Topology 48 (2009) 43–53

Contents lists available at ScienceDirect

Topology

journal homepage: www.elsevier.com/locate/top

The topology of systems of hyperspaces determined by dimension functions[☆]

Taras Banakh[a,b,∗], Natalia Mazurenko[c]

[a] *Department of Mathematics, Ivan Franko University in Lviv, Ukraine*
[b] *Instytut Matematyki, Jan Kochanowski University in Kielce, Poland*
[c] *Precarpatian National University, Ivano-Frankivsk, Ukraine*

ARTICLE INFO

MSC:
57N17
54B20
54F45
55M10
54F65
28A80

Keywords:
Hyperspace
Dimension function
Hausdorff dimension
Hilbert cube
Absorbing system

ABSTRACT

Given a non-degenerate Peano continuum X, a dimension function $\mathsf{D} : 2^X_* \to [0,\infty]$ defined on the family 2^X_* of compact subsets of X, and a subset $\Gamma \subset [0,\infty)$, we recognize the topological structure of the system $\langle 2^X, \mathsf{D}_{\le\gamma}(X)\rangle_{\alpha\in\Gamma}$, where 2^X is the hyperspace of non-empty compact subsets of X and $\mathsf{D}_{\le\gamma}(X)$ is the subspace of 2^X, consisting of non-empty compact subsets $K \subset X$ with $\mathsf{D}(K) \le \gamma$.

1. Introduction

The problem of topological characterization (identification) of topological objects is a central problem in topology. A classical result of this sort is the Curtis–Schori Theorem [1] asserting that for each non-degenerate Peano continuum X the hyperspace 2^X of non-empty compact subsets of X endowed with the Vietoris topology is homeomorphic to the Hilbert cube $Q = [-1,1]^\omega$. A bit later, Curtis [2] characterized the topological spaces X whose hyperspace 2^X is homeomorphic to the pseudointerior $s = (-1,1)^\omega$ of the Hilbert cube as locally connected Polish nowhere locally compact spaces.

In [3] Dobrowolski and Rubin recognized the topology of the subspace $\dim_{\le n}(Q) \subset 2^Q$ consisting of compact subsets of Q having covering dimension $\le n$. They constructed a homeomorphism $h : 2^Q \to Q^\omega$ such that $h(\dim_{\le n}(Q)) = Q^n \times s^{\omega\setminus n}$ for all $n = \{0,\dots,n-1\} \in \omega$. In this case it is said that the system $\langle 2^Q, \dim_{\le n}(Q)\rangle_{n\in\omega}$ is homeomorphic to the system $\langle Q^\omega, Q^n \times s^{\omega\setminus n}\rangle_{n\in\omega}$.

This result was later generalized by Gladdines [4] to products of Peano continua. Finally, Cauty [5] has characterized the spaces X for which the system $\langle 2^X, \dim_{\le n}(X)\rangle_{n\in\omega}$ is homeomorphic to $\langle Q^\omega, Q^n \times s^{\omega\setminus n}\rangle_{n\in\omega}$ as Peano continua whose any non-empty open subset contains compact subsets of arbitrary high finite dimension.

In [6] given a metric space X the second author initiated the study of the subspace $HD_{\le\gamma}(X) \subset 2^X$ of compact subsets of X whose Hausdorff dimension is $\le \gamma$. Unlike the (integer-valued) topological dimension, the Hausdorff dimension of a metric compactum can take on any non-negative real value γ. So, the system $\langle 2^X, HD_{\le\gamma}(X)\rangle_{\gamma\in[0,\infty)}$ that naturally appears in this situation is uncountable. In [7] it was proved that for a finite-dimensional cube $X = [0,1]^n$ the system $\langle 2^X, HD_{\le\gamma}(X)\rangle_{\gamma\in[0,n)}$

☆ This work was supported by the State Fund of fundamental research of Ukraine, project F25.1/099.
∗ Corresponding author at: Department of Mathematics, Ivan Franko University in Lviv, Ukraine.
E-mail addresses: tbanakh@yahoo.com (T. Banakh), mnatali@ukr.net (N. Mazurenko).

0040-9383/$ – see front matter
doi:10.1016/j.top.2009.11.002

is homeomorphic to the system $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\rangle_{\gamma\in[0,n)}$ (by $\mathbb{Q}$ we denote the space of rational numbers). Here for a subset $A \subset \mathbb{R}$ and a real number γ we put

$$A_{\le\gamma} = \{a \in A : a \le \gamma\}, \qquad A_{\ge\gamma} = \{a \in A : a \ge \gamma\}$$
$$A_{<\gamma} = \{a \in A : a < \gamma\}, \qquad A_{>\gamma} = \{a \in A : a > \gamma\}.$$

Both the (topological) covering dimension and the (metric) Hausdorff dimension are particular cases of dimension functions defined as follows.

Definition 1. A function $\mathsf{D} : 2^X_* \to [0,\infty]$ defined on the family 2^X_* of compact subsets of a topological space X is called a *dimension function* if:

1. $\mathsf{D}(\emptyset) = 0$;
2. D is *monotone* in the sense that $\mathsf{D}(A) \le \mathsf{D}(B)$ for any compact subsets $A \subset B$ of X;
3. D is *finitely additive* in the sense that $\mathsf{D}(F \cup A \cup B) \le \max\{\mathsf{D}(A), \mathsf{D}(B)\}$ for any finite subset $F \subset X$ and disjoint compact subsets $A, B \subset X$;
4. D is *ω-additive* in the sense that each non-empty open subset $U \subset X$ contains non-empty open sets $U_n \subset U$, $n \in \omega$, such that each compact subset $K \subset \mathrm{cl}_X(\bigcup_{n\in\omega} U_n)$ has dimension $\mathsf{D}(K) \le \sup_{n\in\omega} \mathsf{D}(K \cap \overline{U}_n)$.

Given a dimension function $\mathsf{D} : 2^X_* \to [0,\infty]$ on X and a subset $\Gamma \subset [0,\infty)$, for every $\gamma \in \Gamma$ consider the subspace

$$\mathsf{D}_{\le\gamma}(X) = \{F \in 2^X : \mathsf{D}(F) \le \gamma\}$$

in the hyperspace 2^X. Our aim is to recognize the topological structure of the system $\langle 2^X, D_{\le\gamma}(X)\rangle_{\gamma\in\Gamma}$.

In the sequel, by a Γ-system $\langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ we shall understand a pair consisting of a set X and a family $\langle X_\gamma\rangle_{\gamma\in\Gamma}$ of subsets of X, indexed by the elements of an index set Γ. Two Γ-systems $\langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle Y, Y_\gamma\rangle_{\gamma\in\Gamma}$ are *homeomorphic* if there is a homeomorphism $h : X \to Y$ such that $h(X_\gamma) = Y_\gamma$ for all $\gamma \in \Gamma$.

The following theorem describes the topological structure of the Γ-system $\langle 2^X, \mathsf{D}_{\le\gamma}(X)\rangle_{\gamma\in\Gamma}$ for a dimension function $\mathsf{D} : 2^X_* \to [0,\infty]$ taking values in the half-line with attached infinity (that is assumed to be larger than any real number). In that theorem we shall refer to the subsets $(\gamma]_\Gamma$ defined for $\Gamma \subset \mathbb{R}$ and $\gamma \in \Gamma$ as follows:

$$(\gamma]_\Gamma = \begin{cases} (\gamma, \inf(\Gamma_{>\gamma})] & \text{if } \gamma < \inf(\Gamma_{>\gamma});\\ (\sup(\Gamma_{<\gamma}), \gamma] & \text{if } \Gamma \ni \sup(\Gamma_{<\gamma}) < \gamma = \inf(\Gamma_{>\gamma});\\ [\sup(\Gamma_{<\gamma}), \gamma] & \text{in all other cases.}\end{cases}$$

In this definition we assume that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = +\infty$.

Theorem 1. *Let X be a topological space and $\mathsf{D} : 2^X_* \to [0,\infty]$ be a dimension function. For every subset $\Gamma \subset [0,\infty)$ the Γ-system $\langle 2^X, \mathsf{D}_{\le\gamma}(X)\rangle_{\gamma\in\Gamma}$ is homeomorphic to the Γ-system $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\rangle_{\gamma\in\Gamma}$ if and only if*

1. *X is a non-degenerate Peano continuum,*
2. *each subspace $\mathsf{D}_{\le\gamma}(X)$, $\gamma \in \Gamma$, is of type G_δ in 2^X, and*
3. *each non-empty open set $U \subset X$ for every $\gamma \in \Gamma$ contains a compact subset $K \subset U$ with $\mathsf{D}(K) \in (\gamma]_\Gamma$.*

First, we apply this theorem to integer-valued dimension functions. We identify each natural number n with the set $\{0, \dots, n-1\}$. Also we put $\overline{\omega} = \omega \cup \{\omega\}$.

Corollary 1. *Let X be a topological space and $\mathsf{D} : 2^X_* \to \overline{\omega}$ be a dimension function. For every $n \in \overline{\omega}$ the n-system $\langle 2^X, \mathsf{D}_{\le k}(X)\rangle_{k\in n}$ is homeomorphic to the n-system $\langle Q^\omega, Q^k \times s^{\omega\setminus k}\rangle_{k\in n}$ if and only if*

1. *X is a non-degenerate Peano continuum,*
2. *each subspace $\mathsf{D}_{\le k}(X)$, $k \in n$, is of type G_δ in 2^X, and*
3. *each non-empty open set $U \subset X$ for every $k \in n$ contains a compact subset $K \subset U$ with $\mathsf{D}(K) = k$.*

The covering dimension dim and the cohomological dimension $\dim_G$ for an arbitrary Abelian group G are examples of integer-valued dimension functions. Therefore Corollary 1 implies the following theorem of Cauty [5] that was mentioned above.

Theorem 2 (*Cauty*). *For any non-degenerate Peano continuum X the ω-systems $\langle 2^X, \dim_{\le n}(X)\rangle_{n\in\omega}$ is homeomorphic to $\langle Q^\omega, Q^n \times s^{\omega\setminus n}\rangle_{n\in\omega}$ if and only if each non-empty open set $U \subset X$ contains an compact subset of arbitrary finite dimension.*

In [5] Cauty notices, that this theorem holds also for the cohomological dimension $\dim_G$ or any other dimension function in the sense of [3]. It does not demand any modifications of arguments in the proof.

Applying Theorem 1 to the half-interval $\Gamma = [0,b) \subset [0,\infty)$, we obtain:

Corollary 2. *Let X be a topological space and $\mathsf{D} : 2^X_* \to [0,\infty]$ be a dimension function. For every $b \in [0,\infty]$ the $[0,b)$-system $\langle 2^X, \mathsf{D}_{\le\gamma}(X)\rangle_{\gamma\in[0,b)}$ is homeomorphic to the $[0,b)$-system $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\rangle_{\gamma\in[0,b)}$ if and only if*

1. *X is a non-degenerate Peano continuum,*

2. *each subspace* $\mathsf{D}_{\le\gamma}(X)$, $\gamma \in [0, b)$, *is of type* G_δ *in* 2^X, *and*
3. *each non-empty open set* $U \subset X$ *for every* $\gamma \in [0, b)$ *contains a compact subset* $K \subset U$ *with* $\mathsf{D}(K) = \gamma$.

Applying Corollary 2 to the Hausdorff dimension $\dim_H$ we obtain the following theorem whose partial case for $X = \mathbb{I}^n$ was proved in [7].

Theorem 3. *For a number* $b \in (0, \infty]$ *and a non-degenerate metric Peano continuum* X *the system* $\langle 2^X, HD_{\le\gamma}(X)\rangle_{\gamma\in[0,b)}$ *is homeomorphic to the system* $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\rangle_{\gamma\in[0,b)}$ *if and only if each non-empty open subset* $U \subset X$ *has Hausdorff dimension* $\dim_H(U) \ge b$.

To derive this theorem from Corollary 2, we need to check the conditions (2) and (3) for the Hausdorff dimension. The condition (2) was established in [6] while (3) follows from the subsequent Mean Value Theorem for Hausdorff dimension, which will be proved in Section 6.

Theorem 4. *Let* X *be a separable complete metric space* X. *For every non-negative real number* $d < \dim_H(X)$ *the space* X *contains a compact subset* $K \subset X$ *of Hausdorff dimension* $\dim_H(K) = d$.

A similar Mean Value Theorem holds for topological dimension: each regular space X with finite inductive dimension $\mathrm{ind}(X)$ contains a closed subspaces of any dimension $k \le \mathrm{ind}(X)$, see [8, 1.5.1]. However, (in contrast to the Hausdorff dimension) this theorem does not hold for infinite-dimensional spaces: there is an infinite-dimensional compact metrizable space X containing no subspace of positive finite dimension [8, 5.2.23].

2. Absorbing systems in the Hilbert cube

Theorem 1 is proved by the technique of absorbing systems created and developed in [9,4]. So, in this section we start by recalling some basic information related to absorbing systems.

From now on all topological spaces are metrizable and separable, all maps are continuous. By $\mathbb{I}$ we denote the unit interval $[0, 1]$, by $\mathbb{Q}$ the space of rational numbers, by $Q = [-1, 1]^\omega$ the *Hilbert cube*, by $s = (-1, 1)^\omega$ its *pseudointerior* and by $B(Q)$ its *pseudoboundary*. By *a Hilbert cube* we understand any topological space homeomorphic to the Hilbert cube Q. In particular, for each at most countable set A the power Q^A is a Hilbert cube; $B(Q^A) = Q^A \setminus s^A$ will stand for its pseudoboundary.

Given two maps $f, g : X \to Y$ and a cover $\mathcal{U}$ of Y we write $(f, g) \prec \mathcal{U}$ and say that f, g are $\mathcal{U}$*-near* if for every point $x \in X$ there is a set $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$.

A closed subset A of an ANR-space X is a called a Z*-set* if for each map $f : Q \to X$ and an open cover $\mathcal{U}$ of X there is a map $g : Q \to X \setminus A$ such that $(f, g) \prec \mathcal{U}$. A subset $A \subset X$ is called a σZ*-set* if A can be written as the countable union of Z-sets. It is known [10] that a closed σZ-set in a Polish ANR-space is a Z-set. An embedding $f : K \to X$ is called a Z*-embedding* if the image $f(K)$ is a Z-set in X.

It is well known that each map $f : K \to Q$ defined on a compact space can be approximated by Z-embeddings, see [11, 10].

Let Γ be a set. By a Γ*-system* $\mathscr{X} = \langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ we shall understand a pair consisting of a space X and an indexed collection $\langle X_\gamma\rangle_{\gamma\in\Gamma}$ of subsets of X. Given a map $f : Z \to X$ and a set $K \subset X$ let $f^{-1}(\mathscr{X}) = \langle f^{-1}(X), f^{-1}(X_\gamma)\rangle_{\gamma\in\Gamma}$ and $K \cap \mathscr{X} = \langle K \cap X, K \cap X_\gamma\rangle_{\gamma\in\Gamma}$.

From now on, $\mathfrak{C}_\Gamma$ is a fixed class of Γ-systems.

Generalizing the standard concept of a strongly universal pair [12, Section 1.7] to Γ-systems we get an important notion of a strongly $\mathfrak{C}_\Gamma$-universal Γ-system.

Definition 2. A Γ-system $\mathscr{X} = \langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ is defined to strongly $\mathfrak{C}_\Gamma$*-universal* if for any open cover $\mathcal{U}$ on X, any Γ-system $\mathscr{C} = \langle C, C_\gamma\rangle_{\gamma\in\Gamma} \in \mathfrak{C}_\Gamma$, and a map $f : C \to X$ whose restriction $f|B : B \to X$ to a closed subset $B \subset C$ is a Z-embedding with $(f|B)^{-1}(\mathscr{X}) = B \cap \mathscr{C}$ there exists a Z-embedding $\tilde{f} : C \to X$ such that $(\tilde{f}, f) \prec \mathcal{U}$, $\tilde{f}|B = f|B$, and $\tilde{f}^{-1}(\mathscr{X}) = \mathscr{C}$.

The strong universality is the principal ingredient in the notion of a $\mathfrak{C}_\Gamma$-absorbing system, generalizing the notion of an absorbing pair, see [12, Section 1.6].

Definition 3. A Γ-system $\mathscr{X} = \langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ is defined to $\mathfrak{C}_\Gamma$*-absorbing* if

(i) $\mathscr{X}$ is strongly $\mathfrak{C}_\Gamma$-universal;
(ii) there is a sequence $\langle Z_n\rangle_{n\in\omega}$ of Z-sets in X such that $\bigcup_{\gamma\in\Gamma} X_\gamma \subset \bigcup_{n\in\omega} Z_n$ and $Z_n \cap \mathscr{X} \in \mathfrak{C}_\Gamma$ for all $n \in \omega$.

A remarkable feature of $\mathfrak{C}_\Gamma$-absorbing system in the Hilbert cube is their topological equivalence. We define two Γ-systems $\langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle Y, Y_\gamma\rangle_{\gamma\in\Gamma}$ to be *homeomorphic* if there is a homeomorphism $h : X \to Y$ such that $h(X_\gamma) = Y_\gamma$ for $\gamma \in \Gamma$.

The following Uniqueness Theorem can be proved by analogy with Theorem 1.7.6 from [12].

Theorem 5. *Two* $\mathfrak{C}_\Gamma$*-absorbing* Γ*-systems* $\langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ *and* $\langle Y, Y_\gamma\rangle_{\gamma\in\Gamma}$ *are homeomorphic provided* X *and* Y *are homeomorphic to a manifold modeled on* Q *or* s.

By a *manifold* modeled on a space E we understand a metrizable separable space M whose any point has an open neighborhood homeomorphic to an open subset of the model space E.

3. Characterizing model absorbing systems

In this section, given a subset $\Gamma \subset \mathbb{R}$ we characterize the topology of the model Γ-system $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\rangle_{\gamma\in\Gamma}$. In fact, it will be more convenient to work with the complementary Γ-system

$$\Sigma_\Gamma = \langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})\rangle_{\gamma\in\Gamma},$$

where $B(Q^{\mathbb{Q}_{>\gamma}}) = Q^{\mathbb{Q}_{>\gamma}} \setminus s^{\mathbb{Q}_{>\gamma}}$. We shall prove that the latter system is $\sigma\mathfrak{C}_\Gamma$-absorbing for a suitable class $\sigma\mathfrak{C}_\Gamma$ of Γ-systems.

Let $\Gamma \subset \mathbb{R}$. Let us define a Γ-system $\langle A, A_\gamma\rangle_{\gamma\in\Gamma}$ to be

- σ*-compact* if the space A is compact while all subspaces A_γ, $\gamma \in \Gamma$, are σ-compact;
- inf-*continuous* if $A_\gamma = \bigcup_{\beta\in B} A_\beta$ for any subset $B \subset \Gamma$ with $\inf B = \gamma \in \Gamma$.

By $\sigma\mathfrak{C}_\Gamma$ we shall denote the class of σ-compact inf-continuous Γ-systems. Let us observe that each Γ-system $\langle A, A_\gamma\rangle_{\gamma\in\Gamma} \in \sigma\mathfrak{C}_\Gamma$ is decreasing. Indeed, for any real numbers $\alpha < \beta$ in Γ the equality $\alpha = \inf\{\alpha, \beta\}$ implies $A_\alpha = A_\alpha \cup A_\beta \supset A_\beta$.

Each Γ-system $\mathcal{A} = \langle A, A_\gamma\rangle_{\gamma\in\Gamma} \in \sigma\mathfrak{C}_\Gamma$ can be extended to the $\mathbb{R}$-system $\tilde{\mathcal{A}} = \langle A, \tilde{A}_\gamma\rangle_{\gamma\in\mathbb{R}} \in \sigma\mathfrak{C}_\mathbb{R}$ consisting of the sets

$$\tilde{A}_\gamma = \begin{cases} \bigcup\limits_{\alpha\in\Gamma_{\ge\gamma}} A_\alpha & \text{if } \sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\ge\gamma}); \\ A_\alpha & \text{if } \alpha = \sup(\Gamma_{<\gamma}) \in \Gamma \text{ and } \gamma < \inf(\Gamma_{\ge\gamma}), \end{cases}$$

indexed by real numbers γ.

Lemma 1. *The* $\mathbb{R}$*-system* $\tilde{\mathcal{A}} = \langle A, \tilde{A}_\gamma\rangle_{\gamma\in\mathbb{R}}$ *is* σ*-compact,* inf-*continuous and extends the* Γ*-system* $\mathcal{A} = \langle A, A_\gamma\rangle_{\gamma\in\Gamma}$ *in the sense that* $\tilde{A}_\gamma = A_\gamma$ *for all* $\gamma \in \Gamma$.

Proof. To see that the $\mathbb{R}$-system $\tilde{\mathcal{A}}$ is σ-compact, fix any real number γ. The set $\tilde{A}_\gamma$ is clearly σ-compact if $\tilde{A}_\gamma = A_\alpha$ for some $\alpha \in \Gamma$. So, we assume that $\tilde{A}_\gamma \ne A_\alpha$ for all $\alpha \in \Gamma$. In this case $\tilde{A}_\gamma = \bigcup_{\alpha\in\Gamma_{\ge\gamma}} A_\alpha$ and $\inf \Gamma_{\ge\gamma} \notin \Gamma$. Choose any countable dense subset $D \subset \Gamma$ and observe that $\inf D_{\ge\gamma} = \inf \Gamma_{\ge\gamma}$ and hence

$$\tilde{A}_\gamma = \bigcup_{\alpha\in\Gamma_{\ge\gamma}} A_\alpha = \bigcup_{\alpha\in D_{\ge\gamma}} A_\alpha$$

is σ-compact, being the countable union of σ-compact spaces A_α, $\alpha \in D_{\ge\gamma}$.

Observe that for every $\gamma \in \Gamma$ we get $\gamma = \inf(\Gamma_{\ge\gamma})$ and hence $A_\gamma \subset \bigcup_{\alpha\in\Gamma_{\ge\gamma}} A_\alpha = \tilde{A}_\gamma$. The reverse inclusion $\tilde{A}_\gamma = \bigcup_{\alpha\in\Gamma_{\ge\gamma}} A_\alpha \subset A_\gamma$ follows from the decreasing property of the γ-system $\mathcal{A}$. Thus $A_\gamma = \tilde{A}_\gamma$, which means that the $\mathbb{R}$-system $\tilde{\mathcal{A}}$ extends the Γ-system $\mathcal{A}$.

Next, we prove that the $\mathbb{R}$-system $\tilde{\mathcal{A}}$ is decreasing. Given two real numbers $\beta < \gamma$, we need to show that $\tilde{A}_\beta \supset \tilde{A}_\gamma$. We consider four cases:

(1) Both β and γ satisfy the first case of the definition of $\tilde{A}_\beta$ and $\tilde{A}_\gamma$:

$$\big(\sup(\Gamma_{<\beta}) \notin \Gamma \text{ or } \beta = \inf(\Gamma_{\ge\beta})\big) \quad \text{and} \quad \big(\sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\ge\gamma})\big).$$

In this case $\beta < \gamma$ implies $\Gamma_{\ge\beta} \supset \Gamma_{\ge\gamma}$ and thus

$$\tilde{A}_\beta = \bigcup_{\alpha\in\Gamma_{\ge\beta}} A_\alpha \supset \bigcup_{\alpha\in\Gamma_{\ge\gamma}} A_\gamma = \tilde{A}_\gamma.$$

(2) The element β satisfies the first case of the definition of $\tilde{A}_\beta$ while γ satisfies the second case:

$$\big(\sup(\Gamma_{<\beta}) \notin \Gamma \text{ or } \beta = \inf(\Gamma_{\ge\beta})\big) \quad \text{and} \quad \alpha = \sup(\Gamma_{<\gamma}) \in \Gamma \text{ and } \gamma < \inf(\Gamma_{\ge\gamma}).$$

In this case $\beta \le \alpha$. Indeed, assuming conversely that $\alpha < \beta$, we get $\Gamma_{<\beta} = \Gamma_{<\gamma}$ and thus $\alpha = \sup(\Gamma_{<\beta}) \in \Gamma$, which implies that $\beta = \inf(\Gamma_{\ge\beta})$. In this case, $\alpha = \sup(\Gamma_{<\gamma}) \ge \beta$, which is a contradiction. So, $\beta \le \alpha$ and then $\alpha \in \Gamma_{\ge\beta}$ and $\tilde{A}_\beta \supset A_\alpha = \tilde{A}_\gamma$.

(3) The element β satisfies the second case of the definition of $\tilde{A}_\beta$ while γ satisfies the first one:

$$\alpha = \sup(\Gamma_{<\beta}) \in \Gamma \quad \text{and} \quad \beta < \inf(\Gamma_{\ge\beta}) \quad \text{and} \quad \big(\sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\ge\gamma})\big).$$

In this case

$$\tilde{A}_\beta = A_\alpha \supset \bigcup_{\delta\in\Gamma_{\ge\gamma}} A_\delta = \tilde{A}_\gamma.$$

(4) Both β and γ satisfy the second case of the definition of $\tilde{A}_\beta$ and $\tilde{A}_\gamma$:

$$\alpha_\beta = \sup(\Gamma_{<\beta}) \in \Gamma, \quad \beta < \inf(\Gamma_{\ge\beta}), \qquad \alpha_\gamma = \sup(\Gamma_{<\gamma}) \in \Gamma, \quad \gamma < \inf(\Gamma_{\ge\gamma}).$$

In this case $\alpha_\beta \le \alpha_\gamma$ and $\tilde{A}_\beta = A_{\alpha_\beta} \supset A_{\alpha_\gamma} = \tilde{A}_\gamma$. This completes the proof of the decreasing property of the $\mathbb{R}$-system $\tilde{A}$.

Finally, we show that the $\mathbb{R}$-system $\tilde{\mathcal{A}}$ is inf-continuous. Fix any real number γ and a subset $B \subset \mathbb{R}$ with $\gamma = \inf B$. We need to check that $\tilde{A}_\gamma = \bigcup_{\beta \in B} \tilde{A}_\beta$. The decreasing property of $\tilde{A}$ guarantees that $\tilde{A}_\gamma \supset \bigcup_{\beta \in B} \tilde{A}_\beta$. It remains to prove the reverse inclusion, which is trivial if $\gamma \in B$. So, we assume that $\gamma \notin B$. Two cases are possible:

1. $\sup(\Gamma_{<\gamma}) \notin \Gamma$ or $\gamma = \inf(\Gamma_{\geq\gamma})$. In this case $\tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq\gamma}} A_\alpha$. We consider three subcases:

(1a) If $\gamma = \inf(\Gamma_{>\gamma})$, then

$$\tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq\gamma}} A_\alpha = \bigcup_{\alpha \in \Gamma_{>\gamma}} A_\alpha$$

because of the inf-continuity of the system $\mathcal{A}$. Given any point $a \in \tilde{A}_\gamma$, find $\alpha \in \Gamma_{>\gamma}$ such that $a \in A_\alpha$. Since $B \not\ni \gamma = \inf B$, there is a point $\beta \in B \cap (\gamma, \alpha)$. Now the definition of $\tilde{A}_\beta$ implies that $a \in A_\alpha \subset \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta$.

(1b) If $\Gamma \ni \gamma < \inf(\Gamma_{>\gamma})$, then we can find $\beta \in B \cap (\gamma, \inf(\Gamma_{>\gamma}))$ and conclude that $\tilde{A}_\gamma = A_\gamma = \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta$.

(1c) If $\Gamma \not\ni \gamma < \inf(\Gamma_{>\gamma})$, then $\sup(\Gamma_{<\gamma}) \notin \Gamma$. Choose any point $\beta \in B \cap (\gamma, \inf(\Gamma_{>\gamma}))$ and observe that $\Gamma_{\geq\beta} = \Gamma_{\geq\gamma}$, $\sup(\Gamma_{<\beta}) = \sup(\Gamma_{<\gamma}) \notin \Gamma$ and thus

$$\tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq\gamma}} A_\alpha = \bigcup_{\alpha \in \Gamma_{\geq\beta}} A_\alpha = \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta.$$

2. $\alpha = \sup(\Gamma_{<\gamma}) \in \Gamma$ and $\gamma < \inf(\Gamma_{\geq\gamma})$, in which case $\tilde{A}_\gamma = \mathcal{A}_\alpha$. Since $\inf B = \gamma \notin \Gamma$, there is a point $\beta \in B \cap (\gamma, \inf(\Gamma_{\geq\gamma}))$.

(2a) If $\gamma \in \Gamma$, then $\sup(\Gamma_{<\beta}) = \gamma \in \Gamma$ and thus $\tilde{A}_\gamma = A_\gamma = \tilde{A}_\beta \subset \bigcup_{\delta \in B} A_\delta$.

(2b) If $\gamma \notin \Gamma$, then $\Gamma_{<\beta} = \Gamma_{<\gamma}$ and thus $\tilde{A}_\gamma = A_\alpha = \tilde{A}_\beta \subset \bigcup_{\delta \in B} A_\delta$. □

In the following theorem for every subset $\Gamma \subset \mathbb{R}$ we introduce a model $\sigma\mathfrak{C}_\Gamma$-absorbing system Σ_Γ in the Hilbert cube $Q^{\mathbb{Q}}$.

Theorem 6. *For every $\Gamma \subset \mathbb{R}$ the Γ-system*

$$\Sigma_\Gamma = \langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\leq\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})\rangle_{\gamma \in \Gamma}$$

is $\sigma\mathfrak{C}_\Gamma$-absorbing and hence is homeomorphic to any other $\sigma\mathfrak{C}_\Gamma$-absorbing Γ-system $\langle X, X_\gamma\rangle_{\gamma \in \Gamma}$ in a Hilbert cube X.

Proof. First we check that the system Σ_Γ is strongly $\sigma\mathfrak{C}_\Gamma$-universal.

We start defining a suitable metric on the Hilbert cube $Q^{\mathbb{Q}}$. Let $\nu : \mathbb{Q} \to (0, 1)$ be any vanishing function, which means that for every $\varepsilon > 0$ the set $\{q \in \mathbb{Q} : \nu(q) \geq \varepsilon\}$ is finite. Take any metric d generating the topology of the Hilbert cube Q and consider the metric

$$\rho((x_q), (y_q)) = \max_{q \in \mathbb{Q}} \nu(q) \cdot d(x_q, y_q)$$

on the Hilbert cube $Q^{\mathbb{Q}}$.

In order to prove the strong $\sigma\mathfrak{C}_\Gamma$-universality of the system Σ_Γ, fix a Γ-system $\mathscr{A} = \langle A, A_\gamma\rangle_{\gamma \in \Gamma} \in \sigma\mathfrak{C}_\Gamma$ and a map $f : A \to Q^{\mathbb{Q}}$ that restricts to a Z-embedding of some closed subset $K \subset A$ such that $(f|K)^{-1}(\Sigma_\Gamma) = K \cap \mathscr{A}$. Given $\varepsilon > 0$, we need to construct a Z-embedding $\tilde{f} : A \to Q^{\mathbb{Q}}$ such that $\rho(\tilde{f}, f) < \varepsilon$, $\tilde{f}|K = f|K$ and $\tilde{f}^{-1}(\Sigma_\gamma) = A_\gamma$ for all $\gamma \in \Gamma$.

By Lemma 1, the Γ-system $\mathcal{A}$ extends to an $\mathbb{R}$-system $\tilde{\mathcal{A}} = \langle A, A_\gamma\rangle_{\gamma \in \mathbb{R}} \in \sigma\mathfrak{C}_\mathbb{R}$. We shall construct a Z-embedding $\tilde{f} : A \to Q^{\mathbb{Q}}$ such that $\rho(\tilde{f}, f) < \varepsilon$, $\tilde{f}|K = f|K$ and $\tilde{f}^{-1}(\Sigma_\gamma) \setminus K = A_\gamma \setminus K$ for all $\gamma \in \mathbb{R}$.

For every $q \in \mathbb{Q}$ let $\mathrm{pr}_q : Q^{\mathbb{Q}} \to Q$ denote the coordinate projection. Since $f(K)$ is a Z-set in $Q^{\mathbb{Q}}$, we can approximate the map f by a map $f' : A \to Q^{\mathbb{Q}}$ such that $\rho(f', f) < \varepsilon/2$, $f'|K = f|K$ and $f'(A \setminus K) \cap f'(K) = \emptyset$. Using the strong $\sigma\mathfrak{C}_{\{0\}}$-universality of the pair $(Q, B(Q))$, for each $q \in \mathbb{Q}$ we can approximate the map $\mathrm{pr}_q \circ f' : A \to Q$ by a map $\tilde{f}_q : A \to Q$ such that

(a) $d(\tilde{f}_q(x), \mathrm{pr}_q \circ f'(x)) \leq \frac{\varepsilon}{2}\rho(f'(x), f(K))$ for all $x \in A$;
(b) $\tilde{f}_q|A \setminus K$ is injective;
(c) $\tilde{f}_q(A \setminus K)$ is a σZ-set in Q;
(d) $\tilde{f}_q^{-1}(B(Q)) \setminus K = A_q \setminus K$.

Now consider the diagonal product $\tilde{f} = (\tilde{f}_q)_{q \in \mathbb{Q}} : A \to Q^{\mathbb{Q}}$ of the maps $\tilde{f}_q$, $q \in \mathbb{Q}$. It follows from (a) that $\tilde{f}|K = f'|K = f|K$, $\rho(\tilde{f}, f) \leq \rho(\tilde{f}, f') + \rho(f', f) < \varepsilon$ and $\tilde{f}(A \setminus K) \cap f(K) = \emptyset$. Combining this fact with (b) we conclude that the map $\tilde{f} : A \to Q^{\mathbb{Q}}$ is injective and hence an embedding. It follows from (c) that $\tilde{f}(A)$ is a σZ-set in $Q^{\mathbb{Q}}$ and hence a Z-set, see [10, 6.2.2]. Therefore, $\tilde{f}$ is a Z-embedding approximating the map f.

It remains to check that $\tilde{f}^{-1}(\Sigma_\gamma) = A_\gamma$ for every $\gamma \in \Gamma$. Since

$$\tilde{f}^{-1}(\Sigma_\gamma) \cap K = (f|K)^{-1}(\Sigma_\gamma) = K \cap A_\gamma,$$

it suffices to check that $\tilde{f}^{-1}(\Sigma_\gamma) \setminus K = A_\gamma \setminus K$.

It follows that

$$\begin{aligned}\tilde{f}^{-1}(\Sigma_\gamma) \setminus K &= \tilde{f}^{-1}(Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})) \setminus K \\ &= \bigcup_{q\in\mathbb{Q}_{>\gamma}} \tilde{f}_q^{-1}(B(Q)) \setminus K = \bigcup_{q\in\mathbb{Q}_{>\gamma}} A_q \setminus K = A_\gamma \setminus K.\end{aligned}$$

The last equality follows from the inf-continuity of the $\mathbb{R}$-system $\widetilde{\mathscr{A}} = \langle A, A_\gamma\rangle_{\gamma\in\mathbb{R}}$ because $\gamma = \inf \mathbb{Q}_{>\gamma}$. This completes the proof of the strong $\sigma\mathfrak{C}_\Gamma$-universality of the system Σ_Γ.

It remains to check that the Γ-system Σ_Γ satisfies the second condition of Definition 3 of a $\sigma\mathfrak{C}_\Gamma$-absorbing system. It is clear the Γ-system Σ_Γ is σ-compact and decreasing. To show that it is inf-continuous, take any subset $B \subset \Gamma$ with $\gamma = \inf B \in \Gamma$. If $\gamma \in B$, then $\Sigma_\gamma \supset \bigcup_{\beta\in B}\Sigma_\beta \supset \Sigma_\gamma$. So, we assume that $\gamma \notin B$. Since the Γ-system $\langle Q^{\mathbb{Q}}, \Sigma_\gamma\rangle_{\gamma\in\Gamma}$ is decreasing, we get $\Sigma_\gamma \supset \bigcup_{\beta\in B}\Sigma_\beta$. To prove the reverse inclusion, take any point $(x_q)_{q\in\mathbb{Q}} \in \Sigma_\gamma = Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})$ and observe that $x_q \in B(Q)$ for some $q \in \mathbb{Q}_{>\gamma}$. Since $\gamma = \inf B$ the half-interval $[\gamma, q)$ contains a point $\beta \in B$.

Then $(x_q)_{q\in\mathbb{Q}} \in Q^{\mathbb{Q}_{\le\beta}} \times B(Q^{\mathbb{Q}_{>\beta}})$ and thus $(x_q)_{q\in\mathbb{Q}} \in \Sigma_\beta \subset \bigcup_{\alpha\in B}\Sigma_\alpha$. Therefore, $\Sigma_\Gamma \in \sigma\mathfrak{C}_\Gamma$.

Since each space Σ_γ, $\gamma \in \mathbb{Q}$, is a σZ-set in $Q^{\mathbb{Q}}$, so is the countable union

$$\bigcup_{\gamma\in\mathbb{Q}}\Sigma_\gamma = \bigcup_{\gamma\in\mathbb{R}}\Sigma_\gamma.$$

So, we can find a sequence $\langle Z_n\rangle_{n\in\omega}$ of Z-sets in $Q^{\mathbb{Q}}$ such that

$$\bigcup_{n\in\omega} Z_n = \bigcup_{\gamma\in\mathbb{Q}}\Sigma_\gamma.$$

It follows from $\Sigma_\Gamma \in \sigma\mathfrak{C}_\Gamma$ that $Z_n \cap \Sigma_\Gamma \in \sigma\mathfrak{C}_\Gamma$, which completes the proof of the $\sigma\mathfrak{C}_\Gamma$-absorbing property of the system Σ_Γ.

By the Uniqueness Theorem 5, each $\sigma\mathfrak{C}_\Gamma$-absorbing system $\langle X, X_\gamma\rangle_{\gamma\in\Gamma}$ in a Hilbert cube X is homeomorphic to the $\sigma\mathfrak{C}_\Gamma$-absorbing Γ-system Σ_Γ. □

4. Strongly universal systems of hyperspaces

In this section we establish an important Theorem 7 detecting strongly $\mathfrak{C}_\Gamma$-universal Γ-systems in hyperspaces. In this section, Γ is any set and $\mathfrak{C}_\Gamma$ is a class of Γ-systems.

By the *hyperspace* of a topological space X we understand the space 2^X of non-empty compact subsets of X endowed with the Vietoris topology. This topology is generated by the sub-base consisting of the sets

$$\langle V\rangle = \{K \in 2^X : K \subset V\} \quad \text{and} \quad \langle X, V\rangle = \{K \in 2^X : K \cap V \ne \emptyset\}$$

where V is an open subset of X. If the topology of X is generated by a metric d, then the Vietoris topology on 2^X is generated by the Hausdorff metric $d_H(A, B) = \max\{\max_{a\in A} d(a, B), \max_{b\in B} d(b, A)\}$.

In the sequel by $2^X_{<\omega}$ we shall denote the subspace of 2^X consisting of finite non-empty subsets of X. By [4], [10, 8.4.3] for a non-degenerate Peano continuum X the subset $2^X_{<\omega}$ is homotopy dense in 2^X.

We recall that a subset A of a topological space X is *homotopy dense* if there is a homotopy $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, t) \in A$ for all $x \in X$ and $t \in (0, 1]$.

We define a subspace $\mathcal{H} \subset 2^X$ to be *finitely additive* if

- $A \cup F \in \mathcal{H}$ for any $A \in \mathcal{H}$ and any finite subset $F \subset X$;
- $A \sqcup B \in \mathcal{H}$ for any disjoint sets $A, B \in \mathcal{H}$.

The first condition implies that each finite subset of X belongs to the family

$$\mathrm{add}(\mathcal{H}) = \{A \in 2^X : \forall\, B \in \mathcal{H}\ A \cup B \in \mathcal{H}\}.$$

For a Γ-system $\mathscr{H} = \langle 2^X, \mathcal{H}_\gamma\rangle_{\gamma\in\Gamma}$ the intersection

$$\mathrm{add}(\mathscr{H}) = \bigcap_{\gamma\in\Gamma} \mathrm{add}(\mathcal{H}_\gamma) \cap \mathrm{add}(2^X \setminus \mathcal{H}_\gamma)$$

will be called *the additive kernel* of $\mathscr{H}$.

For example, the additive kernel of the ω-system $\langle 2^X, \dim_{\le n}(X)\rangle_{n\in\omega}$ is equal to the subspace $\dim_{\le 0}(X)$ of all zero-dimensional compact subsets of X. The additive kernel of the $[0,\infty)$-system $\langle 2^X, HD_{\le\gamma}(X)\rangle_{\gamma\in[0,\infty)}$ is equal to the subspace $HD_{\le 0}(X) \subset 2^X$ consisting of subsets of X with Hausdorff dimension zero.

The following technical theorem was implicitly proved by Cauty in [5].

Theorem 7. *Let X be a non-degenerate Peano continuum. A Γ-system $\mathscr{H} = \langle 2^X, \mathcal{H}_\gamma\rangle_{\gamma\in\Gamma}$ is strongly $\mathfrak{C}_\Gamma$-universal if:*

(1) *for every $\gamma \in \Gamma$ the subspaces $\mathcal{H}_\gamma$ and $2^X \setminus \mathcal{H}_\gamma$ are finitely additive;*

(2) *for every non-empty open set $U \subset X$ there is a map $\xi : Q \to 2^U \cap \mathrm{add}(\mathscr{H})$ such that for any distinct points $x, x' \in Q$ the symmetric difference $\xi(x) \triangle \xi(x')$ is infinite;*
(3) *for any non-empty open set $U \subset X$ and any Γ-system $\mathscr{C} = \langle C, C_\gamma\rangle_{\gamma\in\Gamma} \in \mathfrak{C}_\Gamma$ there is a map $\varphi : C \to 2^U$ such that $\varphi^{-1}(\mathscr{H}) = \mathscr{C}$.*

5. The strong $\sigma\mathfrak{C}_\Gamma$-universality of Γ-systems of hyperspaces

In this section, we detect strongly $\sigma\mathfrak{C}_\Gamma$-universal systems of the form $\langle 2^X, \mathsf{D}_{>\gamma}(X)\rangle_{\gamma\in\Gamma}$ where $\Gamma \subset [0,\infty)$ and $\mathsf{D} : 2^X_* \to [0,\infty]$ is a dimension function defined on the hyperspace of a non-degenerated Peano continuum X.

First we establish one property of dimension functions which is formally stronger that the ω-additivity.

Lemma 2. *Let X be a metrizable compact space without isolated points and $\mathsf{D} : 2^X_* \to [0,\infty]$ be a dimension function. For every non-empty open set $U \subset X$ there is a disjoint sequence $\langle U_n\rangle_{n\in\omega}$ of non-empty open sets of U such that*

1. *$\langle U_n\rangle_{n\in\omega}$ converges to some point $x_\infty \in U$, which means that each neighborhood $O(x_\infty)$ contains all but finitely many sets U_n;*
2. *for any compact subsets $K_n \subset U_n$, $n \in \omega$, the set $K_\infty = \{x_\infty\}\cup\bigcup_{n\in\omega} K_n$ is compact and has dimension $\mathsf{D}(K_\infty) \le \sup_{n\in\omega} \mathsf{D}(K_n)$.*

Proof. Take any non-empty open subset $V \subset X$ with $\mathrm{cl}(V) \subset U$. The ω-additivity of the dimension function D yields a sequence $\langle V_n\rangle_{n\in\omega}$ of open subsets of V such that for any compact subset $K \subset \mathrm{cl}(\bigcup_{n\in\omega} V_n)$ has dimension

$$\mathsf{D}(K) \le \sup_{n\in\omega} \mathsf{D}(K \cap \overline{V}_n).$$

Replacing the sets V_n by their suitable subsets, we can assume that $\mathrm{diam}(V_n) \to 0$ as $n \to \infty$. In each set V_n pick a point x_n. Since the space X has no isolated point, we can choose the points x_n, $n \in \omega$, to be pairwise distinct. Next, replacing the sets V_n by small neighborhoods of the points x_n, we can make the sets V_n, $n \in \omega$, pairwise disjoint. By the compactness of X, the sequence $\langle x_n\rangle_{n\in\omega}$ contains a subsequence $\langle x_{n_k}\rangle_{k\in\omega}$ that converges to some point $x_\infty \in \mathrm{cl}(V) \subset U$. Since $\mathrm{diam}(V_{n_k}) \to 0$, the sequence $\langle V_{n_k}\rangle_{k\in\omega}$ also converges to x_∞.

It is clear that the sets $U_k = V_{n_k}$, $k \in \omega$, have the desired properties. □

Now we are able to prove the principal ingredient in the proof of Theorem 1. Below $\Gamma \subset [0,\infty)$ and $\sigma\mathfrak{C}_\Gamma$ stands for the class of inf-continuous σ-compact Γ-systems.

Theorem 8. *Let X be a non-degenerate Peano continuum, $\mathsf{D} : 2^X_* \to [0,\infty]$ be a dimension function, and $\Gamma \subset [0,\infty)$. The Γ-system $\langle 2^X, \mathsf{D}_{>\gamma}(X)\rangle_{\gamma\in\Gamma}$ is strongly $\sigma\mathfrak{C}_\Gamma$-universal if and only if each non-empty open set $U \subset X$ for every $\gamma \in \Gamma$ contains a compact subset $K \subset U$ with $\mathsf{D}(K) \in (\gamma]_\Gamma$.*

Proof. To prove the "only if" part, assume that the system $\mathscr{D} = \langle 2^X, \mathsf{D}_{>\gamma}(X)\rangle_{\gamma\in\Gamma}$ is strongly $\sigma\mathfrak{C}_\Gamma$-universal.

Fix any non-empty open set $U \subset X$ and an element $\gamma \in \Gamma$. We need to find a compact subset $K \subset U$ with $\mathsf{D}(K) \in (\gamma]_\Gamma$.

Let $A = \{a\}$ be any singleton and put $A_\alpha = A$ for all $\alpha < \gamma$ and $A_\alpha = \emptyset$ for all $\alpha > \gamma$. Put also $A_\gamma = \emptyset$ if $\gamma = \inf(\Gamma_{>\gamma})$ and $A_\gamma = A$ otherwise.

Observe that the so-defined Γ-system $\mathscr{A} = \langle A, A_\gamma\rangle_{\gamma\in\Gamma}$ belongs to the class $\sigma\mathfrak{C}_\Gamma$. Now using the strong $\sigma\mathfrak{C}_\Gamma$-universality of the Γ-system $\mathscr{D}$, find a map $f : A \to 2^U$ such that $f^{-1}(\mathscr{D}) = \mathscr{A}$.

We claim that the compact subset $K = f(a) \subset U$ has dimension $\mathsf{D}(K) \in (\gamma]_\Gamma$. To prove this inclusion, consider the three cases from the definition of the set $(\gamma]_\Gamma$.

(i) If $\gamma < \inf(\Gamma_{>\gamma})$, then $a \in A_\gamma$ and hence $K = f(a) \in \mathsf{D}_{>\gamma}(X)$ and $\gamma < \mathsf{D}(K)$. On the other hand, for every $\alpha \in \Gamma_{>\gamma}$ we get $a \notin A_\alpha = \emptyset$ and thus $K = f(a) \in 2^X \setminus D_{>\alpha}(X) = D_{\le\alpha}(X)$ and $\mathsf{D}(K) \le \alpha$, which implies $\mathsf{D}(K) \le \inf(\Gamma_{>\gamma})$. Consequently, $\mathsf{D}(K) \in (\gamma, \inf(\Gamma_{>\gamma})] = (\gamma]_\Gamma$.

(ii) $\Gamma \ni \sup(\Gamma_{<\Gamma}) < \gamma = \inf(\Gamma_{>\gamma})$. In this case $a \notin A_\gamma = \emptyset$ and thus $K = f(a) \in \mathsf{D}_{\le\gamma}(X)$. On the other hand, $a \in A_\alpha$ where $\alpha = \sup(\Gamma_{<\gamma}) < \gamma$ and hence $K = f(a) \in \mathsf{D}_{>\alpha}(X)$. Consequently, $\mathsf{D}(K) \in (\sup(\Gamma_{<\gamma}), \gamma] = (\gamma]_\Gamma$.

(iii) If $\gamma = \inf(\Gamma_{>\gamma})$ and $\sup(\Gamma_{<\gamma})$ is equal γ or does not belongs to Γ, then for every $\alpha \in \Gamma_{<\gamma}$, we get $a \in A_\alpha$ and thus $K = f(a) \in \mathsf{D}_{>\alpha}(X)$ ad $\mathsf{D}(K) > \alpha$. Consequently, $\mathsf{D}(K) \ge \sup(\Gamma_{<\gamma})$. On the other hand, $a \notin A_\gamma = \emptyset$ implies $K = f(a) \in \mathsf{D}_{\le\gamma}(X)$ and thus $\mathsf{D}(K) \in [\sup(\Gamma_{<\gamma}), \gamma] = (\gamma]_\Gamma$.

To prove the "only if" part, assume that for every non-empty open set $U \subset X$ and every $\gamma \in \Gamma$ there is a compact subset $K \subset U$ with $\mathsf{D}(K) \in (\gamma]_\Gamma$.

The strong $\sigma\mathfrak{C}_\Gamma$-universality of the system $\mathscr{D}$ will follow as soon as we check the conditions (1)–(3) of Theorem 7 for the class $\sigma\mathfrak{C}_\Gamma$.

1. The monotonicity of the dimension function D implies that the subspace $\mathsf{D}_{>\gamma}(X)$ of 2^X is finitely additive. The finite additivity of the complement $\mathsf{D}_{\le\gamma}(X) = 2^X \setminus \mathsf{D}_{>\gamma}(X)$ follows from the finite additivity of the dimension function D.

2. To establish the condition (2) of Theorem 7, fix any non-empty open set $U \subset X$. Lemma 2 yields a sequence $\langle U_n\rangle_{n\in\omega}$ of non-empty open subsets of U that converge to some point $x_\infty \in U$ and has the property that for any compact subsets $K_n \subset U_n$ the set $K = \{x_\infty\}\cup\bigcup_{n\in\omega} K_n$ is compact and has dimension $\mathsf{D}(K) \le \sup_{n\in\omega} \mathsf{D}(K_n)$. Each set U_n contains a topological copy of the interval $[0,1]$, so we can find a topological embedding $\xi_n : [-1,1] \to U_n$.

Let $\nu : \omega \to \omega$ be any function such that the preimage $\nu^{-1}(n)$ of every $n \in \omega$ is infinite. Define a map $\xi : Q \to 2^U$ assigning to each $\vec{t} = \langle t_n \rangle_{n\in\omega} \in Q$ the compact subset

$$\xi(\vec{t}) = \{x_\infty\} \cup \{\alpha_n(t_{\nu(n)}) : n \in \omega\}$$

of U having a unique non-isolated point x_∞. The equality $\mathsf{D}(\emptyset) = 0$ and the finite additivity of the dimension function D implies that $\mathsf{D}(F) = 0$ for each finite subset $F \subset X$. The choice of the sequence $\langle U_n \rangle$ guarantees that $\mathsf{D}(\xi(\vec{t})) = 0$ and thus

$$\xi(Q) \subset \mathsf{D}_{\le 0}(X) \subset \mathrm{add}(\mathscr{D}).$$

The choice of the function ν guarantees that $\xi(\vec{t}) \triangle \xi(\vec{u})$ is infinite for any distinct vectors $\vec{t}, \vec{u} \in Q$.

3. To check the condition (3) of Theorem 7, fix any non-empty open set $U \subset X$ and a Γ-system $\mathscr{A} = \langle A, A_\gamma \rangle_{\gamma\in\Gamma} \in \sigma\mathfrak{C}_\Gamma$. Each set A_γ, $\gamma \in \Gamma$, being σ-compact, can be written as the countable union $A_\gamma = \bigcup_{n\in\omega} A_{\gamma,n}$ of an increasing sequence $\langle A_{\gamma,n} \rangle_{n\in\omega}$ of compact subsets of A. Let D be a countable subset of Γ meeting each half-interval $[\gamma, \gamma + \varepsilon)$ where $\gamma \in \Gamma$ and $\varepsilon > 0$.

Apply Lemma 2 to find a disjoint family $\langle U_d \rangle_{d\in D}$ of non-empty open subsets of U such that

- $\langle U_d \rangle_{d\in D}$ converges to some point $x_\infty \in U$ in the sense that each neighborhood $O(x_\infty)$ contains all but finitely many sets U_d, $d \in D$;
- for any compact sets $K_d \subset U_d$ the set $K = \{x_\infty\} \cup \bigcup_{d\in D} K_d$ is compact and has dimension $\mathsf{D}(K) \le \sup_{d\in D} \mathsf{D}(K_d)$.

For every $d \in D$ use Lemma 2 once more and find a disjoint family $\langle U_{d,n} \rangle_{n\in\omega}$ of non-empty open subsets of U such that

- $\langle U_{d,n} \rangle_{n\in\omega}$ converges to some point $x_d \in U_d$;
- for any compact sets $K_n \subset U_{d,n}$ the set $K_d = \{x_d\} \cup \bigcup_{n\in\omega} K_n$ is compact and has dimension $\mathsf{D}(K_d) = \sup_{n\in\omega} \mathsf{D}(K_n)$.

By our assumption, for every $d \in D$ and $n \in \omega$ we can find a compact subset $K_{d,n} \subset U_{d,n}$ with $\mathsf{D}(K_{d,n}) \in (d]_\Gamma$. Using the homotopical density of the subspace $2^X_{<\omega}$ of finite subsets in 2^X, construct a map $\kappa_{d,n} : A \to 2^X$ such that $\kappa_{d,n}(a) = K_{d,n}$ for every $a \in A_{d,n}$ and $\kappa_{d,n}(a)$ is a finite subset of $U_{d,n}$ for every $a \in A \setminus A_{d,n}$.

Now for every $a \in A$ and $d \in D$ consider the compact subset

$$\kappa_d(a) = \{x_d\} \cup \bigcup_{n\in\omega} \kappa_{d,n}(a) \subset U_d$$

having dimension

$$\mathsf{D}(\kappa_d(a)) = \sup_{n\in\omega} \mathsf{D}(\kappa_{d,n}(a)).$$

The choice of the sequence $\langle U_d \rangle_{d\in D}$ ensures that

$$\kappa(a) = \{x_\infty\} \cup \bigcup_{d\in D} \kappa_d(a)$$

is a compact subset of U with dimension

$$\mathsf{D}(\kappa(a)) = \sup_{d\in D} \mathsf{D}(\kappa_d(a)) = \sup\{\mathsf{D}(\kappa_{d,n}(a)) : d \in D, n \in \omega\}.$$

It is easy to prove that the map

$$\kappa : A \to 2^U, \kappa : a \mapsto \kappa(a),$$

is continuous. It remains to check that $\kappa^{-1}(\mathsf{D}_{>\gamma}(X)) = A_\gamma$ for all $\gamma \in \Gamma$.

If $a \in A \setminus A_\gamma$, then for every $d \ge \gamma$ in D the inclusion $a \in A \setminus A_d$ implies $\kappa_{d,n}(a) \in 2^X_{<\omega}$. In this case

$$\mathsf{D}(\kappa_d(a)) \le \sup_{n\in\omega} \kappa_{d,n}(a) = 0 \le \gamma.$$

On the other hand, for every $d < \gamma$ the inclusions $D(K_{d,n}) \in (d]_\Gamma \subset [0, \gamma]$, $n \in \omega$, and the choice of the sequence $\langle U_{d,n} \rangle_{n\in\omega}$ imply $\mathsf{D}(\kappa_d(a)) \le \sup_{n\in\omega} D(\kappa_{d,n}(a)) \le \gamma$.

Now the choice of the sequence $\langle U_d \rangle_{d\in D}$ guarantees that

$$\mathsf{D}(\kappa(a)) \le \sup_{d\in D} \mathsf{D}(\kappa_d(a)) \le \gamma$$

and hence

$$\kappa(a) \in \mathsf{D}_{\le\gamma}(X) = 2^X \setminus \mathsf{D}_{>\gamma}(X).$$

Now assume that $a \in A_\gamma$ and hence $a \in A_{\gamma,n}$ for some $n \in \omega$. If $\gamma < \inf(\Gamma_{>\gamma})$, then $\gamma \in D$ and $\mathsf{D}(K_{\gamma,n}) \in (\gamma]_\Gamma = (\gamma, \inf(\Gamma_{>\gamma})]$. Since $K_{\gamma,n} \subset \kappa(a)$, we conclude that $\mathsf{D}(\kappa(a)) \ge \mathsf{D}(K_{\gamma,n}) > \gamma$ and thus $\kappa(a) \in \mathsf{D}_{>\gamma}(X)$.

Next, assume that $\gamma = \inf(\Gamma_{>\gamma})$. In this case $\gamma = \inf(D_{>\gamma})$ and hence $A_\gamma = \bigcup_{d\in D_{>\gamma}} A_d$. It follows that $a \in A_{d,n}$ for some $d \in D_{>\gamma}$ and $n \in \omega$. Since $\kappa(a) \supset K_{d,n}$ and $\mathsf{D}(K_{d,n}) \in (d]_\Gamma \subset (\gamma, +\infty)$, we conclude that $\mathsf{D}(\kappa(a)) \ge \mathsf{D}(K_{d,n}) > \gamma$. So, again $\kappa(a) \in \mathsf{D}_{>\gamma}(X)$. □

The following characterization theorem implies Theorem 1 announced in the Introduction.

Theorem 9. *Let X be a topological space, $\mathsf{D} : 2^X_* \to [0,\infty]$ be a dimension function, and $\Gamma \subset [0,\infty)$ be a subset. The Γ-system $\langle 2^X, \mathsf{D}_{>\gamma}(X)\rangle_{\gamma\in\Gamma}$ is homeomorphic to the model $\sigma\mathfrak{C}_\Gamma$-absorbing Γ-system $\langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})\rangle_{\gamma\in\Gamma}$ if and only if*

1. *X is a non-degenerate Peano continuum,*
2. *each space $\mathsf{D}_{>\gamma}(X)$, $\gamma \in \Gamma$, is σ-compact, and*
3. *each non-empty open set $U \subset X$ for every $\gamma \in \Gamma$ contains a compact subset $K \subset U$ with $D(K) \in (\gamma]_\Gamma$.*

Proof. To prove the "only if" part, assume that the Γ-system $\mathscr{D} = \langle 2^X, \mathsf{D}_{>\gamma}(X)\rangle_{\gamma\in\Gamma}$ is homeomorphic to the model Γ-system $\Sigma_\Gamma = \langle Q^{\mathbb{Q}}, Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})\rangle_{\gamma\in\Gamma}$. Since 2^X is homeomorphic to $Q^{\mathbb{Q}}$, we may apply the Curtis–Schori Theorem [1] and conclude that X is a non-degenerate Peano continuum.

Since each space $\Sigma_\gamma = Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})$, $\gamma \in \Gamma$, is σ-compact, so is its topological copy $\mathsf{D}_{>\gamma}(X)$.

The Γ-system $\mathscr{D}$, being homeomorphic to the model $\sigma\mathfrak{C}_\Gamma$-absorbing Γ-system Σ_Γ, is strongly $\sigma\mathfrak{C}_\Gamma$-universal. Now Theorem 8 guarantees that for every $\gamma \in \Gamma$ each non-empty open subset $U \subset X$ contains a compact subset $K \subset U$ with $\mathsf{D}(K) \in (\gamma]_\Gamma$.

Next, we prove the "if" part. Assume that the conditions (1)–(3) are satisfied. We shall prove that the Γ-system $\mathscr{D}$ is $\sigma\mathfrak{C}_\Gamma$-absorbing. By the Curtis–Schori Theorem [1], the hyperspace 2^X is homeomorphic to the Hilbert cube Q. By Theorem 8, the Γ-system $\mathscr{D}$ is strongly $\sigma\mathfrak{C}_\Gamma$-universal. It is clear that this Γ-system is inf-continuous. By the condition (2), it is σ-compact. Hence $\mathscr{D} \in \sigma\mathfrak{C}_\Gamma$.

Let $D \subset \Gamma$ be a countable subset that meets each half-interval $[\gamma, \gamma+\varepsilon)$ where $\gamma \in \Gamma$ and $\varepsilon > 0$. It follows that $\bigcup_{\gamma\in\Gamma} \mathsf{D}_{>\gamma}(X) = \bigcup_{\gamma\in D} \mathsf{D}_{>\gamma}(X) \subset \mathsf{D}_{>0}(X)$ is a σZ-set in 2^X, being a σ-compact subset of 2^X that has empty intersection with the homotopy dense subset $2^X_{<\omega} \subset \mathsf{D}_{\le 0}(X)$ on 2^X. So, we can find a countable sequence $\langle Z_n\rangle_{n\in\omega}$ of Z-sets in 2^X such that $\bigcup_{n\in\omega} Z_n \supset \bigcup_{\gamma\in\Gamma} \mathsf{D}_{>\gamma}(X)$. Since $\mathscr{D} \in \sigma\mathfrak{C}_\Gamma$, we get $Z_n \cap \mathscr{D} \in \sigma\mathfrak{C}_\Gamma$ for all $n \in \omega$. This completes the proof of the $\sigma\mathfrak{C}_\Gamma$-absorbing property of the Γ-system $\mathscr{D}$. Since 2^X is homeomorphic to the Hilbert cube, Theorem 6 ensures that $\mathscr{D}$ is homeomorphic to the model Γ-system Σ_Γ. □

6. Mean Value Theorem for Hausdorff dimension

In this section we shall prove Theorem 4. First, we recall shortly the definitions of the Hausdorff measure and dimension. Given a complete separable metric space E and two non-negative real numbers s, ε, consider the number

$$\mathscr{H}^s_\varepsilon(E) = \inf_{\mathscr{B}} \sum_{B\in\mathscr{B}} (\mathrm{diam} B)^s,$$

where infimum is taken over all ε-covers $\mathscr{B}$ of E, i.e. cover of E by sets of diameter $\le \varepsilon$. Since X is separable, we can restrict ourselves by countable covers by closed subsets of diameter $\le \varepsilon$.

The limit $\mathscr{H}^s(E) = \lim_{\varepsilon\to 0} \mathscr{H}^s_\varepsilon(E)$ is called the s-dimensional Hausdorff measure of E. It is known that there is a unique finite or infinite number $\dim_H(E)$ called the *Hausdorff dimension* of E and denoted by $\dim_H(E)$ such that $\mathscr{H}^s(E) = \infty$ for all $s < \dim_H(E)$ and $\mathscr{H}^s(E) = 0$ for all $s > \dim_H(E)$, see [13,14].

Let (X, d) be a separable complete metric space. Theorem 4 will be proved as soon as for every positive real number $s < \dim_H(X)$ we shall find a compact subset $K \subset X$ with Hausdorff dimension $\dim_H(K) = s$.

It follows from $s < \dim_H(E)$ that $\mathscr{H}^s(E) = \infty$ and there exists $0 < \delta < 1$ with

$$\mathscr{H}^s_\delta(E) = k_0 > \frac{\delta^s}{2^{s-1}} = \left(\frac{\delta}{2}\right)^s + \left(\frac{\delta}{4}\right)^s + \left(\frac{\delta}{8}\right)^s + \cdots. \tag{0}$$

We define inductively a decreasing sequence $\{E_i\}_{i=1}^\infty$ of closed subsets of E. Let $E_1 = E$. Consider $\mathscr{H}^s_{\delta/2}(E_1)$. Two cases are possible (taking into account the definition of Hausdorff measure):

- $\mathscr{H}^s_{\delta/2}(E_1) = k_0$. In this case we take $E_2 = E_1$.
- $\mathscr{H}^s_{\delta/2}(E_1) > k_0$. Therefore we can choose a closed $\delta/2$-cover $\{U_1, \ldots, U_{m_1}, \ldots\}$ of the set E_1, (without loss of generality assume that this cover is ordered so that $\mathrm{diam}(U_{i+1}) \le \mathrm{diam}(U_i)$ for all i), such that

$$\mathscr{H}^s_{\delta/2}(E_1) \le \sum_i (\mathrm{diam}(U_i))^s < \mathscr{H}^s_{\delta/2}(E_1) + (\delta/2)^s. \tag{1}$$

Find a finite number m_1 such that

$$k_0 \le \sum_{i=1}^{m_1} (\mathrm{diam}(U_i))^s \le k_0 + (\delta/2)^s. \tag{2}$$

Then take $E_2 = \bigcup_{i=1}^{m_1} E_1 \cap U_i$.

Now we need to estimate $\mathscr{H}^s_{\delta/2}(E_2)$ (obviously the second case is interesting). For this we put $E_2' = \bigcup_{i>m_1} E_1 \cap U_i$ and note that

$$\mathscr{H}^s_{\delta/2}(E_1) \le \mathscr{H}^s_{\delta/2}(E_2) + \mathscr{H}^s_{\delta/2}(E_2'). \tag{3}$$

On the other hand

$$\sum_i (\mathrm{diam}(U_i))^s = \sum_{i\le m_1} (\mathrm{diam}(U_i))^s + \sum_{i>m_1} (\mathrm{diam}(U_i))^s. \quad (4)$$

Consider the real numbers

$$\varepsilon_1 = \sum_i (\mathrm{diam}(U_i))^s - \mathcal{H}^s_{\delta/2}(E_1),$$
$$\varepsilon_2 = \sum_{i=1}^{m_1} (\mathrm{diam}(U_i))^s - \mathcal{H}^s_{\delta/2}(E_2),$$
$$\varepsilon_2' = \sum_{i>m_1} (\mathrm{diam}(U_i))^s - \mathcal{H}^s_{\delta/2}(E_2')$$

and observe that $0 \le \varepsilon_1 < (\delta/2)^s$ by (1), and $\varepsilon_2, \varepsilon_2' \ge 0$. Therefore (3) and (4) yield $\varepsilon_1 \ge \varepsilon_2 + \varepsilon_2'$ and hence $0 \le \varepsilon_2 < (\delta/2)^s$. Taking into account (2), we have:

$$k_0 - (\delta/2)^s \le \mathcal{H}^s_{\delta/2}(E_2) \le k_0 + (\delta/2)^s. \quad (5)$$

Now denote $\mathcal{H}^s_{\delta/2}(E_2) = k_1$. From (0) and (5) it follows that $0 < k_1 < \infty$. By the definition of Hausdorff measure we have $0 < \mathcal{H}^s(E_2) \le \infty$, that in turn implies $\dim_H(E_2) \ge s$. It allows us to make the following inductive step.

Consider now $\mathcal{H}^s_{\delta/4}(E_2)$. If $\mathcal{H}^s_{\delta/4}(E_2) = k_1$, then take $E_3 = E_2$. If $\mathcal{H}^s_{\delta/4}(E_2) > k_1$, then similarly to the described above we find a closed $\delta/4$-cover $\{U_1, \dots, U_{m_2}, \dots\}$ of E_2, such that

$$\mathcal{H}^s_{\delta/4}(E_2) \le \sum_i (\mathrm{diam}(U_i))^s < \mathcal{H}^s_{\delta/4}(E_2) + (\delta/4)^s.$$

Find a finite number m_2 such that

$$k_1 \le \sum_{i=1}^{m_2} (\mathrm{diam}(U_i))^s \le k_1 + (\delta/4)^s.$$

Let $E_3 = \bigcup_{i=1}^{m_2} E_2 \cap U_i$. As above, we can to estimate $\mathcal{H}^s_{\delta/4}(E_3)$. We obtain:

$$k_1 - (\delta/4)^s \le \mathcal{H}^s_{\delta/4}(E_3) \le k_1 + (\delta/4)^s.$$

Or, taking into account (5):

$$k_0 - (\delta/2)^s - (\delta/4)^s \le \mathcal{H}^s_{\delta/4}(E_3) \le k_0 + (\delta/2)^s + (\delta/4)^s.$$

Again we can state that $\dim_H(E_3) \ge s$ and continue inductive process by constructing in similar way

$$E_4, E_5, \dots E_n, \dots,$$

for which we obtain in general case the estimate:

$$k_0 - (\delta/2)^s - \dots - (\delta/2^{n-1})^s \le \mathcal{H}^s_{\delta/2^{n-1}}(E_n) \le k_0 + (\delta/2)^s + \dots + (\delta/2^{n-1})^s. \quad (6)$$

It follows that $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ is a decreasing sequence of closed subsets of X with compact intersection $K = \lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$. Using the continuity of the measure $\mathcal{H}^s$ we obtain:

$$\mathcal{H}^s(K) = \lim_{n\to\infty} \mathcal{H}^s(E_n) = \lim_{n\to\infty}\lim_{i\to\infty} \mathcal{H}^s_{\delta/2^{i-1}}(E_n) = \lim_{n\to\infty} \mathcal{H}^s_{\delta/2^{n-1}}(E_n).$$

Additionally using (6) we obtain the estimate:

$$k_0 - \frac{\delta^s}{2^{s-1}} \le \mathcal{H}^s(F) \le k_0 + \frac{\delta^s}{2^{s-1}}.$$

Taking into account (0) we can state that $\dim_H(F) = s$.

References

[1] D.W. Curtis, R.M. Schori, Hyperspaces of peano continua are Hilbert cubes, Fund. Math. 101 (1978) 19–38.
[2] D.W. Curtis, Hyperspaces of noncompact metric spaces, Compos. Math. 40 (1980) 139–152.

[3] T. Dobrowolski, L. Rubin, The hyperspace of infinite-dimensional compacta for covering and cohomological dimension are homeomorphic, Pacific J. Math. 164 (1994) 15–39.
[4] H. Gladdines, Absorbing Systems in Infinite-Dimensional Manifolds and Applications, Vrije Universiteit, Amsterdam, 1994.
[5] R. Cauty, Suites $\mathcal{F}_\sigma$-absorbantes en théorie de la dimension, Fund. Math. 159 (1999) 115–126.
[6] N. Mazurenko, Absorbing sets related to Hausdorff dimension, Visnyk Lviv Univ., Ser. Mech-Math. 61 (2003) 121–128.
[7] N. Mazurenko, Uncountable absorbing systems related to the Hausdorff dimension, Mat. Stud. 31 (2009) 195–203.
[8] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, Lemgo, 1995.
[9] J.J. Dijkstra, J. van Mill, J. Mogilski, The space of infinite-dimensional compacta and other topological copies of $(l_f^2)^\omega$, Pacific J. Math. 152 (1992) 255–273.
[10] J. van Mill, Infinite-Dimensional Topology, Vrije Universiteit, Amsterdam, 1989.
[11] T.A. Chapman, Lectures on Hilbert Cube Manifolds, in: CBMS, vol. 28, AMS, Providence, 1975.
[12] T. Banakh, T. Radul, M. Zarichnyi, Absorbing Sets in Infinite-Dimensional Manifolds, in: Monograph Series, vol. 1, VNTL Publ., Lviv, 1996.
[13] G. Edgar, Measure, Topology and Fractal Geometry, Springer, New York, 1995.
[14] K.J. Falconer, The Geometry of Fractal Sets, Cambridge University Press, 1985.