(s) \leq \phi(s_i) + \frac{2}{s^3} \int_{s_i}^{s} z^2(t) \, dt \leq \phi(s_i) + \frac{\alpha a_m^n z_m^2}{2s_i^2}.
\] (23)

In the same way we conclude from the identity
\[
\phi(s) = \phi(s_{i+1}) - \int_{s_i}^{s_{i+1}} \phi'(t) \, dt
\]
that
\[
\phi(s) \geq \phi(s_{i+1}) - \frac{2}{s_{i+1}^3} \int_{s_i}^{s} z^2(t) \, dt
\]
\[
\geq \phi(s_{i+1}) - \frac{\alpha a_m^n z_m^2}{2s_{i+1}^2}.
\] (24)

The product \( a_m^n z_m^2 \) in the right members of (23) and (24) remains bounded as \( i \to \infty \) (cf. Lemma 2). Therefore, (22), (23), and (24) imply that for any \( \varepsilon > 0 \) there exists an index \( j(\varepsilon) \) such that, if \( i > j(\varepsilon) \), then \( |\phi(s)| < \varepsilon \) for all \( s \in (s_i, s_{i+1}) \). This proves the assertion of the lemma.

2.3. Qualitative Behavior of \( z \) between Successive Zeros

We now turn to a detailed investigation of the qualitative behavior of \( z \) between two successive zeros.

Again, let \( s_i \) and \( s_{i+1} \) be two successive zeros of \( z \). We assume, without loss of generality, that \( z \) is positive in the interval \( (s_i, s_{i+1}) \). Let \( s_m(i) \) be the point in \( (s_i, s_{i+1}) \) where \( z \) reaches its maximum value.

**Lemma 4.** The function \( z \) is strictly increasing on \( (s_i, s_m(i)) \) and strictly decreasing on \( (s_m(i), s_{i+1}) \) for all sufficiently large \( i \).

**Proof.** (We again write \( m \), instead of \( m(i) \).) Suppose \( z \) has a local minimum at some point \( s_c \) between \( s_i \) and \( s_m \). Then \( z'(s_c) = 0 \) and \( z''(s_c) > 0 \).
Evaluating (18) at $s = s_c$, we find

$$\psi(s_c) = s_c^{\alpha} z_2^2 \left[ \frac{a_c z_c^{2/\gamma}}{2 + 2/\gamma} - \frac{1}{8} \left( 1 - \frac{(\gamma \sigma)^2}{s_c^2} \right) \right].$$

Because $\psi$ is positive (cf. Lemma 1), it follows that

$$a_c z_c^{2/\gamma} \geq \frac{1}{4} (1 + 1/\gamma) \left( 1 - \frac{(\gamma \sigma)^2}{s_c^2} \right).$$

The differential equation (14) yields the identity

$$-\frac{z''_c}{z_c} = a_c z_c^{2/\gamma} - \left( \frac{1}{4} + \frac{\alpha}{s_c^2} \right).$$

Combining this identity with the preceding inequality, we find that

$$-\frac{z''_c}{z_c} \geq \frac{1}{4\gamma} \left( 1/4 + 1/\gamma \right) \left( \gamma \sigma \right)^2 + \alpha \frac{s_c^2}{s_c^2}.$$

The expression in the right member is strictly positive if $i$ is sufficiently large, so it would follow that $z''_c < 0$, and we have a contradiction. A local minimum between $s_i$ and $s_m$ is thus ruled out. A local minimum between $s_m$ and $s_{i+1}$ is ruled out by a similar argument. \[\]

It is easy to show that $z'(s_i)$ tends to 0 as $i \to \infty$.

**Lemma 5.** \(\lim_{i \to \infty} z'(s_i) = 0.\)

**Proof.** Evaluating (18) at $s = s_i$, we find $(z'_i)^2 = 2s_i^{-\gamma \sigma} \psi(s_i)$. The expression in the right member tends to zero as $i \to \infty$, because $\psi$ is bounded. \[\]

The following lemma gives specific information about the value of $az_z^{2/\gamma}$ at $s_m(i)$.

**Lemma 6.** \(\lim_{i \to \infty} a(s_m(i)) z^{2/\gamma}(s_m(i)) = \frac{1}{4}(1 + 1/\gamma).\)

**Proof.** (We again write $m$, instead of $m(i)$.) Evaluating (20) at $s = s_m$, we find

$$\phi(s_m) = \left[ \frac{a_m}{2 + 2/\gamma} \frac{z_m^{2 + 2/\gamma}}{Z_m^{2 + 2/\gamma}} - \frac{1}{2} \left( \frac{1}{4} + \frac{\alpha}{s_m^2} \right) \right] a_m z_m^{2/\gamma}.$$

Let $A_m = a_m z_m^{2/\gamma}$. Then

$$A_m \left[ a_m \left( \frac{1}{4} + 1/\gamma \right) \right] = (2 + 2/\gamma) \phi(s_m) + \frac{(1 + 1/\gamma) \alpha a_m z_m^2}{s_m^2}.$$
Both terms in the right member tend to zero as $i \to \infty$, so the quantity $A_m$ either vanishes in the limit or it approaches $\frac{1}{4}(1 + 1/\gamma)$. Because

$$\psi(s_m) = s_m^{\gamma \sigma} 2^2 \left[ \frac{A_m}{2 + 2/\gamma} - \frac{1}{8} \left( 1 - \frac{(\gamma \sigma)^2}{s_m^2} \right) \right]$$

and $\psi$ is positive, it must be the case that

$$A_m > \frac{1}{4} (1 + 1/\gamma) \left( 1 - \frac{(\gamma \sigma)^2}{s_m^2} \right).$$

But $s_m$ grows beyond bounds as $i \to \infty$, so the lower bound is certainly greater than $\frac{1}{4}(1 + 1/\gamma)$ if $i$ is sufficiently large. The possibility that $A_m$ vanishes in the limit is thus ruled out.

2.4. Auxiliary Results on the Variation of $a$

In this subsection we prove an auxiliary result that puts a restriction on the variation of $a$.

**Lemma 7.** Let $s_p$ and $s_q$ be two arbitrary points in $[s_i, s_{i+1}]$ with $s_p < s_q$. If there exists an $\eta > 0$ such that $a(s) z^{2 + 2/\gamma}(s) \geq \eta$ for all $s \in [s_p, s_q]$, then

$$\lim_{i \to \infty} \frac{a(s_p)}{a(s_q)} = 1.$$

**Proof.** Consider the expression (19), which we write as

$$\psi'(s) = -\frac{s^{\gamma \sigma}}{2 + 2/\gamma} \left[ \frac{\sigma}{s} - \frac{a'(s)}{a(s)} \right] a(s) z^{2 + 2/\gamma}(s).$$

Estimating the factor $az^{2 + 2/\gamma}$ by $\eta$ and integrating both sides of the resulting inequality over the interval $(s_p, s_q)$, we obtain

$$\psi(s_p) - \psi(s_q) \geq \frac{\eta}{2 + 2/\gamma} \left[ \frac{1}{\gamma} (s_q^{\gamma \sigma} - s_p^{\gamma \sigma}) - \int_{s_p}^{s_q} \frac{a'(s)}{a(s)} s^{\gamma \sigma} ds \right].$$

We may assume without loss of generality that $s_i$ is greater than 1, so we can ignore the factor $s^{\gamma \sigma}$ under the integral sign. Then we can estimate the integral and obtain the following inequality:

$$\psi(s_p) - \psi(s_q) \geq \frac{\eta}{2 + 2/\gamma} \left[ \frac{1}{\gamma} (s_q^{\gamma \sigma} - s_p^{\gamma \sigma}) + \log \frac{a(s_p)}{a(s_q)} \right].$$

The function $s \to \psi(s)$ is non-increasing and converging to a (non-negative) limit as $s \to \infty$ (cf. Lemma 1). Hence, the expression in the left member is
positive; furthermore, by taking $i$ sufficiently large, we can make it arbitrarily small. The same must then be true for each term in the right member, so the logarithmic term must vanish as $i \to \infty$.

On the interval $[s_i, s_{m(i)}]$, $z$ is increasing, while $a$ is non-increasing, so $az^{2+2/\gamma}$ may be decreasing or increasing there. The following lemma puts a lower bound on the rate of decrease of this function.

**Lemma 8.** Let $s_{p(i)}$ be an arbitrary point in $[s_i, s_{m(i)}]$. If $i$ is sufficiently large, then $a(s) z^{2+2/\gamma}(s) \geq \frac{1}{2} a(s_{p(i)}) z^{2+2/\gamma}(s_{p(i)})$ for all $s \in [s_{p(i)}, s_{m(i)}]$.

**Proof.** (We write $m$ and $p$, instead of $m(i)$ and $p(i)$.) Let $s_q$ be the smallest value of $s \in [s_p, s_m]$ where $a(s) z^{2+2/\gamma}(s) = \frac{1}{2} a(s_p) z^{2+2/\gamma}(s_p)$. Then

$$a(s) z^{2+2/\gamma}(s) \geq \frac{1}{2} a(s_p) z^{2+2/\gamma}(s_p), \quad s \in [s_p, s_q].$$

Applying Lemma 7 with $q = \frac{1}{2} a_p z^{2+2/\gamma}$, we conclude that the ratio $a_p/a_q$ tends to 1 as $i \to \infty$, so by taking $i$ large enough we can certainly achieve the inequality $a_q \geq \frac{2}{16} a_p$. The assumption that $s_q < s_m$ leads to a contradiction, because it would follow that

$$\frac{1}{2} a_p z^{2+2/\gamma} = a_q z^{2+2/\gamma} > a_q z^{2+2/\gamma} \geq \frac{9}{15} a_p z^{2+2/\gamma}.$$ 

Therefore, $s_q \geq s_m$.

On the interval $[s_{m(i)}, s_{i+1}]$, the analog of Lemma 8 is trivial, because both $a$ and $z$ are non-increasing. Thus, if $s_{q(i)}$ is an arbitrary point in $[s_{m(i)}, s_{i+1}]$, then $a(s) z^{2+2/\gamma}(s) \geq a(s_{q(i)}) z^{2+2/\gamma}(s_{q(i)})$ for all $s \in [s_{m(i)}, s_{q(i)}]$.

**Lemma 9.** Let $\eta$ be an arbitrary positive constant. If $s_{p(i)} \in (s_i, s_{i+1})$ is such that $a(s_{p(i)}) z^{2/\gamma}(s_{p(i)}) = \eta$, then

$$\lim_{i \to \infty} \frac{a(s_{p(i)})}{a(s_{m(i)})} = 1.$$ 

**Proof.** (We write $p$ and $m$, instead of $p(i)$ and $m(i)$.) If $s_p < s_m$, then

$$a(s) z^{2+2/\gamma}(s) \geq \frac{1}{2} a_p z^{2+2/\gamma} \geq \frac{1}{2} (a_p z^{2/\gamma})^{1+\gamma}$$

for all $s \in [s_p, s_m]$. (We recall that $0 < a(s) \leq 1$ for all $s$.) If $s_p > s_m$, then

$$a(s) z^{2+2/\gamma}(s) \geq a_p z^{2+2/\gamma} \geq (a_p z^{2/\gamma})^{1+\gamma}$$

for all $s \in [s_m, s_p]$. In either case, the assertion of the lemma follows from Lemma 7.
2.5. Behavior of $z$ Near a Maximum

We now consider in more detail the behavior of $z$ near $s_{m(i)}$, the point in $[s_i, s_{i+1}]$ where $z$ has its maximum. We recall that the value of $az^{2/\gamma}$ at $s_{m(i)}$ approaches $\frac{1}{4}(1 + 1/\gamma)$ as $i \to \infty$ (cf. Lemma 6).

Let $Z$ be the solution of the initial value problem

$$Z'' + Z^{1+2/\gamma} - \frac{1}{4}Z = 0, \quad t > 0; \quad Z(0) = \left(\frac{1}{4}(1 + 1/\gamma)\right)^{1/2}; \quad Z'(0) = 0 \quad (25)$$

and let the definition of $Z$ be extended to negative values of the argument by the identity $Z(t) = Z(-t)$.

As the next lemma shows, $Z$ is the limit of a scaled version of $z$ in the neighborhood of $s_{m(i)}$ as $i \to \infty$.

**Lemma 10.**

$$\lim_{i \to \infty} a^{(1/2)\gamma}(s_{m(i)}) z(s_{m(i)} + \cdot) = Z(\cdot)$$

uniformly on compact intervals containing the origin.

**Proof.** (We write $m$, instead of $m(i)$.) Let the function $\zeta$ be defined by the identity

$$\zeta(s) = a^{(1/2)\gamma}(s_{m(i)}) z(s_{m(i)} + \cdot), \quad s \in (s_i, s_{i+1}].$$

Like $z$, $\zeta$ has a maximum at $s_m$, where its value approaches $\left(\frac{1}{4}(1 + 1/\gamma)\right)^{1/2}$ as $i \to \infty$ (cf. Lemma 6). The function $\zeta$ satisfies the differential equation

$$\zeta'' + \frac{a(s)}{a(s_m)} \zeta^{1+2/\gamma} - \frac{1}{4} \zeta + \left(\frac{\gamma \sigma}{s} \zeta' - \frac{\alpha}{s^2} \zeta\right) = 0. \quad (26)$$

Being the solution of an initial value problem with continuous data, $\zeta$ depends continuously on the coefficients of the differential equation and the initial data. Confining ourselves to compact intervals of the type $[s_m - \mu, s_m + \mu]$, where $\mu$ is a fixed positive constant, we let $i$ tend to infinity. The ratio $a(s)/a(s_m)$ approaches 1 uniformly. Because $\zeta'$ and $\zeta$ are bounded, the expression inside the parentheses tends to 0. Therefore, $\zeta(s_m + \cdot)$ approaches $Z(\cdot)$, the solution of the initial value problem (25), uniformly on $[-\mu, \mu]$ as $i \to \infty$.

Let $s_{r(i)}$ be the smallest value of $s \in [s_i, s_{m(i)}]$ where $a(s) z^{2/\gamma}(s) = \frac{1}{8}(1 + 1/\gamma)$. Similarly, let $s_{n(i)}$ be the greatest value of $s \in [s_{m(i)}, s_{i+1}]$ where $a(s) z^{2/\gamma}(s) = \frac{1}{8}(1 + 1/\gamma)$. The point of the following lemma is that the length of the interval $[s_{r(i)}, s_{n(i)}]$ approaches a fixed value as $i$ tends to infinity.
Lemma 11. There exists a positive constant $\mu$ such that
\[
\lim_{i \to \infty} (s_{m(i)} - s_{l(i)}) = \mu
\]
and
\[
\lim_{i \to \infty} (s_{n(i)} - s_{m(i)}) = \mu.
\]

Proof. (We write $l, m, \text{ and } n$, instead of $l(i), m(i), \text{ and } n(i)$.) From the definitions of $s_l$ and $s_n$ we obtain the identities
\[
\frac{a_l}{a_m} a(s_m) z^{2/\gamma}(s_m - (s_m - s_l)) = \frac{1}{8} (1 + \frac{1}{\gamma})
\]
and
\[
\frac{a_n}{a_m} a(s_m) z^{2/\gamma} (s_m + (s_n - s_m)) = \frac{1}{8} (1 + \frac{1}{\gamma}).
\]
It follows from Lemma 9 that the ratios $a_l/a_m$ and $a_n/a_m$ tend to 1 as $i \to \infty$. Lemma 10 implies that $s_m - s_l$ and $s_n - s_m$ tend to $\mu$, where $Z(\mu) = (\frac{1}{8}(1 + \frac{1}{\gamma}))^{(1/2)\gamma}$.

2.6. Behavior of $z$ Away From a Maximum
In this subsection we analyze the behavior of $z$ a bounded distance away from the point $s_{m(i)}$. As in the foregoing section, we let $s_{l(i)}$ be the smallest value of $s \in [s_i, s_{m(i)}]$ where $a(s) z^{2/\gamma}(s) = \frac{1}{8}(1 + \frac{1}{\gamma})$ and $s_{n(i)}$ the greatest value of $s \in [s_{m(i)}, s_{i+1}]$ where $a(s) z^{2/\gamma}(s) = \frac{1}{8}(1 + \frac{1}{\gamma})$.

We recall that $z$ satisfies the differential equation (14),
\[
z'' + \frac{\gamma \sigma}{s} z' + \left[ a z^{2/\gamma} - \left( \frac{1}{4} + \frac{\alpha}{s^2} \right) \right] z = 0.
\]
A Generalized Sturm Comparison Theorem will play an important role in the following analysis. The theorem, together with its proof, is given in the Appendix.

As we saw in Lemma 5, the derivative $z'$ at a zero $s_i$ of $z$ tends to 0 as $i \to \infty$. The following lemma implies that the logarithmic derivative $z'/z$ at $s_i$ remains bounded away from 0.

Lemma 12. There exists a positive constant $c$ such that
\[
(z'(s))^2 \geq c z^2(s), \quad s \in [s_i, s_{l(i)}] \cup [s_{n(i)}, s_{i+1}].
\]
Proof. The proofs for the two intervals are similar. We restrict ourselves to \([s_i, s_{k(i)}]\).

Let \(s_{k(i)}\) be the smallest value of \(s \in [s_i, s_{k(i)}]\) where \(a(s) z^{2/\gamma}(s) = 0.01\). We write \(k, l,\) and \(m,\) instead of \(k(i), l(i),\) and \(m(i).\)

First we consider the interval \([s_i, s_k]\). By choosing \(i\) large enough, we can certainly achieve the inequality \(a(s)/s \leq 0.01.\) Furthermore, \(az^{2/\gamma} - (\frac{1}{2} + \alpha/s^2) \leq -0.24.\) These observations lead us to compare the solution \(z\) of (27) on \([s_i, s_k]\) with the solution \(w\) of the linear initial value problem

\[
w'^{\prime} + 0.01 w - 0.24 w = 0, \quad s > s_i; \quad w(s_i) = 0; \quad w'(s_i) = z'(s_i).
\]

The solution of (28) is

\[
w(s) = z'(s_i) e^{\lambda_1(s-s_i)} - e^{\lambda_2(s-s_i)} \lambda_1 - \lambda_2,
\]

where \(\lambda_1\) and \(\lambda_2\) are the characteristic roots of the linear equation (28). We note that \(\lambda_1\) and \(\lambda_2\) have opposite signs. The Generalized Sturm Comparison Theorem yields the inequality

\[
\frac{z'(s)}{w(s)} \geq \frac{\lambda_1 e^{\lambda_1(s-s_i)} - \lambda_2 e^{\lambda_2(s-s_i)}}{e^{\lambda_1(s-s_i)} - e^{\lambda_2(s-s_i)}}, \quad s \in [s_i, s_k].
\]

Taking \(\lambda_1\) to be the positive root, we conclude that

\[
(z'(s))^2 \geq \lambda_1^2 z^2(s), \quad s \in [s_i, s_k].
\]

Next, we consider the interval \([s_k, s_i]\). We start from the identity

\[
a_m(z')^2 = a_m^2 \left[ \frac{1}{4} z^2 - \frac{1}{1 + 1/\gamma} az^{2+2/\gamma} + \frac{\alpha}{s^2} z^2 \right] + 2\phi(s),
\]

which follows from the definition (20) of \(\phi,\) and estimate the various terms in the right member.

The ratio \(a_k/a_m\) tends to 1 as \(i \rightarrow \infty;\) by taking \(i\) sufficiently large, we can certainly achieve the inequality \(a_k^2 \geq \frac{9}{10} a_m^2.\) Because \(a\) is non-increasing, it follows that \(a^2(s) \geq a_m^2 \geq \frac{9}{10} a^2(s)\) for all \(s \in [s_k, s_i].\) We use these inequalities to estimate the first and second term. We estimate the third term, which is positive, by 0.

Thus, using the abbreviation \(A = az^{2/\gamma},\) we obtain the inequality

\[
a_m^2(z')^2 \geq A^2 \left[ \frac{9}{40} - \frac{1}{1 + 1/\gamma} A \right] - 2|\phi(s)|.
\]
Because $s$ is bounded away from $s_i$, $A$ is certainly bounded below on $[s_k, s_i]$ by a positive constant. Also, because $s$ is bounded above by $s_i$, $A$ is bounded above by a constant that is certainly less than $\frac{9}{16}(1 + 1/\gamma)$. Consequently, the first term in the lower bound is bounded below by a positive constant on $[s_k, s_i]$. By increasing $i$ if necessary, we can also achieve that $2|\phi(s)|$ is less than this positive lower bound, so there exists a positive constant $\eta$ such that $a_m^2(z')^2 \geq \eta$ on $[s_k, s_i]$. We combine this result with the estimate $az^{2/\gamma} \leq \frac{1}{4}(1 + 1/\gamma)$, which holds everywhere on $[s_i, s_i]$, and use the fact that $a$ is non-increasing. We thus find that

$$\left( z'(s) \right)^2 \geq \eta \left( \frac{8}{1 + 1/\gamma} \right)^\gamma \left( z(s) \right)^2, \quad s \in [s_k, s_i].$$

(30)

The assertion of the lemma follows from (29) and (30).

To conclude this subsection, we show that the length of each of the intervals $[s_i, s_{n(i)}]$ and $[s_{n(i)}, s_{i+1}]$ grows beyond bounds as $i \to \infty$.

**Lemma 13.** $\lim_{i \to \infty} (s_{n(i)} - s_i) = \infty$ and $\lim_{i \to \infty} (s_{i+1} - s_{n(i)}) = \infty$.

**Proof.** The proofs of the two cases are similar. We restrict ourselves to the first case.

Again, let $s_{k(i)}$ be the smallest value of $s \in [s_i, s_{n(i)}]$ where $a(s) z^{2/\gamma}(s) = 0.01$. It suffices to prove that the length of the interval $[s_i, s_{k(i)}]$ grows beyond bounds as $i \to \infty$. (In the remainder of the proof we write $k$ and $m$, instead of $k(i)$ and $m(i)$.)

Consider the differential equation (27) satisfied by $z$. The coefficient of $z'$ is always positive. By taking $i$ sufficiently large, we can achieve that the coefficient of $z$ is at least equal to $-0.26$ on the entire interval $[s_i, s_k]$. These observations lead us to compare the solution $z$ of (27) on $[s_i, s_k]$ with the solution $w$ of the linear initial value problem

$$w'' - 0.26w = 0, \quad w(s_i) = 0; \quad w'(s_i) = z'(s_i),$$

(31)

which is

$$w(s) = z'(s_i) \frac{\sinh((s - s_i) \sqrt{0.26})}{\sqrt{0.26}}.$$

According to the Generalized Sturm Comparison Theorem, we have the inequality

$$z'(s) \leq w'(s), \quad s \in [s_i, s_k].$$
Applying this inequality at $s_k$, we find

$$\frac{z'_k}{z'_i} \leq \cosh((s_k - s_i) \sqrt{0.26}).$$

We claim that the expression in the left member grows beyond bounds as $i \to \infty$.

From the expression for $\phi$ at $s_k$ we obtain the identity

$$a^\gamma_m (z'_k)^2 = a^\gamma_m \left[ \frac{1}{4} z_k^2 - \frac{a_k}{1 + 1/\gamma} z_k^{2 + 2/\gamma} + \frac{\alpha}{s_k^2} z_k^2 \right] + 2\phi(s_k).$$

The last term inside the brackets is positive. The ratio $a_m/a_k$ tends to 1 as $i \to \infty$, so by taking $i$ sufficiently large, we can certainly achieve the inequality $a^\gamma_m \geq \frac{1}{4} a^\gamma_k$. Furthermore, $a_m \leq a_k$. Therefore,

$$a^\gamma_m (z'_k)^2 \geq \frac{1}{2} A_k \left[ \frac{1}{4} - \frac{1}{1 + 1/\gamma} A_k \right] - 2|\phi(s_k)|,$$

where we have used the abbreviation $A_k = a_k z_k^{2/\gamma}$. Here the expression in the right member is certainly bounded below by a positive constant if $i$ is large enough. Because $a_m \leq 1$, it must be the case that $z'_k$ is bounded below by a positive constant. On the other hand, $z'_i$ tends to 0 as $i \to \infty$, according to Lemma 5, so the ratio $z'_k/z'_i$ grows beyond bounds, as claimed.

This result implies that $\cosh((s_k - s_i) \sqrt{0.26})$ and therefore $s_k - s_i$ tends to infinity as $i \to \infty$.

2.7. Monotonicity of $\phi$

We now turn to an investigation of the monotonicity properties of $\phi$. We use the same definitions of $s_{l(i)}$ and $s_{n(i)}$ as in the foregoing subsection.

**Lemma 14.** If $i$ is sufficiently large, then $\phi$ is monotone decreasing on $[s_i, s_{l(i)}]$ and $[s_{n(i)}, s_{i + 1}]$.

**Proof.** We write $l, m$, and $n$, instead of $l(i), m(i)$, and $n(i)$.

We recall that $\phi$ has a downward jump discontinuity at $s_i$. The derivative $\phi'$ is given by the expression

$$\phi'(s) = \left[ -\frac{\gamma}{s} \frac{\sigma (z')^2 + \frac{a'(s)}{2 + 2/\gamma} z^{2 + 2/\gamma} + \frac{\alpha}{s^2} z^2}{s} \right] a^\gamma_m.$$

Ignoring the middle term, which is negative, and using the result of Lemma 12, we see that

$$\phi'(s) \leq \left( \frac{\gamma \sigma - \frac{\alpha}{s^2}}{s} \right) a^\gamma_m \frac{z^2(s)}{s}, \quad s \in [s_i, s_i] \cup [s_n, s_{i + 1}].$$
If \( i \) is sufficiently large, the expression inside the parentheses is positive, so \( \phi'(s) < 0 \) on the intervals considered.

Lemmas \( 11, 13, \) and \( 14 \) show that the functional \( \phi \) is monotone decreasing on \([s_i, s_{i+1}]\), except possibly on a subinterval of finite length near \( s_{m(i)} \). Thus, \( \phi \) behaves almost everywhere like a classical Lyapunov functional for the differential equation (14).

### 2.8. Differential Inequality for \( \phi \)

We now derive a differential inequality for \( \phi \), which holds almost everywhere where \( \phi \) is monotone decreasing, namely on \([s_i, s_{i+1}]\), except for two subintervals, one adjacent to \( s_i \), the other adjacent to \( s_{i+1} \). The lengths of these exceptional subintervals remain bounded as \( i \to \infty \).

As a first step, we derive an estimate for the term \( a'(s_{m(i)})(z')^2 \), which occurs in the expression (21) for \( \phi' \).

**Lemma 15.** There exists a positive constant \( v \), such that

\[
\gamma\sigma a'(s_{m(i)})(z'(s))^2 \geq 2\phi(s), \quad s \in [s_i + v, s_{n(i)}] \cup [s_{n(i)}, s_{i+1} - v].
\]

**Proof.** We treat the two intervals separately.

**Case 1.** \( s \) near \( s_i \). Let \( s_k \) be as in the proof of Lemma 12. On \([s_i, s_k]\), we apply the Generalized Sturm Comparison Theorem, comparing \( z \) with the solution of (28). We conclude that

\[
z'(s) \geq w'(s), \quad s \in [s_i, s_k].
\]

Hence,

\[
\frac{z'(s)}{w'(s)} = \frac{\lambda_1 e^{\lambda_1(s-s_i)} - \lambda_2 e^{\lambda_2(s-s_i)}}{w'(s)} \geq \frac{\lambda_1 e^{\lambda_1(s-s_i)}}{w'(s)}, \quad s \in [s_i, s_k].
\]

If we define \( v_1 \) by the equation \( (\lambda_1 e^{\lambda_1 v_1})^2 = 1/(\gamma\sigma) \), then

\[
\left(\frac{z'(s)}{z'(s_i)}\right)^2 \geq \frac{1}{\gamma\sigma}, \quad s \in [s_i + v_1, s_k].
\]

Consequently,

\[
\gamma\sigma a_m^*(z'(s))^2 \geq a_m^*(z')^2 = 2\phi(s), \quad s \in [s_i + v_1, s_k].
\]

Since \( \phi \) is decreasing on \([s_i, s_k]\), it follows that

\[
\gamma\sigma a_m^*(z'(s))^2 \geq 2\phi(s), \quad s \in [s_i + v_1, s_k].
\]
Next, consider the interval \([s_k, s_i]\). We recall from the proof of Lemma 12 that \(a_m^\gamma(z')^2\) is bounded below by a positive constant \(\eta\) on \([s_k, s_i]\). Since \(\phi(s)\) tends to 0 as \(s \to \infty\), it is certainly the case that
\[
y\sigma a_m^\gamma(z'(s))^2 \geq 2\phi(s), \quad s \in [s_k, s_i],
\]
provided \(i\) is large enough.

**Case 2.** \(s\) near \(s_{i+1}\). Consider the function \(F\),
\[
F(p) = 1 - \frac{1 - 4p}{(1 + 4p)^2}, \quad p \geq 0.
\]
Since \(F(0) = 0\), there exists an \(\epsilon > 0\) such that \(F(\epsilon) < \gamma\sigma\). Let \(\epsilon\) be so chosen. Let \(s_{q(i)}\) be the (unique) point in \([s_{n(i)}, s_{i+1}]\) where \(a(s) z^{2/\gamma}(s) = \epsilon\). We consider the intervals \([s_{q(i)}, s_{i+1}]\) and \([s_{q(i)}, s_{i+1}]\) separately. We write \(n\) and \(q\), instead of \(n(i)\) and \(q(i)\).

We begin by considering the interval \([s_{q(i)}, s_{i+1}]\). If \(z\) satisfies the differential equation (14), then \(\tilde{z}\), defined by the expression
\[
\tilde{z}(\bar{s}) = z(s_{i+1} - \bar{s}), \quad \bar{s} \geq 0,
\]
is a solution of
\[
\tilde{z}'' - \frac{\gamma\sigma}{\bar{s}} \tilde{z}' + \left[ a(s) z^{2/\gamma}(s) - \left(\frac{1}{4} + \frac{\alpha}{s^2}\right)\right] \tilde{z} = 0. \tag{33}
\]
(Here ' denotes differentiation with respect to \(\bar{s}\), and \(s = s_{i+1} - \bar{s}\).)

By taking \(i\) sufficiently large, we can certainly achieve the inequalities \(\gamma\sigma/\bar{s} \leq \epsilon\) and \(\alpha/\bar{s}^2 \leq \frac{1}{6}\). These observations lead us to compare the solution \(\tilde{z}\) of (33) with the solution \(\tilde{w}\) of the linear initial value problem
\[
\tilde{w}'' - (\frac{1}{4} + \frac{1}{6}) \tilde{w} = 0, \quad \tilde{w}(0) = 0; \quad \tilde{w}'(0) = \tilde{z}'(0), \tag{34}
\]
which is
\[
\tilde{w}(\bar{s}) = \tilde{z}'(0) \frac{e^{\lambda_1 \bar{s}} - e^{\lambda_2 \bar{s}}}{\lambda_1 - \lambda_2}.
\]
Here, \(\lambda_1\) and \(\lambda_2\) are the characteristic roots of the linear differential equation (34). We observe that \(\lambda_1\) and \(\lambda_2\) have opposite signs. The Generalized Sturm Comparison Theorem yields the inequality
\[
\frac{\tilde{z}'(\bar{s})}{\tilde{z}(\bar{s})} \leq \frac{\tilde{w}'(\bar{s})}{\tilde{w}(\bar{s})} = \frac{\lambda_1 e^{\lambda_1 \bar{s}} - \lambda_2 e^{\lambda_2 \bar{s}}}{e^{\lambda_1 \bar{s}} - e^{\lambda_2 \bar{s}}}.
\]
As $s$ increases, the quantity in the right member approaches $\lambda_1$, the larger root of the characteristic equation, which is equal to $\frac{1}{2} + \varepsilon$. There exists therefore a $\nu_2 > 0$ such that

$$\frac{w'(\tilde{s})}{w(\tilde{s})} < \frac{1}{2} + 2\varepsilon, \quad \tilde{s} \geq \nu_2.$$ 

Let $\nu_2$ be so chosen, then

$$\frac{z'(s)}{z(s)} = \frac{z'(\tilde{s})}{z(\tilde{s})} < \frac{1}{2} + 2\varepsilon, \quad s \in [s_q, s_{i+1} - \nu_2],$$

so

$$\left( \frac{z(s)}{z'(s)} \right)^2 > \frac{4}{(1 + 4\varepsilon)^2}, \quad s \in [s_q, s_{i+1} - \nu_2].$$

Given this inequality, we estimate $\phi$ as follows:

$$2\phi(s) \leq \left[ (z')^2 + \frac{a(s)}{1 + 1/\gamma} z^{2 + 2/\gamma} - \frac{1}{4} z^2 \right] a_m^r$$

$$\leq a_m^r (z')^2 \left[ 1 - \left( \frac{1}{4} - \frac{a(s)}{1 + 1/\gamma} z^{2/\gamma} \right) \left( \frac{z}{z'} \right)^2 \right]$$

$$\leq a_m^r (z')^2 \left[ 1 - \frac{1 - 4\varepsilon}{(1 + 4\varepsilon)^2} \right]$$

$$\leq \gamma \sigma a_m^r (z')^2, \quad s \in [s_q, s_{i+1} - \nu_2]. \quad (35)$$

On the interval $[s_n, s_q]$ we start from the identity

$$a_m^r (z')^2 = a_m^r \left[ \frac{1}{4} z^2 - \frac{a(s)}{1 + 1/\gamma} z^{2 + 2/\gamma} + \frac{\alpha}{s^2} z^2 \right] + 2\phi(s)$$

and proceed as in the proof of Lemma 12 (the part dealing with the interval $[s_k, s_l]$), to show that the quantity $a_m^r (z')^{2/\gamma}$ is bounded below by a positive constant $\eta$. Since $\phi(s)$ tends to 0 as $s \to \infty$, it is therefore certainly the case that

$$\gamma \sigma a_m^r (z'(s))^2 \geq 2\phi(s), \quad s \in [s_n, s_q], \quad (36)$$

provided $i$ is large enough.

The assertion of the lemma follows from (32), (35), and (36), with $v = \max\{v_1, v_2\}$. \qed
We return to the expression (21) for $\phi'$,
\[
\phi'(s) = \left[ -\frac{\gamma\sigma}{s} (z')^2 + \frac{a'(s) + \alpha s^2}{2 + 2/\gamma} \times \frac{\alpha}{s^3} z^2 \right] \psi(s, s_{m(i)}), \quad s \in (s_i, s_{i+1}].
\]
Because $a$ is non-increasing, we can ignore the middle term to obtain the estimate
\[
\phi'(s) \leq \left[ -\frac{\gamma\sigma}{s} (z')^2 + \frac{\alpha}{s^3} z^2 \right] \psi(s, s_{m(i)}), \quad s \in (s_i, s_{i+1}]. \tag{37}
\]
We decompose the interval $(s_i, s_{i+1}]$ into two disjoint intervals,
\[
(s_i, s_{i+1}] = I_i \cup J_i,
\]
where
\[
J_i = (s_i, s_i + v) \cup [s_{m(i)}, s_{m(i)}] \cup [s_{m(i)}, s_{m(i)}] \cup [s_{i+1} - v, s_{i+1}]
\]
and
\[
I_i = (s_i, s_i + 1] - J_i.
\]
It follows from Lemmas 11 and 15 that the length of the subinterval $J_i$ remains bounded; in fact, $\lim_{i \to \infty} |J_i| = 2\mu + 2v$. The length of the subinterval $I_i$ on the other hand grows beyond bounds as $i \to \infty$.

On $I_i$ we have the inequality
\[
\gamma \sigma a^i(s_{m(i)})(z'(s))^2 \geq 2\phi(s), \quad s \in I_i.
\]
On $J_i$, we use the trivial estimate
\[
\gamma \sigma a^i(s_{m(i)})(z'(s))^2 \geq 0, \quad s \in J_i.
\]
We conclude that there exists a positive constant $M$, which does not depend on $i$, such that
\[
\phi'(s) \leq -\frac{2}{s} \phi(s) + \frac{M}{s^2}, \quad s \in I_i, \tag{38}
\]
\[
\phi'(s) \leq \frac{M}{s^2}, \quad s \in J_i. \tag{39}
\]
In the following subsection we combine these inequalities for $i = 0, 1, \ldots$ into one single inequality and derive an asymptotic estimate for $\phi$. 
2.9. Asymptotic Behavior of $\phi$

Let $I$ and $J$ be the union of the intervals $I_i$ and $J_i$ over all $i = 0, 1, \ldots$,

$$I = \bigcup_i I_i, \quad J = \bigcup_i J_i,$$

and let the function $f$ be defined by the expression

$$f(s) = \frac{2}{s}, \quad s \in I; \quad f(s) = 0, \quad s \in J. \quad (40)$$

We prove the following lemma.

**Lemma 16.** The functional $\phi$ satisfies the inequality

$$\phi(s) \leq \Phi(s),$$

where

$$\Phi(s) = \frac{1}{F(s)} \left[ \Phi(s_0) + M \int_{s_0}^s \frac{F(t)}{t^3} \, dt \right], \quad (41)$$

with

$$F(s) = \exp \int_{s_0}^s f(t) \, dt. \quad (42)$$

**Proof.** It follows from the inequalities (38) and (39) and the definition of $f$ that

$$\phi'(s) \leq -f(s) \phi(s) + \frac{M}{s^3}. \quad (43)$$

Comparing the solution of this differential inequality with the solution $\Phi$ of the initial value problem

$$\Phi'(s) = -f(s) \Phi(s) + \frac{M}{s^3}, \quad s \geq s_0; \quad \Phi(s_0) = \phi(s_0). \quad (44)$$

we conclude that

$$\phi(s) \leq \Phi(s), \quad s \geq s_0. \quad (45)$$

The initial value problem (44) is linear and can be solved explicitly for $\Phi$. The solution is given by (41).
Our next task is to estimate the function $\Phi$ and thus obtain an estimate for $\phi$.

**Lemma 17.** Let $\varepsilon$ be an arbitrarily small positive constant. There exists a positive constant $C$, which depends on $\varepsilon$ but not on $s$, such that

$$\phi(s) \leq C s^{-2-\varepsilon}$$

for all sufficiently large $s$.

**Proof.** Let $\varepsilon > 0$ be given. According to Lemma 16, $\phi$ is majorized by $\Phi$, so it suffices to prove the assertion of the lemma for the function $\Phi$.

Since $f(s) \leq 2/s$, we have $F(s) \leq (s/s_0)^2$ and therefore

$$\Phi(s_0) + M \int_{s_0}^s \frac{F(t)}{t^3} \, dt \leq \Phi(s_0) + \frac{M}{s_0^2} \log \frac{s}{s_0} \leq C_0 s^{\varepsilon/2}$$

for all sufficiently large $s$. Here $C_0$ is a positive constant which does not depend on $s$.

Next, we estimate $F(s)$ from below. Without loss of generality, we make two simplifying assumptions.

First, we recall that $F$ is, in fact, an integral over $I$, where $I$ is the union of intervals $[\alpha_i, \beta_i]$, $i = 1, 2, ..., \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ...$. The length of each interval $[\alpha_i, \beta_i]$ grows beyond bounds as $i \to \infty$, while the length of each gap $(\beta_i, \alpha_{i+1})$ approaches either $2\mu$ or $2\nu$. Since $F$ is the integral of a positive function, we obtain a lower bound for $F$ by assuming that the length of each gap tends to the same constant $\rho$, where $\rho$ is the larger of $2\mu$ and $2\nu$.

Second, since we are interested in the asymptotic behavior of $\Phi$, we may assume that the inequalities

$$\alpha_{i+1} - \beta_i \leq 2\rho, \quad \beta_i \geq (8\rho/\varepsilon) i$$

are satisfied for all $i$. If necessary, we absorb the contributions from those (finitely many) intervals where the inequalities are not satisfied in the multiplicative constants and renumber the remaining intervals.

Let $s$ be any point in $[\alpha_{n+1}, \beta_{n+1}]$. Then

$$\int_{\alpha_1}^s f(t) \, dt = \sum_{i = 1}^n \int_{\alpha_i}^{\beta_i} \frac{2}{t} \, dt + \int_{\alpha_n}^s \frac{2}{t} \, dt = 2 \log \frac{\beta_1 \cdots \beta_n s}{\alpha_1 \cdots \alpha_n \alpha_{n+1}},$$

so

$$F(s) = \left( \frac{\beta_1 \cdots \beta_n s}{\alpha_1 \cdots \alpha_n \alpha_{n+1}} \right)^2.$$
We write the ratio in the form

\[ \frac{\beta_1 \cdots \beta_n s}{\alpha_1 \cdots \alpha_n \alpha_{n+1}} = \frac{s}{\alpha_1} \left( \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \right) \]

It follows from (47) that \( \frac{\alpha_{i+1} - \beta_i}{\beta_i} \leq \varepsilon/(4i) \) for all \( i \), whence we conclude, first, that \( \frac{\alpha_{i+1} - \beta_i}{\beta_i} < 1 \) for all \( i \) and, second, that the infinite series \( \sum_{i=1}^{\infty} \left( \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right)^2 \) converges. Then it follows from [20, Chap. VII, Theorem 10] that the product \( \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \) is asymptotically equivalent with the sum \( \sum_{i=1}^{n} \frac{\alpha_{i+1} - \beta_i}{\beta_i} \) as \( n \to \infty \). There exist therefore positive constants \( C_1 \) and \( C_2 \), such that

\[ C_1 \exp \left( \sum_{i=1}^{n} \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \leq \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \leq C_2 \exp \left( \sum_{i=1}^{n} \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \]

for all sufficiently large \( n \). Hence,

\[ \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \leq C_2 \exp \left( \frac{\varepsilon}{4} \sum_{i=1}^{n} \frac{1}{i} \right) \leq C_2 \left( \exp \sum_{i=1}^{n} \frac{1}{i} \right)^{\varepsilon/4} \]

for all sufficiently large \( n \).

Next, we recall the definition of Euler's constant \( \gamma \),

\[ \gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \log n \right) \]

(cf. [21, Sect. 6.1]). The definition implies that there exists a constant \( C_3 \) such that

\[ \exp \sum_{i=1}^{n} \frac{1}{i} \leq C_3 n \]

for all sufficiently large \( n \). Therefore

\[ \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \leq C_4 n^{\varepsilon/4} \]

for some constant \( C_4 \). If \( s \) is in the interval \( [\alpha_{n+1}, \beta_{n+1}] \), as assumed, then \( s > \beta_n \geq (8\rho/\varepsilon) n \), so \( n \leq (\varepsilon/(8\rho)) s \). There exists therefore a constant \( C_5 \) such that

\[ \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \leq C_5 s^{\varepsilon/4} \]
for all sufficiently large $s$. Hence, there exists a constant $C_6$ such that
\[
\frac{\beta_1 \cdots \beta_n s}{\alpha_1 \cdots \alpha_n \alpha_{n+1}} = \frac{s}{\alpha_1} \left( \prod_{i=1}^{n} \left( 1 + \frac{\alpha_{i+1} - \beta_i}{\beta_i} \right) \right)^{-1} \geq C_6 s^{1-\epsilon/4}
\]
for all sufficiently large $s$. Consequently,
\[
F(s) \geq C_6^2 s^{2-\epsilon/2}
\]
(48)
for all sufficiently large $s$. This is the desired lower bound for $F(s)$.

Combining the definition (41) of $\Phi$ with the estimates (46) and (48), we find that there exists a constant $C$ such that
\[
\Phi(s) \leq C s^{-2+\epsilon}
\]
for all sufficiently large $s$. This proves the assertion of the lemma. 1

2.10. Final Contradiction

We now have all the ingredients necessary to complete the proof of the theorem.

We recall our basic supposition that the differential equation (7) has a non-trivial oscillatory solution $x$. This solution gives rise to a non-trivial oscillatory solution $z$ of (14) via the transformations (9), (11), and (13). Let $s_0, s_1, \ldots$ be the zeros of $z$. If $z$ is oscillatory, this sequence continues indefinitely. Let the functional $\phi$ be defined in terms of $z$ by (20).

Consider the interval $(s_i, s_{i+1})$ between two consecutive zeros of $z$. Assume that $z$ is positive between $s_i$ and $s_{i+1}$, so $s_{m(i)}$ is the point in $(s_i, s_{i+1})$ where $z$ achieves its maximum. According to Lemma 6, the value of $az^{2\gamma}$ approaches $\frac{1}{4}(1 + 1/\gamma)$ as $i \to \infty$. Let $s_{l(i)}$ be the smallest value of $s$ in $[s_i, s_{m(i)}]$ where $a(s)z^{2\gamma}(s) = \frac{1}{4}(1 + 1/\gamma)$ and, similarly, let $s_{n(i)}$ be the greatest value of $s$ in $[s_{m(i)}, s_{i+1}]$ where $a(s)z^{2\gamma}(s) = \frac{1}{4}(1 + 1/\gamma)$. In the remainder of the proof we write $l$ and $n$, instead of $l(i)$ and $n(i)$.

Integrating the expression (21) for $\phi'$ from $s_{l(i)}$ to $s_{m(i)}$, ignoring the term involving the derivative of $a$ (which is non-positive), we obtain the following estimate for $\phi(s_n)$:
\[
\phi(s_n) \leq \phi(s_i) - \frac{\gamma \sigma}{s_n - s_l} \int_{s_l}^{s_n} a_n(z(s))^2 \, ds + \frac{\alpha}{s_i - s_l} \int_{s_l}^{s_n} a_m(z(s))^2 \, ds.
\]
(49)

Let $\epsilon > 0$ be given. According to Lemma 17, there exists a positive constant $C_1$, which depends on $\epsilon$ but not on $i$, such that
\[
\phi(s_i) \leq C_1 s_i^{-2+\epsilon}
\]
(50)
for all sufficiently large \(i\). We assume that \(i\) has been chosen sufficiently large for (50) to hold.

Let \(Z\) be the solution of the initial value problem (25), symmetrically extended to negative values of the argument. According to Lemma 11, there exists a positive constant \(\mu\) and positive constants \(C_2\) and \(C_3\) such that

\[
\int_{s_i}^{s_n} a_i^{\mu}(z'(s))^2 \, ds \geq \frac{1}{2} \int_{-\mu}^{\mu} (Z'(s))^2 \, ds = C_2
\]

(51)

and

\[
\int_{s_i}^{s_n} a_i^e(z(s))^2 \, ds \leq 2 \int_{-\mu}^{\mu} (Z(s))^2 \, ds = C_3.
\]

(52)

Combining the estimates (50), (51), and (52) with (49), we obtain the inequality

\[
\phi(s_n) \leq \frac{C_1}{s_n^2 - \varepsilon} - \frac{C_2}{s_n} + \frac{C_3}{s_n^2}.
\]

(53)

Clearly, as \(i\) increases, the middle term in the right member of (53) will dominate the two other terms, so eventually the expression in the right member, and therefore \(\phi(s_n)\), will be negative. We may assume that this is indeed the case on the interval under consideration; if necessary, we increase \(i\) further.

According to Lemma 14, the functional \(\phi\) is decreasing on the interval \((s_n, s_{i+1})\), so once \(\phi\) is negative at \(s_n\) it remains negative on the entire interval. In particular, \(\phi\) will be negative at \(s_{i+1}\). But now we have a contradiction, as the definition of \(\phi\) is such that \(\phi\) is positive at every zero of \(z\). Thus, the supposition that there is an infinite sequence of zeros of \(z\) is ruled out. In other words, if \(z\) is a non-trivial solution of (14), it must be non-oscillatory. This completes the proof of the theorem.

**APPENDIX A: A GENERALIZED STURM COMPARISON THEOREM**

In this appendix we state and prove a Generalized Sturm Comparison Theorem that is used at several places in the proof of Theorem 1.

**THEOREM 1.** Let \(u\) and \(v\) satisfy the initial value problems

\[
u'' + p(s) u' + q(s) u = 0, \quad s > a; \quad u(a) = \alpha
\]

(A.1)

and

\[
v'' + P(s) v' + Q(s) v = 0, \quad s > a; \quad v(a) = \alpha,
\]

(A.2)
respectively, where \( \alpha \geq 0 \). Suppose that \( u, u', v, \) and \( v' \) are (strictly) positive on some interval \((a, b)\) of positive length. If

\[
p(s) \leq P(s), \quad q(s) \leq Q(s), \quad s \in [a, b],
\]

then the inequality

\[
u'(a) \geq v'(a)
\]

implies

\[
u(s) \geq v(s), \quad u'(s) \geq v'(s), \quad \frac{u'(s)}{u(s)} \geq \frac{v'(s)}{v(s)}, \quad s \in [a, b].
\]

**Proof.** We introduce the functions \( r = u'/u \) and \( R = v'/v \). Both \( r \) and \( R \) are (strictly) positive on \((a, b)\). Since \( u \) and \( v \) satisfy Eqs. (A.1) and (A.2), \( r \) and \( R \) are solutions of the Riccati equations

\[
r' = -(r^2 + pr + q), \quad s > a \tag{A.6}
\]

and

\[
R' = -(R^2 + PR + Q), \quad s > a, \tag{A.7}
\]

respectively. Because of (A.3), Eq. (A.7) yields the following differential inequality for \( R \):

\[
R' \leq -(R^2 + pR + q), \quad s > a. \tag{A.8}
\]

If (A.4) holds, then \( R(a) < r(a) \), so it follows from the theory of differential inequalities that

\[
R(s) \leq r(s), \quad s \in [a, b]. \tag{A.9}
\]

Hence,

\[
u'(s)/u(s) \geq v'(s)/v(s), \quad s \in [a, b]. \tag{A.10}
\]

This proves the third inequality in (A.5).

Because of the positivity of \( u \) and \( v \) and their derivatives on \((a, b)\), we can integrate both sides of the inequality (A.10) over any interval \([a, s]\) with \( s \in (a, b) \). Since \( u \) and \( v \) assume the same initial value \( \alpha \) at \( a \), we obtain the inequality

\[
\log \frac{u(s)}{\alpha} \geq \log \frac{v(s)}{\alpha}, \quad s \in [a, b]. \tag{A.11}
\]
Hence,

\[ u(s) \geq v(s), \quad s \in [a, b]. \quad (A.12) \]

This proves the first inequality in (A.5).

Finally, the second inequality in (A.5) follows from (A.9) and (A.12),

\[ u'(s) \geq v'(s) \frac{u(s)}{v(s)} \geq v'(s), \quad s \in [a, b]. \quad (A.13) \]

This completes the proof of the theorem.

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