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Transition Systems, Event Structures, and Unfoldings

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A subclass of transition systems called elementary transition systems can be identified with the help of axioms based on a structural notion called regions. Elementary transition systems have been shown to be *the* transition system model of a basic system model of net theory called elementary net systems. Here we show that by smoothly strengthening the regional axioms for elementary transition systems, one obtains a subclass called occurrence transition system. We then prove that occurrence transition systems are *the* transition system model of yet another basic model of concurrency, namely, prime event structures. We then propose an operation of unfolding elementary transition systems into occurrence transition systems. We prove that it is "correct" in a strong categorical sense.

INTRODUCTION

Elementary transition systems were introduced in Nielsen *et al.* (1992). They were proved to be, in a strong categorical sense, the transition system version of elementary net systems. The question arises whether the notion of a region and the axioms (mostly based on regions) imposed on ordinary transition systems to obtain elementary transition systems were simply "tuned" to obtain the correspondence with elementary net systems. Stated differently, one could ask whether elementary transition systems could also play a role in characterizing other models of concurrency.

We show here that by smoothly strenghtening the axioms of elementary transition systems one obtains a subclass called occurrence transition systems which turn out to be categorically equivalent to the well-known model of concurrency called prime event structures. Thus there is more to elementary transition systems than just their (co-reflective) relationship to a basic model of net theory, namely, elementary net systems.

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Next, we turn to the problem of unfolding elementary transition systems into occurrence transition systems. Prime event structures have been studied extensively in the literature. One of the first results concerning these objects was a characterization in terms of so-called occurrence nets (Nielsen *et al.*, 1981). Within net theory, occurrence nets are used to describe the behaviour of net systems, via a notion of unfolding. One of the main results from this theory is Winskel's beautiful characterization (Winskel, 1987) of the unfolding operation categorically as the right adjoint to the inclusion functor from occurrence nets to 1-safe net systems (with the co-unit as the folding morphism). This construction was adapted in Nielsen *et al.* (1990) to elementary net systems. Combining our characterization of prime event structures with the result of Nielsen *et al.* (1981) we know that occurrence transition systems are exactly the case-graphs of occurrence nets. Also, from Nielsen *et al.* (1992) we know that elementary transition systems are exactly the case-graphs of elementary net systems. So, this raises the natural question whether in our more abstract setting we may characterize the unfolding of elementary transition systems into occurrence transition systems, in the spirit of Winskel. Indeed, we prove that the inclusion functor from occurrence transition systems to elementary transition systems does have a right adjoint, *the* unfolding functor, the definition of which is guided by the unfolding operation from net theory.

In the next section, a brief review—and a convenient reformulation—of the category of elementary transition systems ETS is provided. Section 2 contains a quick introduction to the category of prime event structures, PES, due to Winskel (1987). In the subsequent section, we identify the subcategory of occurrence transition systems, OTS, by a smooth strengthening of the regional axioms for elementary transition systems. We then proceed to establish a few

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properties of occurrence transition systems. Using these properties, we show in Section 4 that OTS and PES are equivalent categories. Thus, in some sense, occurrence transition systems are *the* transition system model of prime event structures (in the same sense that prime algebraic, coherent domains, are *the* domain model of prime event structures; see Winskel, 1987). In Section 5, we show that occurrence transition systems can be used to define the unfoldings of elementary transition systems. Exploiting some technical results from the theory of trace languages, we show that the unfold operation, when applied to the objects in ETS, yields objects in OTS. Moreover, we prove that this unfold operation uniquely extends to a functor which is the right adjoint to the inclusion functor from OTS to ETS. This result mirrors the strong result due to Winskel (1987) on the side of net theory which established the "correctness" of the unfolding of elementary net systems (and, in fact, 1-safe Petri nets) into occurrence nets proposed in Nielsen *et al.* (1981).

1. ELEMENTARY TRANSITION SYSTEMS

The purpose of this section is to recall (and rephrase!) the main concepts and results from Nielsen *et al.* (1992).

DEFINITION 1.1. A *transition system* is a four-tuple $TS = (S, E, T, s^{in})$ where

S is the set of *states*,

E is the set of *events*,

$T \subseteq S \times E \times S$ is the set of *transitions*, and

$s^{in} \in S$ is the *initial* state.

DEFINITION 1.2. A *region* of a transition system $TS = (S, E, T, s^{in})$ is a subset of states, $R \subseteq S$, satisfying:

$$\forall (s_0, e, s'_0), (s_1, e, s'_1) \in T.$$
$$(s_0 \in R \wedge s'_0 \notin R) \Leftrightarrow (s_1 \in R \wedge s'_1 \notin R)$$
$$\text{and} \quad (s_0 \notin R \wedge s'_0 \in R) \Leftrightarrow (s_1 \notin R \wedge s'_1 \in R).$$

We shall use the following notations for a qiven a transition system $TS = (S, E, T, s^{in})$.

— R_{TS} is the set of nontrivial (proper, nonempty subsets of S) regions of TS.

— R_s, where $s \in S$, is the set of nontrivial regions containing s; formally

$$R_s \stackrel{\text{def}}{=} \{R \in R_{TS} \mid s \in R\}.$$

— ${}^\circ R$, R°, where $R \in R_{TS}$, is the set of events entering/leaving R resp.; formally,

$${}^\circ R \stackrel{\text{def}}{=} \{e \in E \mid \exists (s, e, s') \in T.\, s \notin R \wedge s' \in R\}, \quad \text{and}$$
$$R^\circ \stackrel{\text{def}}{=} \{e \in E \mid \exists (s, e, s') \in T.\, s \in R \wedge s' \notin R\}.$$

— ${}^\circ e$, e°, where $e \in E$, is the set of pre- and post-regions of e resp., i.e., the set of regions that e is (consistently) leaving/entering; formally,

$${}^\circ e \stackrel{\text{def}}{=} \{R \in R_{TS} \mid \exists (s, e, s') \in T.\, s \in R \wedge s' \notin R\}, \quad \text{and}$$
$$e^\circ \stackrel{\text{def}}{=} \{R \in R_{TS} \mid \exists (s, e, s') \in T.\, s \notin R \wedge s' \in R\}.$$

PROPOSITION 1.3. *Let $TS = (S, E, T, s^{in})$ be a transition system. Then*

(i) *$R \subseteq S$ is a region iff $S \backslash R$ is a region,*

(ii) *$\forall e \in E.\, e^\circ = \{S \backslash R \mid R \in {}^\circ e\}$,*

(iii) *$\forall (s, e, s') \in T.\, R_s \backslash R_{s'} = {}^\circ e$ and $R_{s'} \backslash R_s = e^\circ$ and consequently $R_{s'} = (R_s \backslash {}^\circ e) \cup e^\circ$.*

Given a transition system $TS = (S, E, T, s^{in})$ we shall use the following notation.

— For every $e \in E$, $\xrightarrow{e} \subseteq S \times S$, where $(s, s') \in \xrightarrow{e} \Leftrightarrow (s, e, s') \in T$.

— Let $\rho \in E^*$, $\rho = e_1 e_2 \cdots e_n$, $n \geq 1$. Then $\Rightarrow^\rho \subseteq S \times S$ where $(s, s') \in \Rightarrow^\rho$ iff $\exists s_0, s_1, ..., s_n$ such that $s = s_0 \xrightarrow{e_1} s_1 \cdots s_{n-1} \xrightarrow{e_n} s_n = s'$. By convention, $\Rightarrow^\Lambda = \{(s, s) \mid s \in S\}$, where Λ denotes the null string.

— The set of *computations* of TS is defined as

$$C_{TS} = \{\rho \in E^* \mid \stackrel{\rho}{\Longrightarrow} \cap (\{s^{in}\} \times S) \neq \varnothing\},$$

and the set of nonempty computations of TS is defined as

$$C_{TS^+} = C_{TS} \cap E^+.$$

— $\rightarrow \subseteq S \times S$, where $\rightarrow =^{\text{def}} \bigcup_{e \in E} \xrightarrow{e}$, and

— $\xrightarrow{*}$ is the transitive and reflexive closure of $\rightarrow$.

— For every $s \in S$, $\uparrow s =^{\text{def}} \{s' \in S \mid (s, s') \in \xrightarrow{*}\}$.

So $\uparrow s$ denotes the set of states reachable from s via the transitions of TS.

The results of Nielsen *et al.* (1992) show that the category of elementary transition systems, ETS, introduced below is *the* category of the (sequential) case graphs of elementary net systems. We recall that elementary net systems is a basic system model of net theory in which fundamental behavioural aspects of distributed systems such as causality, concurrency, conflict, and confusion can be made transparent (Thiagarajan, 1987). We also recall that there is a natural way of associating a transition system with an elementary net system using the notion of a sequential case-graph which explicates the operational behaviour of elementary net system (Rozenberg, 1987).

We present the definition of ETS as it was stated in Nielsen *et al.* (1992).

DEFINITION 1.4 (ETS-Objects). A Transition System $TS = (S, E, T, s^{in})$ is said to be *elementary* iff it satisfies the following axioms:

(S1) $\uparrow s^{in} = S$ (every state reachable from s^{in}).

(S2) $\forall s, s' \in S.\ R_s = R_{s'} \Rightarrow s = s'$ (regional separability of states).

(T1) $\forall s \in S,\ e \in E.\ [{}^{\circ}e \subseteq R_s \Rightarrow \exists s' \in S.\ (s, e, s') \in T]$ (enabling of events).

(T2) $\forall (s, e, s') \in T.\ s \neq s'$ (i.e., $\xrightarrow{e}$ irreflexive for every $e \in E$).

(T3) $\forall (s, e_1, s_1), (s, e_2, s_2) \in T.\ [s_1 = s_2 \Rightarrow e_1 = e_2]$

$$(\text{i.e., } e_1 \neq e_2 \Rightarrow \xrightarrow{e_1} \cap \xrightarrow{e_2} = \varnothing).$$

(E) $\forall e \in E.\ \exists (s, e, s') \in T.$ (i.e., $\xrightarrow{e}$ nonempty).

DEFINITION 1.5 (ETS-morphisms). Let $TS_i = (S_i, E_i, T_i, s_i^{in})$ for $i = 0, 1$ be a pair of transition systems. A *morphism* from TS_0 to TS_1 is a pair (f, η) where

$f: S_0 \to S_1$ is a total function from S_0 to S_1, and

$\eta: E_0 \to E_1$ is a partial function from E_0 to E_1 such that

(1) $f(s_0^{in}) = s_1^{in}$,

(2) $\forall (s_0, e_0, s_0') \in T_0.$

$$\begin{cases} f(s_0) = f(s_0'), & \text{if } \eta(e_0) \text{ undefined}, \\ (f(s_0), \eta(e_0), f(s_0')) \in T_1, & \text{if } \eta(e_0) \text{ defined}. \end{cases}$$

Composition of morphisms is componentwise composition of the total/partial functions and identity is the pair of identity functions. Note that isomorphisms are identities up to names of states and events.

We let ETS denote the category of objects and morphisms as defined in Definitions 1.4 and 1.5. In Nielsen *et al.* (1992), a category ENS of elementary net systems as objects and suitably defined behaviour preserving net-morphisms is introduced. We recall the main result from this work.

THEOREM 1.6. *There exists a co-reflection between ETS and ENS, where the right adjoint is the well-known case-graph construction from net systems, and the left adjoint constructs an elementary net system from an ETS-object, in which the regions play the role of local states (conditions in net theory).*

As stated earlier, the importance of this result is that the axioms from Definition 1.4 identify a transition system based model of "true concurrency"—*not* by adding structure, but by imposing the six axioms of Definition 1.4. The reader will have noticed that the notion of regions plays a central role in the axiomatization (S2,T1), but that the axiomatization also contains structural/syntactical axioms like T2, T3, and E. For the purpose of the following sections we provide here an almost purely regional axiomatization of elementary transition systems.

THEOREM 1.7. *A transition system* $TS = (S, E, T, s^{in})$ *is elementary iff it satisfies axioms* S1, S2, T1 *from Definition* 1.4, *and*

(E1) $\forall e \in E.\ {}^{\circ}e \neq \varnothing$.

(E2) $\forall e, e' \in E.\ {}^{\circ}e = {}^{\circ}e' \Rightarrow e = e'$ *(regional separability of events)*.

Proof. ($\Rightarrow$) The fact that E1 and E2 follow from the original ETS-axioms is immediate from (the proof of Proposition 4.2 in) Nielsen *et al.* (1992).

($\Leftarrow$) Assume *TS* satisfies E1. Let $R \in {}^{\circ}e$. By the definition of ${}^{\circ}e$ this implies that we must have $(s, e, s') \in T$ such that $s \in R$ and $s' \notin R$. Hence axiom E follows from E1. Further assume $(s, e, s') \in T$ for some $s, s' \in S$. This implies $s \in R$, $s' \notin R$, i.e., $s \neq s'$, and hence T2 also follows from E1. Assume *TS* satisfies E2 and that $(s, e_1, s'), (s, e_2, s') \in T$. Clearly this means that $\forall R \in R_{TS}.\ [R \in {}^{\circ}e_1 \Leftrightarrow (s \in R \text{ and } s' \notin R) \Leftrightarrow R \in {}^{\circ}e_2]$. By E2, $e_1 = e_2$ and hence T3 holds. ∎

It may be worth noting that the "if part" of the proof above shows that T2, T3 and E (the old structural axioms) follow from E1 and E2 (the new regional axioms). The other direction of this implication does *not* hold (the proof of the "only if part" from Nielsen *et al.* (1992) makes use of axioms S2 and T1!).

2. PRIME EVENT STRUCTURES

In this section, we briefly introduce one of the fundamental models of concurrency called prime event structures originally introduced in Nielsen *et al.* ((1981) and since then studied extensively by primarily Winskel (1987). It is important to realize, that event structures is basically a model of concurrency on the behavioural level; i.e., events represent unique temporal occurrences of actions. In contrast the models mentioned in the previous section, ETS and ENS, are basically models on the system level, in which events may have repeated occurrences at different times in different contexts. We now introduce the category of prime event structures, PES.

DEFINITION 2.1 (PES-Objects). A prime event structure is a triple $ES = (E, \leqslant, \#)$ where

E is a set of *events*,

$\leqslant\ \subseteq E \times E$ is a partial order (*causality*),

$\# \subseteq E \times E$ is a symmetric relation (*conflict*), where

(A1) $\forall e_0, e_1, e_2 \in E.\ e_0 \# e_1 \leqslant e_2 \Rightarrow e_0 \# e_2$ (conflict inheritance),

(A2) $\forall e \in E.\ [e] =^{\text{def}} \{e' \in E \mid e' \leqslant e\}$ is finite and for all $e', e'' \in [e]$ not $(e' \# e'')$.

Given ES as above—the *configurations* of ES are defined as

$$C(ES) \overset{\text{def}}{=} \{c \subseteq E \mid (\forall e, e' \in c.\ \text{not}\ (e \# e'))$$
$$\text{and } \forall e, e' \in E.\ e' \leqslant e \in c \Rightarrow e' \in c\}.$$

So, configurations of ES are the downwards (w.r.t. $\leqslant$) closed and conflict-free subsets of E. We use the notation $FC(ES)$ for the set of finite configurations of ES. In particular, for all $e \in E$, $[e] \in FC(ES)$.

Definition 2.2 (PES-Morphisms). Let $ES_i = (E_i, \leqslant_i, \#_i)$ for $i = 0, 1$ be two Prime Event Structures. A morphism from ES_0 to ES_1 is a partial function η from E_0 to E_1 satisfying (when extended pointwise to sets of events)

$\forall c \in C(ES_0)$.

(*) $[\eta(c) \in C(ES_1)$ and $\forall e, e' \in c$.

$[\eta(e) = \eta(e')$ (and both defined) $\Rightarrow e = e']]$.

Composition of morphisms is normal composition of partial functions, and the identity is the identity function. Note that isomorphisms are identities up to names of events.

We refer the reader to Winskel (1987) for intuition, detailed explanations and results concerning the category PES of prime event structures with objects and morphisms defined in Definitions 2.1 and 2.2. We only wish to mention here that the configurations of a prime event structure may be thought of as the states of a distributed system, where the state is identified with the "events having occurred" at the given state. The fundamental notions of causality (or rather causal dependence) and conflict (exclusion/choice among events) are captured directly by the relations $\leqslant$ and $\#$ in the definition of a prime event structure. The notion of concurrency (or independence) between events may be derived as follows:

$$e \text{ co } e' \overset{\text{def}}{\Leftrightarrow} \text{not } (e \leqslant e' \text{ or } e' \leqslant e \text{ or } e \# e').$$

We shall use the notation $c_0 \overset{e}{\longrightarrow\!\!<} c_1$ for a structure evolving from c_0 to c_1 through the occurrence of event e, i.e., for a prime event structure, ES, as in Definition 2.1. Actually it is sufficient to consider just finite configurations. $\longrightarrow\!\!< \subseteq FC(ES) \times E \times FC(ES)$ is given by

$$(c_0, e, c_1) \in \longrightarrow\!\!< \text{ iff } c_0 \subset c_1 = c_0 \cup \{e\}.$$

As usual, we will often write $c_0 \overset{e}{\longrightarrow\!\!<} c_1$ instead of $(c_0, e, c_1) \in \longrightarrow\!\!<$. We shall use the following facts about prime event structures.

Proposition 2.3. *Let $ES = (E, \leqslant, \#)$ be a prime event structure. Then for every $c \in FC(ES)$, and for every linearization $e_0, e_1, ..., e_n$ (i.e., every listing such that $e_i \leqslant e_j \Rightarrow i \leqslant j$) of the events belonging to c, there exist configurations $c_0, c_1, ..., c_n$ such that*

$$\varnothing \overset{e_0}{\longrightarrow\!\!<} c_0 \overset{e_1}{\longrightarrow\!\!<} c_1 \overset{e_2}{\longrightarrow\!\!<} \cdots c_{n-1} \overset{e_n}{\longrightarrow\!\!<} c_n = c.$$

Proof. See Winskel (1987). ∎

Lemma 2.4. *Let ES_i be two prime event structures as in Definition 2.2, and let η be a partial function from E_0 to E_1. Then η is a morphism from ES_0 to ES_1 iff the condition (*) of Definition 2.2 is satisfied for all finite configurations c of ES_0.*

Proof. The "only if" part of the Lemma is trivial, so we concentrate on the nontrivial "if part." Let η satisfy (*) for all finite configurations and let c be a (infinite) configuration of ES_0.

We first prove that $\eta(c) \in C(ES_1)$. Assume $e_1 \in \eta(c)$ and $e'_1 \leqslant_1 e_1$. $e_1 \in \eta(c)$ implies that we must have e_0 such that $\eta(e_0) = e_1$ and since from definition $[e_0] \in FC(ES_0)$, we have from our assumption $\eta([e_0]) \in FC(ES_1)$. Now, from this we have $e'_1 \in \eta([e_0])$, and hence there must exist $e'_0 \in [e_0]$ such that $\eta(e'_0) = e'_1$. Since c is downwards closed, $e'_0 \in c$, and hence $e'_1 \in \eta(c)$, i.e., $\eta(c)$ is downwards closed.

Assume $\eta(e_0), \eta(e'_0) \in \eta(c)$, $e_0, e'_0 \in c$. Then it follows that $[e_0] \cup [e'_0] \in FC(ES_0)$, hence $\eta([e_0] \cup [e'_0]) \in FC(ES_1)$ (from the assumption of Lemma), and hence not $(\eta(e_0) \# \eta(e'_0))$, i.e., $\eta(c)$ is conflict free.

Finally, let $e_0, e'_0 \in c$ and $\eta(e_0) = \eta(e'_0)$ and both defined. Then again, since $[e_0] \cup [e'_0] \in FC(ES_0)$, we get from the assumption of the lemma, that not only is $\eta([e_0] \cup [e'_0])$ a configuration of ES_1, but also $e_0 = e'_0$. ∎

3. OCCURRENCE TRANSITION SYSTEMS

In this section, we introduce a (full) subcategory of ETS, called the category of occurrence transition systems, OTS, and prove some properties of this subcategory. The main point is that OTS is defined as a simple strengthening of the axiomatization of ETS-objects, and it will be proved in the next section that OTS is (categorically) equivalent to the category of prime event structures. In this section we only prove some technical lemmas for OTS to be used in the proofs of the main results of the next sections.

Definition 3.1. (OTS Category). Let OTS denote the category consisting of

— *objects*: transition systems $TS = (S, E, T, s^{in})$ satisfying axioms S1, S2, and T1 of Definition 1.4 and

$$\text{axiom O: } \forall e \in E. \exists s \in S. [\uparrow s \in R_{TS} \text{ and } {}^{\circ}\uparrow s = \{e\}],$$

— *morphisms*: transition system morphisms as defined in Definition 1.5.

PROPOSITION 3.2. *OTS is a full subcategory of ETS.*

Proof. Follows immediately from Theorem 1.7, because axiom O trivially implies (by Proposition 1.3) E1 and E2. ▮

One might say that OTS is obtained from ETS by a strengthening of axioms E1 and E2. E1 and E2 may be interpreted as "each event is characterized by its nonempty set of pre-regions (or, of course, equivalently its set of post-regions)". Axiom O may be interpreted as "each event is charaterized by one single post-region (or equivalently pre-region) of a particularly simple form (equal to $\uparrow s$ for some $s \in S$)." However, this seemingly innocent strengthening implies some dramatic restrictions on the kind of allowable transition systems.

LEMMA 3.3. *Let $TS = (S, E, T, s^{in})$ be an OTS-object. Then $\xrightarrow{*} \subseteq S \times S$ is a partial order with s^{in} as the least element.*

Proof. Transitivity and reflexivity of $\xrightarrow{*}$ follow from definition. We must only prove antisymmetry. Take any $(s', e, s'') \in T$. From axiom O we have a region $R = \uparrow s$ for some $s \in S$ such that ${}^{\circ}R = \{e\}$, i.e., $s' \notin R$, $s'' \in R$. Antisymmetry now follows from the observation: $(s'' \in \uparrow s$ and $s' \notin \uparrow s) \Rightarrow s' \notin \uparrow s''$. Minimality of s^{in} w.r.t. $\xrightarrow{*}$ follows directly from axiom S1. ▮

LEMMA 3.4. *Let $TS = (S, E, T, s^{in})$ be an OTS-object. Assume $\uparrow s$ is a region of TS such that ${}^{\circ}\uparrow s = \{e\}$. Then there exists $s' \in S$ such that $(s', e, s) \in T$. Furthermore, assume also for some s'', ${}^{\circ}\uparrow s'' = \{e\}$. Then $s = s''$.*

Proof. Consider $\uparrow s$. Since $\uparrow s \in R_{TS}$, $s \neq s^{in}$ and from Lemma 3.3 and S1 we get $s^{in} \notin \uparrow s$. But $s \in \uparrow s$ and so, because $\{e\} = {}^{\circ}\uparrow s$, there exist $\bar{s}, \bar{\bar{s}} \in S$ such that $\bar{s} \notin \uparrow s$, $\bar{\bar{s}} \in \uparrow s$, and $s^{in} \xrightarrow{*} \bar{s} \xrightarrow{e} \bar{\bar{s}} \xrightarrow{*} s$. From ${}^{\circ}\uparrow s = \{e\}$ we get $\bar{\bar{s}} \in \uparrow s$, and hence from Lemma 3.3, $\bar{\bar{s}} = s$. This proves the first part of the lemma. Furthermore, if $\{e\} = {}^{\circ}\uparrow s''$, it must be that $s'' \xrightarrow{*} s$. By symmetric arguments we get $s \xrightarrow{*} s''$. Hence, by Lemma 3.3, $s = s''$. ▮

So, from Lemma 3.4, we may talk about *the* state s satisfying the property of axiom O for a given e of an OTS-object. We shall use the notation s_e, $e \in E$, for this particular state. Obviously from the definition of ${}^{\circ}R$, this association is injective in the sense that $s_e = s_{e'} \Rightarrow e = e'$. So, we may think of s_e as "the state representation of e."

As an example, in Fig. 3.1. the elementary transition systems (a) and (b) do not satisfy axiom O (with respect to event e—left to the reader to check). (c), however, is an occurrence transition system, and we have indicated the unique state representations for the three events.

Based on this, one may ask if there is also a natural way to talk about the states of an OTS-object in terms of its events. One obvious idea seems to be to associate with a state s the set of events e for which s belongs to the characteristic region $\uparrow s_e$.

DEFINITION 3.5. Let $TS = (S, E, T, s^{in})$ be an OTS-object. Let past: $S \to 2^E$ be the function defined as $past(s) = \{e \mid s \in \uparrow s_e\}$.

In Fig. 3.1c, $past(s) = \{e_1, e_2\}$, $past(s_{e_3}) = \{e_2, e_3\}$.

The use of the word "past" is justified by the following lemma.

LEMMA 3.6. *Let $TS = (S, E, T, s^{in})$ be an OTS-object.*

(a) *$past(s^{in}) = \varnothing$ and*

(b) *for every $(s, e, s') \in T$, $past(s) \subset past(s') = past(s) \cup \{e\}$.*

For every computation of the form

$$s^{in} = s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} s_2 \cdots \xrightarrow{e_n} s_n = s,$$

we have

(c) $1 \leqslant i < j \leqslant n \Rightarrow e_i \neq e_j$, *and*

(d) $\{e_i \mid 1 \leqslant i \leqslant n\} = past(s)$.

Proof. Clearly (c) and (d) follow from (a) and (b). Assume $s^{in} \in \uparrow s_e$. From axiom S1 we get $\uparrow s_e = S$, contradicting $\uparrow s_e$ being a nontrivial region. Hence we conclude (a).

Consider an arbitrary $(s, e, s') \in T$.

Obviously $s' \in \uparrow s$, and so $past(s) \subseteq past(s')$.

Since $\uparrow s_e$ is a region such that ${}^{\circ}\uparrow s_e = \{e\}$, $s \notin {}^{\circ}\uparrow s_e$ and $s' \in {}^{\circ}\uparrow s_e$. Hence $e \in past(s') \backslash past(s)$.

Now let $e' \in past(s')$ be such that $e \neq e'$. Since $e' \in past(s')$, $s' \in \uparrow s_{e'}$. Since $e \neq e'$ and ${}^{\circ}\uparrow s_{e'} = \{e'\}$, it must be that $s \in \uparrow s_e$ (due to T3) which implies that $e' \in past(s)$. Consequently $past(s') \backslash past(s) = \{e\}$, and so (b) holds. ▮

LEMMA 3.7. *Let $TS = (S, E, T, s^{in})$ be an OTS-object. The function past from Definition 3.5 is injective.*

Proof. Let $s \in S$, and let R be any region of TS. Then from Lemma 3.6 (c) and (d) we get

$$(*) \quad s \in R \text{ iff either } (s^{in} \in R \text{ and } |R^{\circ} \cap past(s)| = |{}^{\circ}R \cap past(s)|)$$
$$\text{or } (s^{in} \notin R \text{ and } |R^{\circ} \cap past(s)| + 1 = |{}^{\circ}R \cap past(s)|),$$

where $|M|$ denotes the cardinality of a set M. From this we clearly get for two states s and s' that

$$past(s) = past(s') \Rightarrow \forall R \in R_{TS}. [s \in R \text{ iff } s' \in R].$$

But then by axiom S2 (Definition 1.4), we conclude that $s = s'$. ∎

4. EQUIVALENCE BETWEEN OTS AND PES

In this section, we prove that there is a very strong relationship between the two categories OTS and PES; they are basically one and the same thing in the sense that they are categorically equivalent. So, one might conclude that the axioms of OTS-objects identify *the* transition system version of prime event structures.

It was indicated already in Nielsen *et al.* (1981) that one may view a PES-object as a transition system, where the states correspond to configurations, and transitions to the $\overset{e}{-\!\!<}$-relations mentioned previously. We start by proving that this idea may be formalized in the form of a functor T: PES → OTS.

THEOREM 4.1. *T defined as follows is a functor from PES to OTS*:

— *On objects*: $T(ES = (E, \leqslant, \#)) =^{\text{def}} (FC(ES), E, -\!\!<, \varnothing)$.

— *On morphisms*: *Let η be a* PES-*morphism from ES_0 to ES_1. Then $T(\eta) = (f, \eta)$, where $\forall c_0 \in FC(ES_0). f(c_0) = \eta(c_0)$.*

Proof. The only non-trivial part is to see that $T(ES)$ as defined satisfies the axioms for OTS objects.

(S1) $\uparrow\varnothing = FC(ES)$ in $T(ES)$. Follows from Proposition 2.3.

(S2) $R_c = R_{c'} \Rightarrow c = c'$ in $T(ES)$, where $c, c' \in FC(ES)$. Assume $c \neq c'$, e.g., there exists $e \in c$, $e \notin c'$. It is easy to see that $R_e =^{\text{def}} \{x \in FC(ES) \mid e \in x\}$ is a region of $T(ES)$ (such that ${}^\circ R_e = \{e\}$ and $R_e^\circ = \varnothing$). Clearly $c \in R_e$, $c' \notin R_e$.

(T1) ${}^\circ e \subseteq R_c \Rightarrow \exists c'. [c \overset{e}{-\!\!<} c'$ in $T(ES)]$, $e \in E$, $c \in FC(ES)$. Obviously all one must prove is that from the assumption $c \in FC(ES)$ and ${}^\circ e \subseteq R_c$ in $T(ES)$ we get $c \cup \{e\} \in FC(ES)$. (From ${}^\circ e \subseteq R_c$ and the fact that $FC(ES) \backslash R_c$ is a region we at once get $e \notin c$ (R_e is the region constructed above)).

$c \cup \{e\}$ can fail to be a configuration for two reasons.

Case 1. $c \cup \{e\}$ is not downwards closed, i.e., there exists $e' < e$ such that $e' \notin c \cup \{e\}$, i.e., $e' \notin c$. From $e' < e$ it is easy to see that $R = \{x \in FC(ES) \mid e' \in x, e \notin x\}$ is a region of $T(ES)$ such that $R \in {}^\circ e$. But we have also $R \notin R_c$ (since $e' \notin c$). Thus we get contradiction to our assumption ${}^\circ e \subseteq R_c$.

Case 2. $c \cup \{e\}$ is not conflict free, i.e., there exists $e' \in c$ such that $e \;\#\; e'$ (remember c is configuration). From $e \;\#\; e'$, it is again easy to see that $R = \{x \in FC(ES) \mid e \notin x$ and $e' \notin x\}$ is a region of $T(ES)$ such that $R \in {}^\circ e$ (and $R \in {}^\circ e'$). But since $e' \in c$ we also have $R \notin R_c$, and hence again a contradiction to our assumption ${}^\circ e \subseteq R_c$.

(O) $\forall e \in E. \exists c \in FC(ES). [(\uparrow c \in R_{T(ES)}$ and ${}^\circ\!\uparrow c = \{e\})$ in $T(ES)]$. Given $e \in E$, define $c_e =^{\text{def}} [e]$. It follows immediately that $[e] \in FC(ES)$. It follows from Proposition 2.3 that $\uparrow[e] = \{x \in FC(ES) \mid e \in x\} = R_e$ as in the proof of S2 above. Now we have already seen that $R_e \in R_{T(ES)}$ and ${}^\circ R_e = \{e\}$. ∎

As an example, the reader may verify that applying T to the following PES-object yields an OTS-object isomorphic to the one from Fig. 3.1(c): $ES = (\{e_1, e_2, e_3\}, \leqslant, \#)$, where $< = \{(e_2, e_3)\}$ and $\# = \{(e_1, e_3), (e_3, e_1)\}$.

THEOREM 4.2. *The functor T determines an equivalence of categories between PES and OTS.*

Proof. It follows from MacLane (1971, theorem 4.4.1) that it is sufficient to prove that T is full and faithful, and that for every OTS-object TS there exists a PES-object ES such that TS is isomorphic to $T(ES)$. These three facts are proved in three separate lemmas in the following. ∎

LEMMA 4.3. *T is full.*

Proof. Given two prime event structures $ES_i = (E_i, \leqslant_i, \#_i)$, $i = 0, 1$ and an OTS-morphism (f, η) from $T(ES_0)$ to

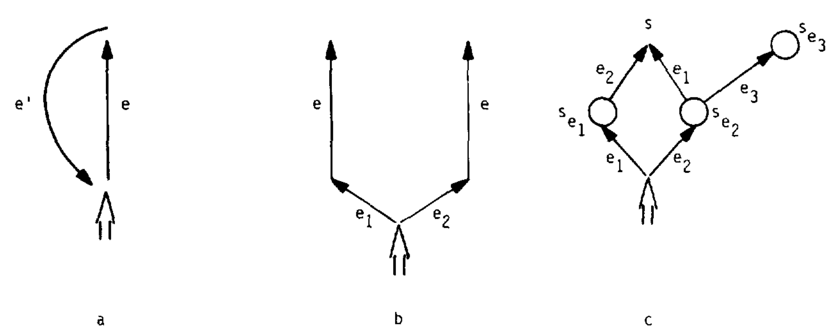

FIGURE 3.1

$T(ES_1)$, we must prove that there exists a PES-morphism $\hat{\eta}$ from ES_0 to ES_1 such that $T(\hat{\eta}) = (f, \eta)$. Since (f, η) is an OTS-morphism, we have from the definition of T that η is a partial function from E_0 to E_1. Suppose η is itself a PES-morphism from ES_0 to ES_1. Then, once again by the definition of T, $T(\eta) = (g, \eta)$ is an OTS-morphism from $T(ES_0)$ to $T(ES_1)$ where g: $FC(ES_0)$ to $FC(ES_1)$ is given by $g(c) = \eta(c)$ for every $c \in FC(ES_0)$. But from [NRT2], it follows that if (f_1, η_1) and (f_2, η_2) are a pair of OTS-morphisms from TS to TS' then $\eta_1 = \eta_2$ implies $f_1 = f_2$. Now (f, η) and (g, η) are a pair of morphisms from $T(ES_0)$ to $T(ES_1)$. Hence we can conclude that $f = g$ and this would establish the fullness of T.

Thus it suffices to prove that η is a PES-morphism from ES_0 to ES_1. So, to prove that η must be a PES-morphism from ES_0 to ES_1, we make use of Lemma 2.4; i.e., we show that property $(*)$ of Definition 2.2 is satisfied for every $c \in FC(ES_0)$. By simple induction on the size of c, we can show that $f(c) = \eta(c)$ and since f: $FC(ES_0) \to FC(ES_1)$ we have that $\eta(c) \in C(ES_1)$. Second, assume $e, e' \in c$, $e \neq e'$, and that $\eta(e)$ and $\eta(e')$ are both defined. From Proposition 2.3 we may assume configurations c, c'' such that $c' \xrightarrow{e} c''$ and $e' \in c'$. From the arguments above, we have $f(c') = \eta(c')$, i.e., $\eta(e') \in f(c')$. But since (f, η) is a morphism and $\eta(e)$ defined we have $f(c') \xrightarrow{\eta(e)} f(c'')$ in ES_1, but this implies $\eta(e) \notin f(c')$, i.e., $\eta(e) \neq \eta(e')$ as required. ∎

LEMMA 4.4. *T is faithful.*

Proof. Let η, η' be two PES-morphisms from ES_0 to ES_1. We must prove that $\eta \neq \eta'$ implies that $T(\eta) \neq T(\eta')$. But this follows from the definition of T. ∎

LEMMA 4.5. *For every OTS-object TS there exists a PES-object ES such that TS and $T(ES)$ are isomorphic.*

Proof. Given an OTS-object $TS = (S, E, T, s^{\text{in}})$ we define $\zeta(TS) = (E, \leqslant, \#)$ where $\forall e, e' \in E$.

$$e \leqslant e' \quad \text{iff} \quad s_e \xrightarrow{*} s_{e'} \text{ in } TS \text{ and}$$

$$e \# e' \quad \text{iff} \quad (\uparrow s_e \cap \uparrow s_{e'}) = \varnothing \text{ in } TS,$$

where s_e and $s_{e'}$ are the unique states associated with e and e', respectively according to Lemma 3.4. First, we must prove that $\zeta(TS)$ is a prime event structure. Lemma 3.3 tells us that $\leqslant$ is a partial order and from the definition we get that $\#$ is a symmetric relation such that $\leqslant \cap \# = \varnothing$. $\#$ is also clearly inherited by $\leqslant$ in the sense of A1 of Definition 2.1. Finally, A2 of Definition 2.1 follows from Lemma 3.6 and the fact that $[e]$ from the definition equals $past(s_e)$. So, $\zeta(TS)$ *is* a prime event structure.

Next we prove that $(past, id_E)$ is the required isomorphism between TS and $T(\zeta(TS))$, where past: $S \to 2^E$ is defined in Definition 3.5, and id_E: $E \to E$ is the identity function. Clearly, past as defined is a function from S to $FC(\zeta(TS))$ (left for the reader to see) and it follows from Lemma 3.6 that $(past, id)$ is a TS-morphism. From Lemma 3.7, it follows that past is injective, and hence has a partial inverse $past^{-1}$. From Lemma 4.6 (to follow) we conclude that $past^{-1}$ is a total function on $FC(\zeta(TS))$ and that $(past^{-1}, id)$ is the categorical inverse of $(past, id)$. This concludes the proof of Lemma 4.5. ∎

As an example, applying the construction of the proof of Lemma 4.5 to the OTS-object from Fig. 3.1c gives exactly the PES object mentioned following Theorem 4.1!

LEMMA 4.6. *Let $TS = (S, E, T, s^{\text{in}})$ be an OTS object. Then for every $c \in FC(\zeta(TS))$*

(a) *$past^{-1}(c)$ is defined and*

(b) *for every $c' \xrightarrow{e'} c$ in $\zeta(TS)$, $(past^{-1}(c'), e', past^{-1}(c)) \in T$.*

Proof. We prove the lemma by induction on the size of c.

$c = \varnothing$. Clearly, $past(s^{\text{in}}) = \varnothing$, and (b) is trivially satisfied.

$c \neq \varnothing$. We distinguish here between two cases.

Case 1. $\exists e \in c.\ c = \{e' \in E \mid e' \leqslant e\}$.

(a) In this case, we get immediately from the definition that $past(s_e) = c$, where s_e is the state representative associated with e from Lemma 3.4.

(b) Take any $c' \xrightarrow{e'} c$ in $\zeta(TS)$. In this case we must have $e = e'$ and $c' = c \backslash \{e\}$ (remember c' is a configuration). From Lemma 3.4, we get for some $s' \in S$, $(s', e', s_e) \in T$, and from Lemma 3.6 we get $past(s') = past(s_e) \backslash \{e\} = c \backslash \{e\} = c'$. This proves (b) in Case 1.

Case 2. $\forall e \in c.\ c \neq \{e' \in E \mid e' \leqslant e\}$.

(a) In this case we must have at least two maximal elements in c (with respect to $\leqslant$). Take any such two elements e_1 and e_2. It follows from Proposition 2.3 that in $\zeta(TS)$ we have configurations related as in Fig. 4.1.

So, from the induction hypothesis, we must have states s_1, s_2, and s_3 in S such that $past(s_i) = c_i$, $i = 1, 2, 3$, and transitions in T as in Fig. 4.2. Now from our assumption we have that e_1 and e_2 are neither related by $\leqslant$ or $\#$, and hence from Lemma 4.7 (to follow) we get $({}^\circ e_1 \cup e_1^\circ) \cap ({}^\circ e_2 \cup e_2^\circ) = \varnothing$. Furthermore, from $s_3 \xrightarrow{e_1}$ we have ${}^\circ e_1 \subseteq R_{s_3}$, and hence from Proposition 1.3(iii), ${}^\circ e_1 \subseteq R_{s_1}$. From axiom T1, we get that there must exist a state $s \in S$ such that $(s_1, e_1, s) \in T$.

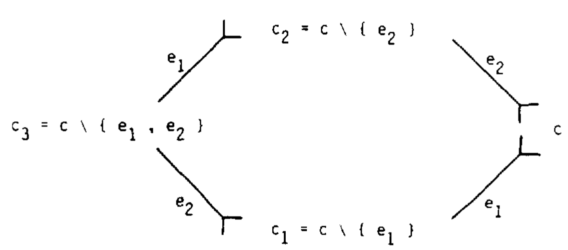

FIGURE 4.1

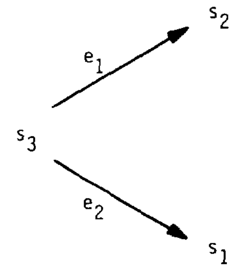

FIGURE 4.2

But now clearly from Lemma 3.6, we get $past(s) = past(s_1) \cup \{e\} = c_1 \cup \{e\} = c$.

(b) Take any $c' \stackrel{e'}{-\!\!<} c$. e' must be a maximal element in c. But now choosing $e_1 = e'$ in the proof of (a) above, we get the conclusion of (b) directly. ▮

LEMMA 4.7. *Let $TS = (S, E, T, s^{in}) \in OTS$, and let $e_0, e_1 \in E$ be two events not related by either $\leqslant$ or $\#$ as defined in the proof of Lemma 4.5. Then $({}^\circ e_0 \cup e_0^\circ) \cap ({}^\circ e_1 \cup e_1^\circ) = \varnothing$ in TS.*

Proof. Since e_0 and e_1 are not $\#$-related, $\uparrow s_e \cap \uparrow s_{e'} \neq \varnothing$ by the definition given in the proof of Lemma 4.5. Hence we have a state s such that $s_{e_0} \xrightarrow{*} s$ and $s_{e_1} \xrightarrow{*} s$ in TS, where s_{e_0}, s_{e_1} are the unique states associated with e_0, e_1 from Lemma 3.4. Choose s to be a minimal (w.r.t. *past*) state satisfying this property. We want to argue that we must have states s_0 and s_1 such that the situation shown in Fig. 4.3 obtains. Since $s_{e_0} \xrightarrow{*} s$, we have from Lemma 3.6 that any computation in TS from s^{in} to s must contain exactly one e_0-occurrence. Now, consider any computation of the form $s^{in} \xrightarrow{*} s_{e_1} \xrightarrow{*} s$. Such a computation cannot have an e_0-occurrence before s_{e_1} since this would imply $s_{e_0} \xrightarrow{*} s_{e_1}$, contradicting our assumption that e_0 and e_1 are not $\leqslant$-related. So, we must have states s' and s'' such that $s^{in} \xrightarrow{*} s_{e_1} \xrightarrow{*} s' \xrightarrow{e_0} s'' \xrightarrow{*} s$. But now $s_{e_1} \xrightarrow{*} s''$ and also from axiom O, $s_{e_0} \xrightarrow{*} s''$, and hence from the minimality of s, we get $s = s''$. By a symmetric argument applied to e_1, we have the situation as shown in Fig. 4.3.

Now, based on Fig. 4.3, we want to argue for the conclusion of the lemma. Assume $R \in ({}^\circ e_0 \cup e_0^\circ) \cap ({}^\circ e_1 \cup e_1^\circ)$.

Case 1. $R \in ({}^\circ e_0 \cap e_1^\circ) \cup (e_0^\circ \cap {}^\circ e_1)$. This assumption leads to the immediate contradiction $s \in R \Leftrightarrow s \notin R$.

Case 2. $R \in {}^\circ e_0 \cap {}^\circ e_1$. From Lemma 3.4, we must have for some s', $(s', e_0, s_{e_0}) \in T$, and hence (from the assumption

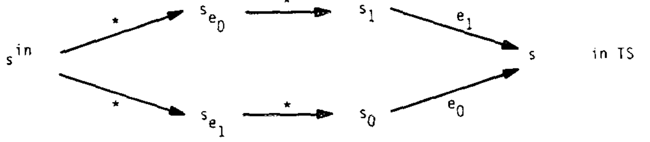

FIGURE 4.3

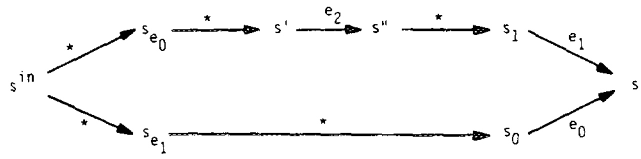

FIGURE 4.4

$R \in {}^\circ e_0$), $s_{e_0} \notin R$. Also (from the assumption $R \in {}^\circ e_1$), $s_1 \in R$ and hence there exist some $s', s'' \in S$ and e_2 such that $R \in e_2^\circ$ and the situation shown in Fig. 4.4 obtains. Now, from axiom O we know that $s'' \in \uparrow s_{e_2}$ and hence $s \in (\uparrow s_{e_0} \cap \uparrow s_{e_1} \cap \uparrow s_{e_2})$, so we can have neither $e_2 \# e_0$ nor $e_2 \# e_1$. But from the existence of $R \in e_2^\circ \cap ({}^\circ e_0 \cap {}^\circ e_1)$ we must have from Case 1 of this proof that e_2 must be $\leqslant$-related to both e_0 and e_1.

Assume $e_2 \leqslant e_0$. This implies from definition, $s_{e_0} \in \uparrow s_{e_2}$ and hence $s' \in \uparrow s_{e_2}$ contradicting the fact that $\uparrow s_{e_2}$ is a post-region of e_2. So, we must have $e_0 < e_2$.

Assume $e_1 \leqslant e_2$. This implies from definition $s_{e_2} \in \uparrow s_{e_1}$, and hence, since $s'' \in \uparrow s_{e_2}$ (because $\uparrow s_{e_2}$ is post-region of e_2) and also $s_1 \in \uparrow s''$ (see Fig. 4.4), we get $s_1 \in \uparrow s_{e_1}$, contradicting the fact that $\uparrow s_{e_1}$ is a post-region of e_1. So, we must have $e_2 < e_1$.

But now obviously $e_0 < e_2$ and $e_2 < e_1$ imply $e_0 < e_1$ contradicting our assumption that e_0 and e_1 are not $\leqslant$-related. All in all, we have contradicted the assumption of Case 2.

Case 3. $R \in e_0^\circ \cap e_1^\circ$. In this case,we would have $\bar{R}$ (the complement of R) belonging to ${}^\circ e_0 \cap {}^\circ e_1$—thus this case is reduced to Case 2.

Since these three cases exhaust the assumption $R \in ({}^\circ e_0 \cup e_0^\circ) \cap ({}^\circ e_1 \cup e_1^\circ)$, we have proved Lemma 4.7 and hence our main Theorem 4.2.

5. UNFOLDINGS OF ELEMENTARY TRANSITION SYSTEMS

One of the nice aspects of net theory is that it provides a uniform formalism in which both distributed *systems* and their *behaviours* can be defined. For instance, one may define the behaviour of an elementary net system in terms of its unfolding (Nielsen *et al.*, 1990). The unfolding is simply an elementary net system called an occurrence net. Hence occurrence nets can be defined as a subcategory of the category of elementary net systems. Furthermore, the operation of unfolding of an elementary net system (extended in a natural way to a functor) was shown by Winskel to be not an arbitrary functor (from the category of elementary net systems to the subcategory of occurrence nets) but, in fact, the right adjoint to the inclusion functor from occurrence nets to elementary net systems. A categorical result like this provides a good deal of insight. Originally, the notion of unfolding was just introduced as one of many possible ways of associating occurrence nets with net systems (Nielsen *et. al.*, 1981). But Winskel's result proved that this construction was actually unique among all in the following sense.

Given a net system, N, a notion of unfolding consists of two parts. One telling what the unfolded object looks like, $UF(N)$, and another telling how the elements (conditions and events) of $UF(N)$ are related to the elements of N, usually called the folding, *fold*: $UF(N) \to N$. This folding is nothing but a special case of a general notion of simulation morphism, setting up a category of nets. In this categorical setting, the uniqueness of the particular construction of $UF(N)$ and *fold* is expressed elegantly by Winskel as a universal property: for any occurrence net ON and a simulation morphism f: $ON \to N$ there is a unique g: $ON \to UF(N)$ such that the diagram shown in Fig. 5.1 commutes. So, $UF(N)$ is in a formal sense "maximal" among all possible notions of unfolding, and uniquely defined with this property (up to isomorphism). This result was originally proved by Winskel for 1-safe net systems (Winskel, 1987) and later adapted to elementary net systems (Nielsen *et al.*, 1990).

This is a nice example of the power of using the language of category theory. Note how morphisms come in naturally because fold is a morphism! It should be mentioned that the notion of simulation morphism adapted by Winskel has independent motivation, in the sense that many constructs from process algebra may be understood as universal constructs in the category like product (parallellism) and coproduct (choice). The same holds for our choice of category for elementary transition systems, as shown in Nielsen *et al.* (1992).

So, we are looking for a similar universal notion of unfolding of elementary transition systems into occurrence transition systems. The question is very natural, since we know that the occurrence transition systems are exactly the case-graphs of occurrence nets and the elementary transition systems are exactly the case graphs of elementary nets systems. The first fact follows from Winskel's established coreflection between prime event structures and occurrence nets combined with our results from Section 4. This is pictured in Fig. 5.2. The second fact follows from Nielsen *et al.* (1992) as shown in Fig. 5.3.

It is tempting to try to fit these two pictures together and try to establish our required co-reflection based on some

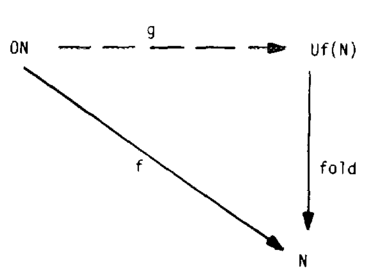

FIGURE 5.1

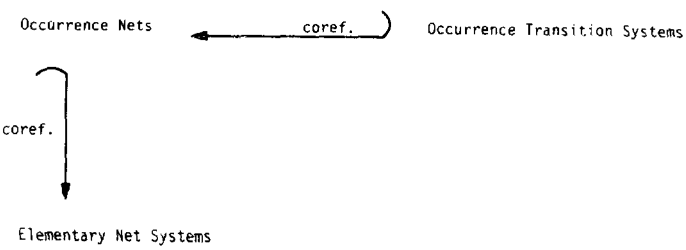

FIGURE 5.2

general grounds. Irrespective of whether such grounds exist (note that our co-reflection does not follow directly from glueing the two pictures together) there are some major obstacles even in relating the two pictures. The point is that the notion of net morphisms used in Fig. 5.2 is different from the net morphisms used in Fig. 5.3. And this is not just a coincidence.

A Winskel morphism from a net system N_0 to another net system N_1 consists of a partial function from the N_0-events to the N_1-events, and a relation between the conditions of the two nets satisfying certain axioms which imply that "N_1 may simulate N_0 with respect to the partial function on events." In Nielsen *et al.* (1992), we worked with a restricted type of such morphisms, requiring the condition part to be a partial function from conditions of N_1 to conditions of N_0. As a matter of fact, historically these morphisms were the first type of morphisms to be considered by Winskel but were generalized exactly to cope with unfoldings! Let us illustrate this with an example. In Fig. 5.4 we have shown a net system N and a small part of the folding from $UF(N)$ to N. In this folding, each condition of N has to be related to all of its occurrences in $UF(N)$. On the other hand, the coreflection from Nielsen *et al.* (1992) shown in Fig. 5.3 does not hold with this liberal view of net morphism as illustrated by the example in Fig. 5.5. The point of the example is that there is exactly one morphism from the case graph of N_0 to the case graph of N_1 satisfying $\eta(e_1) = \eta(e_2) = e$ and $\eta(e') = e'$, whereas there are at least two Winskel morphisms from N_0 to N_1 satisfying this requirement (one relating b to b_1 and b_2, another relating b just to b_3).

Another important observation concerning Figs. 5.2 and 5.3 is the fact that the co-reflection from Nielsen *et al.* (1992), shown in Fig. 5.3, does *not* cut down to the co-reflection between the respective (occurrence) subcategories in Fig. 5.2. As a matter of fact the left adjoint of Fig. 5.3 when applied to occurrence transition systems does not in general produce occurrence nets (e.g., the saturated set of regions will in general introduce cycles in the nets constructed). So, also on objects the glueing of the pictures is not as easy as one might hope. Thus there seems to be no general implication of

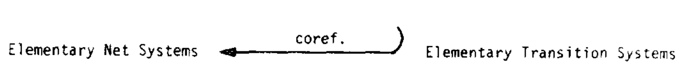

FIGURE 5.3

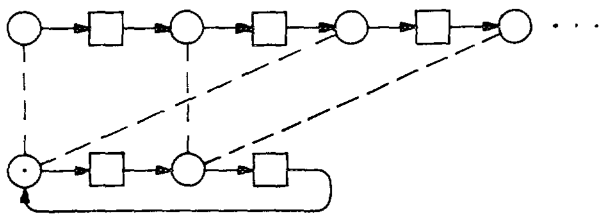

FIGURE 5.4

the existence of our required universal unfolding based on existing results from net theory.

However, based on the results from Figs. 5.2 and 5.3, despite the problems indicated above, one may actually read off a particular way of unfolding a given elementary transition system TS in the following way. First construct the (saturated) net associated with it according to Fig. 5.3. Unfold this net and finally construct the occurrence transition system of the unfolded net—both constructions following Fig. 5.2. This, at least, gives us one way of associating occurrence transition systems with elementary transition systems. What we shall prove in the following is that this construction on objects extends uniquely to a functor which is indeed the right adjoint to the inclusion functor. So, in the abstract setting of transition systems the problems with the variety of notions of morphisms disappear and the derived notion of unfolding from net theory is *the* unfolding in the standard category of transition systems.

We shall start by presenting this unfolding of elementary transition systems in terms of objects. A formal definition following the lines above would be very heavy, and require the introduction of a lot of technical machinery from net theory and existing work of Winskel (1987) and Nielsen *et al.* (1992). Hence we have chosen to give a direct definition of the construction close to the theory of transition systems and leave it for the (interested) reader to check in the relevant literature that our definition is indeed the one derived from net theory as indicated. For the sake of completeness, we do include a sketch of a proof that our construction does produce occurrence transition systems.

We shall use the theory of trace languages—originating from the work of Mazurkiewicz (1989)—to define unfoldings of elementary transition systems.

We will show that this "unfold" map produces occurrence transition systems and it can be smoothly extended to become a functor from ETS to OTS. More importantly, we will prove that this functor is the right adjoint of the inclusion functor from OTS to ETS.

In the literature, a number of authors have independently shown that a strong relationship exists between trace languages and prime event structures (Bednarczyk, 1988; Shields, 1988; Rozoy and Thiagarajan, 1991). In what follows, we will appeal to a number of technical results that arise in the process of establishing that trace languages yield prime event structures. We will not give detailed proofs of these results since they can be found in or can be easily extracted from Rozoy and Thiagarajan (1991). For background material on trace languages, the reader is referred to Aalbersberg and Rozenberg (1986) and Mazurkiewicz (1989).

Until further notice, fix an elementary transition system $TS = (S, E, T, s^{in})$. Then FS_{TS}, the set of *firing sequences* of TS and the relation $[\ \rangle_{TS} \subseteq \{s^{in}\} \times FS_{TS} \times S$ are given inductively by:

- $\Lambda \in FS_{TS}$ and $s^{in}[\Lambda\rangle_{TS}\, s^{in}$.
- If $\rho \in FS_{TS}$, $s^{in}[\rho\rangle_{TS}\, s$ and $(s, e, s') \in T$, then $\rho e \in FS_{TS}$ and $s^{in}[\rho e\rangle_{TS}\, s'$.

When TS is clear from the context, we will write FS instead of FS_{TS} and $[\ \rangle$ instead of $[\ \rangle_{TS}$. In fact, we will follow this convention for a number of relations that we will soon define relative to TS. The independence relation $I_{TS} \subseteq E \times E$ associated with TS is given by

$$I_{TS} = \{(e_1, e_2) \mid ({}^\circ e_1 \cup e_1^\circ) \cap ({}^\circ e_2 \cup e_2^\circ) = \emptyset\}.$$

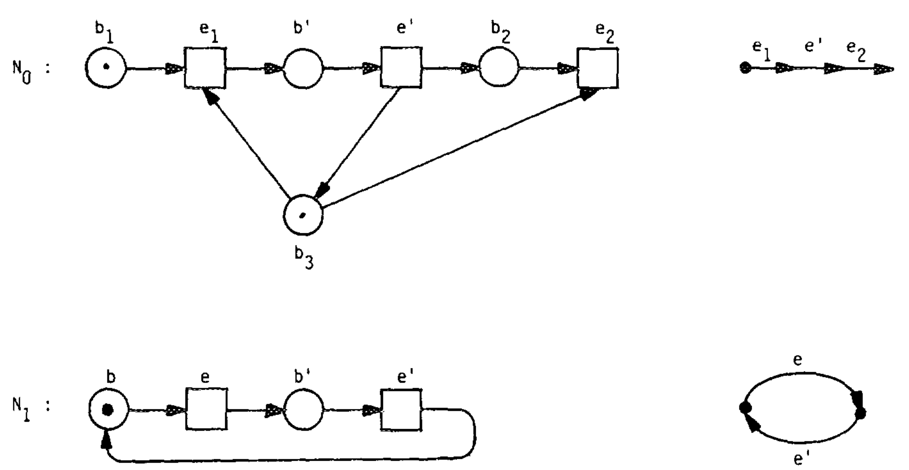

FIGURE 5.5

Clearly I_{TS} is irreflexive and symmetric and hence induces an equivalence relation (see Mazurkiewicz, 1987) over E^*. This equivalence relation will in fact be a congruence w.r.t. the operation of concatenation over E^*. To be specific, $\doteq_{I_{TS}}$ (written for convenience as $\doteq_{TS}$) is the subset of $E^* \times E^*$ given by

$$\sigma \doteq_{TS} \sigma' \text{ iff } \exists \sigma_1, \sigma_2 \in E^*. \exists (e, e') \in I_{TS}. [\sigma = \sigma_1 ee'\sigma_2 \text{ and } \sigma' = \sigma_1 e'e\sigma_2].$$

The equivalence relation we want is denoted as $=_{TS}$ and it is the reflexive transitive closure of $\doteq_{TS}$. In other words, $=_{TS} = (\doteq_{TS})^*$. For $\sigma \in E^*$ we let $[\sigma]_{TS}$ denote the equivalence class containing σ and call it a *trace*. Formally, $[\sigma]_{TS} = \{\sigma' \mid \sigma =_{TS} \sigma'\}$. As remarked earlier, we will often write $[\sigma]$ instead of $[\sigma]_{TS}$. Unless otherwise stated, in what follows we let ρ, ρ', ρ'' with or without subscripts range over FS; we let $\sigma, \sigma', \sigma''$ with or without subscripts range over E^*; we let e, e', e'', e_1, e_2 range over E. The result we mention next is a well-known and very useful characterization of the relation $=_{TS}$ (see, for instance, Aalbersberg and Rozenberg, 1986, for a proof).

In stating the result we will use the following notations. For $e \in E$, $\#_e(\sigma)$ is the number of times the symbol e appears in σ. For $X \subseteq E$, $Proj_X(\sigma)$ is the sequence obtained by erasing from σ all appearences of non-members of X. In other words,

- $Proj_X(\Lambda) = \Lambda$.
- $Proj_X(\sigma e) = \begin{cases} Proj_X(\sigma)\, e, & \text{if } e \in X, \\ Proj_X(\sigma), & \text{otherwise.} \end{cases}$

Proposition 5.1. *$\sigma_1 =_{TS} \sigma_2$ if the following two conditions are satisfied:*

(i) $\forall e \in E.\ \#_e(\sigma_1) = \#_e(\sigma_2)$.

(ii) $\forall (e, e') \in (E \times E) - I_{TS}.\ Proj_{\{e, e'\}}(\sigma_1) = Proj_{\{e, e'\}}(\sigma_2)$.

Next we recall the standard ordering over the traces generated by $=_{TS}$. $[\sigma] \leqslant_{TS} [\sigma']$ iff $\exists \sigma''. \sigma\sigma'' =_{TS} \sigma'$. It is easy to check that $\leqslant$ is well defined and a partial ordering relation with $[\Lambda] = \{\Lambda\}$ as the least element. $[\sigma] \sqcup [\sigma']$ will denote the least upper bound of $[\sigma]$ and $[\sigma']$ under $\leqslant$, if it exists.

Given our purposes, a relation closely related to $\leqslant$ and denoted as $\xrightarrow[TS]{}$ will turn out to very useful to have around. $\xrightarrow[TS]{} \subseteq E^* \times E^*$ is given by

$$\sigma \xrightarrow[TS]{} \sigma' \quad \text{iff} \quad \exists \sigma''. \sigma\sigma'' =_{TS} \sigma'.$$

The next set of observations are easy to verify.

Proposition 5.2. (i) *$[\sigma] \leqslant [\sigma']$ iff $\sigma \to \sigma'$. Thus $\to$ is a pre-order the equivalence relation induced by which is exactly $=_{TS}$.*

(ii) $\forall \rho \in FS.\ [\rho] \subseteq FS$.

(iii) *Suppose $\rho e, \rho e' \in FS$ with $(e, e') \in I_{TS}$. Then $\rho ee', \rho e'e \in FS$.*

Part (iii) of this result leans on the fact that TS, being elementary, satisfies the axiom T1.

The set $\{[\rho] \mid \rho \in FS\}$ will serve as the set of states of $Uf(TS)$, the unfolding of TS, that we wish to construct. To identify the events of $Uf(TS)$ we must work with the *prime intervals* generated by TS denoted as PI_{TS}. It is the subset of $E^* \times E^*$ given by $PI_{TS} = \{(\sigma, \sigma') \mid \exists e \in E. \sigma e =_{TS} \sigma'\}$. Next we define the map $\varphi_{TS}: PI \to E$ as follows: $\forall (\sigma, \sigma') \in PI.\ \varphi(\sigma, \sigma') = e$ provided $\sigma e =_{TS} \sigma'$. (For convenience, we will write $\varphi(x, y)$ instead of $\varphi((x, y))$).

Now suppose that $\sigma e =_{TS} \sigma e'$. Then according to Proposition 5.1, $e = e'$. Hence φ is well-defined. This map—or more precisely, our extension of this map to certain equivalence classes of prime intervals—will turn out to be crucial for linking up the behaviour of $Uf(TS)$ to that of TS; but we still need to identify the events of $Uf(TS)$.

To this end, define the relation $\propto_{TS} \subseteq PI \times PI$ by

$$(\sigma_1, \sigma'_1) \propto_{TS} (\sigma_2, \sigma'_2) \text{ iff } \exists \sigma. [\sigma_1\sigma =_{TS} \sigma_2 \text{ and } \sigma'_1\sigma =_{TS} \sigma'_2].$$

Set $\approx_{TS} = (\propto_{TS} \cup (\propto_{TS})^{-1})^*$. Clearly $\approx_{TS}$ is an equivalence relation over PI. In what follows, we denote by $\langle \sigma, \sigma' \rangle_{TS}$ the equivalence class of prime intervals containing the prime interval (σ, σ'). Again using Proposition 5.1 and the definitions, the next set of observations is easy to verify.

Proposition 5.3. (i) *$\propto_{TS}$ is a pre-order.*

(ii) *Suppose $(\sigma_1, \sigma'_1), (\sigma_2, \sigma'_2) \in PI$. Then*

$$[(\sigma_1, \sigma'_1) \propto_{TS} (\sigma_2, \sigma'_2), \text{ and } (\sigma_2, \sigma'_2) \propto_{TS} (\sigma_1, \sigma'_1)]$$
$$\text{iff } [\varphi(\sigma_1, \sigma'_1) = \varphi(\sigma_2, \sigma'_2) \text{ and } \sigma_1 =_{TS} \sigma_2].$$

(iii) *Suppose $(\sigma_1, \sigma'_1) \propto_{TS} (\sigma_2, \sigma'_2)$. Then $\varphi(\sigma_1, \sigma'_1) = \varphi(\sigma_2, \sigma'_2)$.*

Extend φ to $\approx_{TS}$-equivalence classes of prime intervals as follows (by abuse of notation, this extension will also be denoted as φ):

$$\forall (\sigma_1, \sigma'_1) \in PI.\ \varphi(\langle \sigma_1, \sigma'_1 \rangle) = \varphi(\sigma_1, \sigma'_1).$$

According to Proposition 5.3(iii), this extension of φ is also well-defined. Some of the equivalence classes of prime intervals will serve as the events of $Uf(TS)$.

Definition 5.4. *$Uf(TS)$*, the *unfolding* of TS, is the transition system $Uf(TS) = (\hat{S}, \hat{E}, \hat{T}, \hat{s}_{in})$ where

$\hat{S} = \{[\rho] \mid \rho \in FS\}$,

$\hat{E} = \{\langle \rho, \rho' \rangle \mid \rho, \rho' \in FS \text{ and } (\rho, \rho') \in PI\}$,

$\hat{T} = \{([\rho], \langle \rho, \rho' \rangle, [\rho']) \mid \langle \rho, \rho' \rangle \in \hat{E}\}$, and

$\hat{s}_{\text{in}} = [\Lambda]$.

In Fig. 5.6, we have shown two small examples of unfolding, illustrating that the notion involves a certain kind of horizontal and vertical unfolding. Only a few selected states in the unfolded systems are labeled with their definition. We claimed above that our definition of unfolding is derived from net theory. Let us be a bit more precise. Let H and J be the two functors from Nielsen *et al.* (1992) forming a co-reflection as in Fig. 5.3.

$$\text{Elementary Net Systems} \underset{J}{\overset{H}{\rightleftarrows}} \text{Elementary Transition Systems}$$

In Nielsen *et al.* (1990), a version of Winskel's unfolding was defined—a mapping associating with each elementary net system N its unfolding $UF(N)$ as an occurrence net. Our claim is now the following:

Conjecture. For every elementary transition system TS, $Uf(TS)$ as defined in Definition 5.4 is isomorphic to $H(UF(J(TS)))$.

We do not prove this claim here (and hence we have termed it a conjecture!), partly because it would involve quite a lot of extra technical machinery not relevant for the rest of the paper, partly because the interested rader may find a more or less complete proof in Nielsen *et al.* (1990). Here we limit ourselves to proving that $Uf(TS)$ is an occurrence transition system. In doing so, we shall appeal to a number of technical results from Rozoy and Thiagarajan (1991). However, we will provide sufficient information so that an enterprising reader can work out the details for herself/himself.

LEMMA 5.5. (i) *Suppose* $\sigma e_1 \sigma_1 =_{TS} \sigma e_2 \sigma_2$ *with* $e_1 \neq e_2$. *Then* $(e_1, e_2) \in I_{TS}$. *Moreover there exists* σ' *such that* $\sigma e_1 \sigma_1 =_{TS} \sigma e_1 e_2 \sigma' =_{TS} \sigma e_2 e_1 \sigma' =_{TS} \sigma e_2 \sigma_2$. *Consequently*, $[\sigma e_1] \sqcup [\sigma e_2] = [\sigma e_1 e_2]$.

(ii) *Suppose* $\sigma_1 \to \sigma$ *and* $\sigma_2 \to \sigma$. *Then* $[\sigma_1] \sqcup [\sigma_2]$ *exists.*

(iii) *Suppose* $\rho \to \sigma$ *and* $\rho' \to \sigma$ (*with* $\rho, \rho' \in FS$, $\sigma \in E^*$).

Then $[\rho] \sqcup [\rho'] \in \hat{S}$.

The property captured in part (i) of this result is the so-called forward diamond property. The relevant situation is shown in Fig. 5.7. The proof follows easily by repeated applications of Proposition 5.1. Part (ii) of the result follows by repeated applications of part (i) of the result. Part (iii) of the result follows from part (ii) and repeated applications of Proposition 5.2.

LEMMA 5.6. *Suppose* $\sigma_1 e_1 =_{TS} \sigma_2 e_2$ *with* $e_1 \neq e_2$. *Then* $(e_1, e_2) \in I_{TS}$. *Moreover, there exists* σ *such that* $\sigma e_2 =_{TS} \sigma_1$ *and* $\sigma e_1 =_{TS} \sigma_2$.

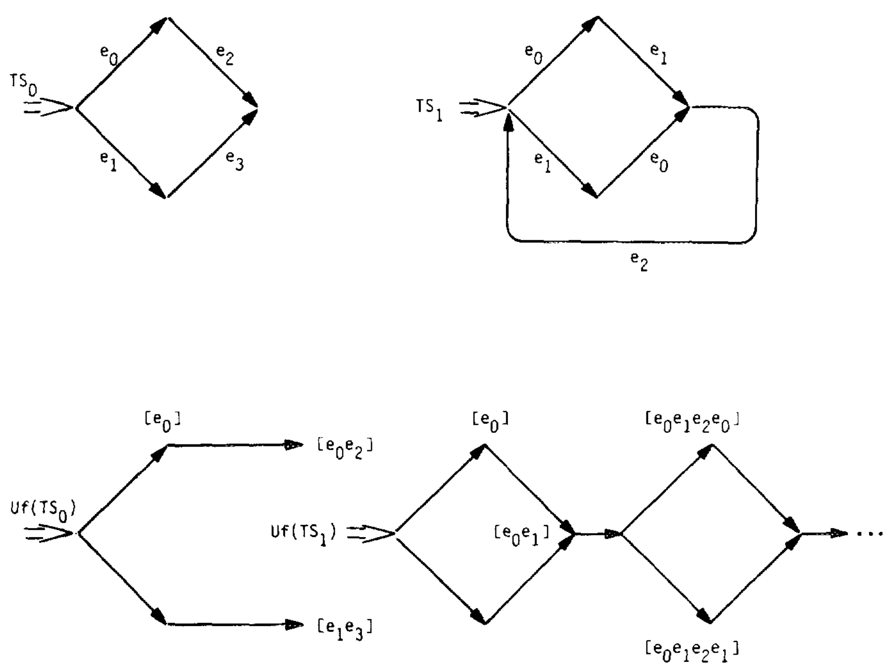

FIGURE 5.6

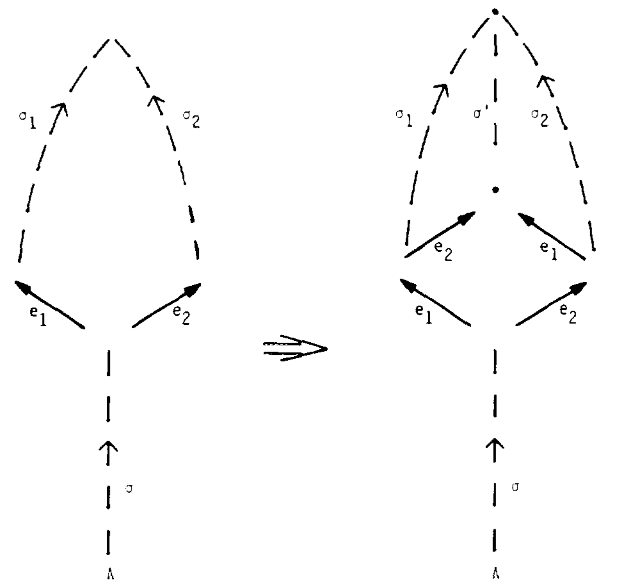

FIGURE 5.7

This is the so-called backward diamond property. This result also follows easily through repeated applications of Proposition 5.1. The relevant situation is shown in Fig. 5.8.

For introducing the next result, we need a notation. This notation will be used extensively in the sequel. Let $(\sigma, \sigma') \in PI$. Then Base $(\langle \sigma, \sigma' \rangle) \subseteq \langle \sigma, \sigma' \rangle$ is the set

$$\{(\sigma_0, \sigma'_0) \mid (\sigma_0, \sigma'_0) \in \langle \sigma, \sigma' \rangle \text{ and } \forall (\sigma_1, \sigma'_1) \in \langle \sigma, \sigma' \rangle \cdot (\sigma_0, \sigma'_0) \propto_{TS} (\sigma_1, \sigma'_1)\}.$$

Recall that according to Proposition 5.2,

$$\text{if } (\sigma_1, \sigma'_1), (\sigma_2, \sigma'_2) \in Base(\langle \sigma, \sigma' \rangle), \text{ then } \sigma_1 =_{TS} \sigma_2 \text{ and } \sigma'_1 =_{TS} \sigma'_2.$$

Hence $Base(\langle \sigma, \sigma' \rangle)$ identifies in some sense the "least" elements of $\langle \sigma, \sigma' \rangle$ under $\propto_{TS}$ modulo the equivalence relation $=_{TS}$.

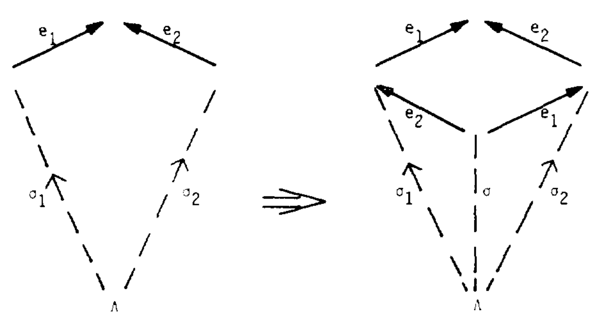

FIGURE 5.8

LEMMA 5.7. (i) $\forall (\sigma, \sigma') \in PI.\ Base(\langle \sigma, \sigma' \rangle) \neq \emptyset$.

(ii) $\forall \hat{e} \in \hat{E}.\ Base(\hat{e}) \subseteq FS \times FS$.

The first part of the result follows leans on Lemma 5.6. The main observation exploiting Lemma 5.6 (and the definition of $\approx_{TS}$) can be depicted graphically as shown in Fig. 5.9.

The second part of the result follows from the first part and the observation that FS is prefix-closed. Thanks to Lemma 5.7 we can *injectively* associate with each element of $\hat{S}$ (in $Uf(TS)$) a set of events in $\hat{E}$; the events that have "occurred so far." To see this, define $Ev: FS \to P(\hat{E})$ (to be soon extended to $\hat{S}$!) as: $\forall \rho \in FS.\ Ev(\rho) = \{\hat{e} \mid \exists (\rho_1, \rho'_1) \in \hat{e}.\ \rho'_1 \to \rho\}$. To be precise, we must define $Ev(\rho)$ as $\{\hat{e} \mid \exists (\sigma_1, \sigma'_1) \in \hat{e}.\ \sigma'_1 \to \rho\}$. But, once again, the fact that FS is prefix-closed guarantees that our definition captures the intended meaning. Ev is extended to a map—also denoted as Ev by abuse of notation—from $\hat{S}$ to $P(\hat{E})$ via

$$\forall \rho \in FS.\ Ev([\rho]) = Ev(\rho).$$

It is easy to verify that this extension is well-defined.

LEMMA 5.8. (i) *Suppose* $\rho e \in FS$. *Then* $\langle \rho, \rho e \rangle \notin Ev(\rho)$. *Moreover*, $Ev(\rho e) = Ev(\rho) \cup \{\langle \rho, \rho e \rangle\}$.

(ii) $\forall \rho, \rho' \in FS.\ \rho \to \rho'$ *iff* $Ev(\rho) \subseteq Ev(\rho')$. *Hence* $\rho =_{TS} \rho'$ *iff* $Ev(\rho) = Ev(\rho')$. *Thus* $Ev: \hat{S} \to P(\hat{E})$ *is injective.*

(iii) *Suppose* $([\rho], \hat{e}, [\rho']) \in \hat{T}$ (*in* $Uf(TS)$). *Then* $\hat{e} \notin Ev([\rho])$. *Moreover*, $Ev([\rho']) = Ev([\rho]) \cup \{\hat{e}\}$.

(iv) *Suppose* $[\sigma] \sqcup [\sigma']$ *exists. Then* $Ev([\sigma] \sqcup [\sigma']) = Ev([\sigma]) \cup Ev([\sigma'])$.

This result follows from Lemma 5.5 and Lemma 5.7. The details are a bit tedious but straightforward. This completes the chain of technical results we shall borrow from the literature. We now turn to the task of proving that $Uf(TS)$ is an occurrence transition system.

Here, we find it technically convenient to exploit the main result of Section 4. Recall the functor T going from PES to OTS. We will show that there exists a prime event structure

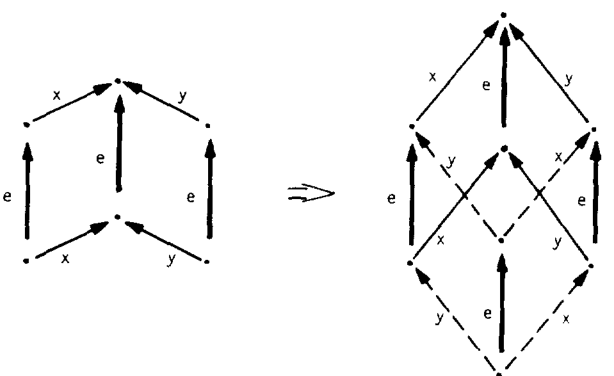

FIGURE 5.9

ES such that $T(ES)$ and $Uf(TS)$ are isomorphic transition systems (relative to the notion of morphisms specified in Definition 1.5). Since $T(ES)$ is an OTS-object we would have then established that $Uf(TS)$ is also an OTS-object.

Define $ES = (\hat{E}, \leqslant, \#)$ where $\leqslant, \# \subseteq \hat{E} \times \hat{E}$ are defined as follows:

(i) $\hat{e}_1 \leqslant \hat{e}_2$ iff $\hat{e}_1 \in Ev(\rho')$, where $(\rho, \rho') \in Base(\hat{e}_2)$.

(ii) $\hat{e}_1 \# \hat{e}_2$ iff there does *not* exist $\rho \in FS$ such that $\hat{e}_1 \in Ev(\rho)$ and $\hat{e}_2 \in Ev(\rho)$.

It is easy to verify that ES is indeed a prime event structure in the sense of Definition 2.1. Recall that $T(ES) = (FC(ES), \hat{E}, \longrightarrow\!\!\!<, \varnothing)$.

The proof of the fact that $T(ES)$ and $Uf(TS)$ are isomorphic can be split into two steps.

LEMMA 5.9. *Let $\hat{e}_1, \hat{e}_2 \in \hat{E}$ be such that not $(\hat{e}_1 \leqslant \hat{e}_2$ or $\hat{e}_2 \leqslant \hat{e}_1$ or $\hat{e}_1 \# \hat{e}_2)$. Then $(\varphi(\hat{e}_1), \varphi(\hat{e}_2)) \in I_{TS}$.*

Proof. Let $(\rho_i, \rho'_i) \in Base(\hat{e}_i)$ and $\varphi(\hat{e}_i) = e_i$ for $i = 1, 2$. Since neither $\hat{e}_1 \leqslant \hat{e}_2$ nor $\hat{e}_2 \leqslant \hat{e}_1$, it must be the case that $[\rho'_1]$ and $[\rho'_2]$ are incomparable. Since it is not the case that $\hat{e}_1 \# \hat{e}_2$, there exists $\rho \in FS$ such that $\hat{e}_1, \hat{e}_2 \in Ev(\rho)$. Consequently $\rho'_1 \to \rho$ and $\rho'_2 \to \rho$. Hence $[\rho_1] \sqcup [\rho_2]$, $[\rho_1] \sqcup [\rho'_2]$, $[\rho'_1] \sqcup [\rho_2]$ and $[\rho'_1] \sqcup [\rho'_2]$ all exist by Lemma 5.5(ii). Let $\rho_{11} \in [\rho_1] \sqcup [\rho_2]$, $\rho'_{12} \in [\rho'_1] \sqcup [\rho_2]$, $\rho'_{21} \in [\rho_1] \sqcup [\rho'_2]$ and $\rho_{22} \in [\rho'_1] \sqcup [\rho'_2]$. It is easy to verify the following:

(i) $Ev(\rho'_{12}) = Ev(\rho_{11}) \cup \{\hat{e}_1\}$ and $Ev(\rho'_{21}) = Ev(\rho_{11}) \cup \{\hat{e}_2\}$,

(ii) $Ev(\rho'_{12}) \cup \{\hat{e}_2\} = Ev(\rho_{11}) \cup \{\hat{e}_1, \hat{e}_2\} = Ev(\rho'_{21}) \cup \{\hat{e}_1\}$,

(iii) $\rho_{11} e_1 =_{TS} \rho'_{12}$ and $\rho_{11} e_2 =_{TS} \rho'_{21}$ and $\rho'_{12} e_2 =_{TS} \rho_{22} =_{TS} \rho'_{21} e_1$.

From (iii), it follows at once that $\rho_{11} e_1 e_2 =_{TS} \rho_{11} e_2 e_1$, which leads to $(e_1, e_2) \in I_{TS}$. ▮

LEMMA 5.10. *The map $Ev: \hat{S} \to P(\hat{E})$ is a bijection from $\hat{S}$ to $FC(ES)$.*

Proof. From the definition of ES it follows easily that $Ev([\rho]) \in FC(ES)$ for every $\rho \in FS$. This map is injective according to Lemma 5.8. Let $c \in FC(ES)$. We must show that there exists $\rho \in FS$ such that $Ev(\rho) = c$. We proceed by induction on $k = |c|$.

$k = 0$. Then $c = \varnothing$ and we can set $\rho = \Lambda$.

$k > 1$. Suppose there exits $\hat{e} \in c$ such that $\hat{e}_1 \leqslant \hat{e}$ for every $\hat{e}_1 \in c$. (In other words, c has a unique maximal element). Let $(\rho', \rho) \in Base(\hat{e})$. Then it is easy to check, using the definition of ES, that $Ev(\rho) = c$.

So assume that c contains (at least) two distinct maximal elements $\hat{e}_1$ and $\hat{e}_2$. Let $c_0 = c \setminus \{\hat{e}_1, \hat{e}_2\}$, $c_1 = c \setminus \{\hat{e}_2\}$, and $c_2 = c \setminus \{\hat{e}_1\}$. Then by the induction hypothesis there exist $\rho_i \in FS$ such that $Ev(\rho_i) = c_i$ for $i = 0, 1, 2$. It is also clear from Lemma 5.8 that $\rho_0 e_1 =_{TS} \rho_1$ and $\rho_0 e_2 =_{TS} \rho_2$ where $\varphi(\hat{e}_1) = e_1$ and $\varphi(\hat{e}_2) = e_2$. Clearly not $(\hat{e}_1 \leqslant \hat{e}_2$ or $\hat{e}_2 \leqslant \hat{e}_1$ or $\hat{e}_1 \# \hat{e}_2)$ holds. Hence, by the previous lemma $(e_1, e_2) \in I_{TS}$. According to Proposition 5.2, $\rho_0 e_1 e_2, \rho_0 e_2 e_1 \in FS$. It is now straightforward to verify that $Ev(\rho_0 e_1 e_2) = c$. ▮

THEOREM 5.11. *$Uf(TS)$ is an occurrence transition system.*

Proof. We know that $T(ES) = (FC(ES), \hat{E}, \longrightarrow\!\!\!<, \varnothing)$ is an occurrence transition system where ES is as constructed above. Consider the pair of maps (Ev, id) where id is the identity map over $\hat{E}$. By the previous lemma, Ev is a bijection. From Lemma 5.8(iii) and the proof of Lemma 5.10 it is easy to verify that $([\rho], \hat{e}, [\rho']) \in \hat{T}$ iff $Ev(\rho) \overset{\hat{e}}{\longrightarrow\!\!\!<} Ev(\rho')$. From this it follows that (Ev, id) is a transition system morphism, and hence is in fact an isomorphism. From this it follows that $Uf(TS)$ is also an occurrence transition system. ▮

To proceed towards the main result we next define the notion of folding as a morphism from $Uf(TS)$ to TS. This map will turn out to be the co-unit of the co-reflection between OTS and ETS that we are trying to establish.

Let TS and $Uf(TS)$ be as defined previously. Let $fold_{TS} = (f, \eta)$ be given by

(i) $f: \hat{S} \to S$ is such that $\forall \rho \in FS. f([\rho]) = s$, where $s^{in}[\rho\rangle s$ in TS.

(ii) $\eta: \hat{E} \to E$ is such that $\forall \langle \rho, \rho' \rangle \in \hat{E}. \eta(\langle \rho, \rho' \rangle) = \varphi(\langle \rho, \rho' \rangle)$.

PROPOSITION 5.12. *$fold_{TS}$ is a transition system morphism from $Uf(TS)$ to TS.*

Proof. It follows easily from the fact that TS is an ETS-object and that f and η are well-defined total functions. It is then routine to verify that $fold_{TS}$ is indeed a morphism. ▮

The following lemma will turn out to be useful for proving the main theorem of this section.

LEMMA 5.13. *Let $TS_0 = (S_0, E_0, T_0, s_0)$ be an occurrence transition system and $TS = (S, E, T, s^{in})$ be an elementary transition system. Let (g, μ) be a morphism from TS_0 to TS. Suppose $s_0[\rho\rangle s$ and $s_0[\rho'\rangle s$ in TS_0 (i.e., ρ and ρ' are two computations—firing sequences—leading to a common state s). Then $\mu(\rho) =_{TS} \mu(\rho')$.*

Proof. By Lemma 3.6, we know that $|\rho| = |\rho'|$. We now proceed by induction on $k = |\rho|$.

$k = 0$. Then clearly $\mu(\rho) = \Lambda = \mu(\rho')$.

$k > 0$. Let $\rho = \rho_1 e$ and $\rho' = \rho'_1 e'$. Assume $s_0[\rho_1\rangle s_1$ and $s_0[\rho'_1\rangle s'_1$. Suppose $e = e'$. Then once again from Lemma 3.6 it follows that $past(s_1) = past(s'_1)$ and hence by Lemma 3.7, it must be the case that $s_1 = s'_1$. Now by the induction hypothesis, $\mu(\rho_1) =_{TS} \mu(\rho'_1)$. Clearly, it now follows that $\mu(\rho_1 e) =_{TS} \mu(\rho'_1 e')$, since we have assumed $e = e'$.

So suppose that $e \neq e'$. Let $s_0[\rho_1\rangle\, s_1$ and $s_0[\rho'_1\rangle\, s_1$ as before. Consider the PES-object $\zeta(TS)$ defined in the proof of Lemma 4.5. It now follows directly from the properties of the function past that there must exist a state s' in TS_0 such that the situation shown in Fig. 5.10 obtains.

Let $s_0[\rho''\rangle\, s'$ as indicated in Fig. 5.10. By the induction hypothesis, $\mu(\rho_1) = \mu(\rho'' e')$ and $\mu(\rho'_1) = \mu(\rho'' e)$. Suppose $\mu(e)$ is undefined. Consequently $\mu(\rho_1 e) = \mu(\rho_1) =_{TS} \mu(\rho'' e') = \mu(\rho'' e e') =_{TS} \mu(\rho'_1 e')$. By a symmetric argument the result follows if $\mu(e')$ is undefined.

So suppose that both $\mu(e)$ and $\mu(e')$ are defined. First suppose that $\mu(e) = \mu(e')$. Then in TS, we would get $g(s') \xrightarrow{\mu(e)} g(s'_1) \xrightarrow{\mu(e)} g(s)$. This is impossible since TS is elementary (see Nielsen *et al.* 1992). Thus $\mu(e) \neq \mu(e')$. But then this at once would imply, once again by the fact that TS is elementary that $(e, e') \in I_{TS}$. Hence $\mu(\rho'' e e') =_{TS} \mu(\rho'' e' e)$, and from the induction hypothesis, we get, $\mu(\rho_1 e) =_{TS} \mu(\rho'' e' e) =_{TS} \mu(\rho'' e e') =_{TS} \mu(\rho'_1 e')$. ∎

We are now prepared to prove the main result. According to MacLane (1971), proving that unfold is the right adjoint to the inclusion functor from OTS to ETS boils down to establishing the following result.

THEOREM 5.14. *Let TS be an elementary transition system and $Uf(TS)$ and $fold_{TS} = (f, \eta)$ be as defined previously. Suppose $TS_0 = (S_0, E_0, T_0, s_0)$ is an occurrence transition system and (g, μ) is a morphim from TS_0 to TS. Then there exists a unique morphism (h, θ) from TS_0 to $Uf(TS)$ so that the diagram in Fig. 5.11 commutes.*

Proof. We propose the following definition (h, θ):

$h\colon S_0 \to \hat{S}$ is given by $\forall s \in S_0.\, h(s) = [\mu(\rho)]_{TS}$ where $s_0[\rho\rangle\, s$ (in TS_0).

$\theta\colon E_0 \to \hat{E}$ is given by

$$\forall e \in E_0.\ \theta(e) = \begin{cases} \text{undefined,} \\ \quad \text{if } \mu(e) \text{ is undefined,} \\ \langle \mu(\rho), \mu(\rho e)\rangle, \\ \quad \text{otherwise where } s_0[\rho e\rangle\, s_e \text{ in } TS_0. \end{cases}$$

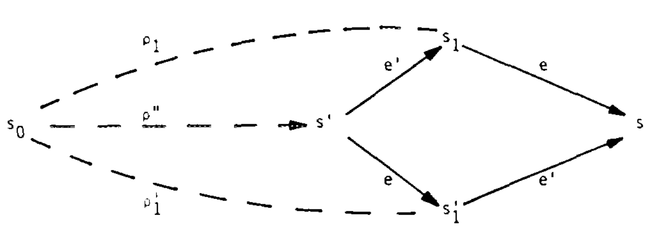

FIGURE 5.10

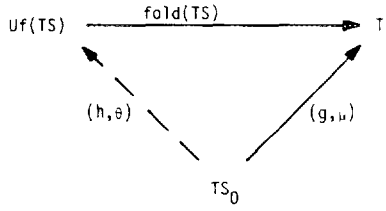

FIGURE 5.11

Recall that s_e is the unique state in TS_0 with the property that $\uparrow s_e$ is a non-trivial region with ${}^\circ(\uparrow s_e) = \{e\}$. By the previous lemma, h and θ are well-defined total and partial functions respectively. We need to prove

(i) (h, θ) is a morphism from TS_0 to $Uf(TS)$,

(ii) $(f, \eta) \circ (h, \theta) = (g, \mu)$, and

(iii) (h, θ) is unique w.r.t. the properties (i) and (ii).

Proof of (i). Let $e \in E_0$ and $(s', e, s) \in T_0$. Suppose $\theta(e)$ is undefined. We must show that $h(s) = h(s')$. Assume that $s_0[\rho'\rangle\, s'$ in TS_0. Then $\mu(\rho') =_{TS} \mu(\rho' e)$. Hence $h(s') = [\mu(\rho')] = [\mu(\rho' e)] = h(s)$ as required.

So suppose that $\theta(e)$ is defined. We then prove that $(h(s'), \theta(e), h(s)) \in \hat{T}$. Let $s_0[\rho_0 e\rangle\, s_e$ in TS_0. Then $\theta(e) = \langle \mu(\rho_0), \mu(\rho_0 e)\rangle$. Since $e \in past(s)$, we must have $\rho'' \in E_0^*$ such that $s_e[\rho''\rangle\, s$ in TS_0. We now proceed by induction on $k = |\rho''|$.

$k = 0$. Then $s = s_e$ and therefore $s_0[\rho_0 e\rangle\, s$ in TS. Consequently $h(s') = [\mu(\rho_0)]$, $h(s) = [\mu(\rho_0 e)]$, and $\theta(e) = \langle \mu(\rho_0), \mu(\rho_0 e)\rangle$. Clearly $(h(s'), \theta(e), h(s)) \in \hat{T}$.

$k > 0$. Let $\rho'' = \rho_1 e_1$. Then from Lemma 4.7, it follows that $({}^\circ e \cup e^\circ) \cap ({}^\circ e_1 \cup e_1^\circ) = \varnothing$ in TS_0 and there exist states s_1 and s'_1 such that the situation shown in Fig. 5.12 obtains.

By the induction hypothesis, $(h(s_1), \theta(e), h(s'_1)) \in \hat{T}$. If $\theta(e_1)$ is undefined, then by the previous argument dealing with the case $\theta(e)$ undefined, we must have $h(s_1) = h(s')$ and $h(s'_1) = h(s)$. Thus $(h(s'), \theta(e), h(s)) \in \hat{T}$ as required.

So suppose that $\theta(e_1)$ is defined. Then from the fact that (g, μ) is a morphism from TS_0 into the elementary transition system TS, we at once get $(\mu(e), \mu(e_1)) \in I_{TS}$. Therefore by the definition of the equivalence relation on prime intervals, we get $\langle \mu(\rho_0 \rho_1 e_1), \mu(\rho_0 \rho_1 e_1 e)\rangle = \langle \mu(\rho_0 \rho_1), \mu(\rho_0 \rho_1 e)\rangle = \theta(e)$ (induction hypothesis). This leads, by the definition of (h, θ) to the desired conclusion that $(h(s'), \theta(e), h(s)) \in \hat{T}$.

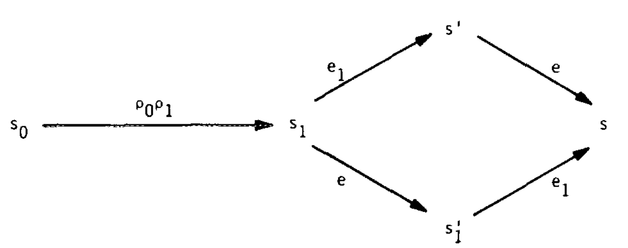

FIGURE 5.12

Proof of (ii). As observed in Nielsen *et al.* (1992), to prove that $(f, \eta) \circ (h, \theta) = (g, \mu)$, it suffices to prove that $f \circ h = g$. But this identity follows immediately from the definitions of f and h.

Proof of (iii). Let (h', θ') be some morphism from TS_0 to $Uf(TS)$ which also satisfies the properties (i) and (ii). As observed above, it suffices to show that $h = h'$, because by Nielsen *et al.* (1992) this would imply $\theta = \theta'$. Let $s \in S_0$ and $s_0[\rho\rangle s$ in TS_0. We proceed by induction on $k = |\rho|$.

$k = 0$. Then $s = s_0$ and the two morphisms (h, θ) and (h', θ') must satisfy $h(s_0) = [\Lambda] = h'(s_0)$.

$k > 0$. Let $\rho = \rho_1 e_1$ and $s_0[\rho_1\rangle s_1$ in TS_0. Then $(s_1, e_1, s) \in T_0$. By the induction hypothesis $h(s_1) = h'(s_1)$.

Suppose $\mu(e_1)$ is undefined. Since $\eta \circ \theta = \mu$ and η is total, it must be the case that $\theta(e_1)$ is undefined. Similarly from $\eta \circ \theta' = \mu$ and the totality of η we can conclude that $\theta'(e_1)$ is also undefined. Now (h, θ) being a morphism, we must have $h(s_1) = h(s)$ and similarly $h'(s_1) = h'(s)$. Thus $h(s) = h'(s)$.

So suppose that $\mu(e_1)$ is defined. Then once again from $\eta \circ \theta = \mu = \eta \circ \theta'$ we conclude that both $\theta(e_1)$ and $\theta'(e_1)$ are defined. Since (h, θ) and (h', θ') are morphisms we get $(h(s_1), \theta(e_1), h(s)) \in \hat{T}$ and $(h'(s_1), \theta'(e_1), h'(s)) \in \hat{T}$. Now $h(s_1) = [\mu(\rho_1)]$ by the definition of h and $h(s) = [\mu(\rho_1)\, \mu(e_1)]$. From property (ii) and the definition of $Uf(TS)$, it now follows that $h(s) = [\mu(\rho_1)\, \eta(\theta(e_1))]$ and $h'(s) = [\mu(\rho_1)\, \eta(\theta'(e_1))]$. But $\eta(\theta(e_1)) = \eta(\theta'(e_1))$ at once implies that $h(s) = h'(s)$ as required. ▮

THEOREM 5.15. *The map unfold uniquely extends to a functor which is the right adjoint of the inclusion functor from OTS to ETS, i.e., OTS is a co-reflective full subcategory of ETS.*

Proof. Follows easily from the previous theorem according to MacLane (1971). ▮

6. DISCUSSION

Elementary transition systems were introduced in Nielsen *et al.* (1992) where they were shown to be *the* transition system version of elementary net systems, a basic system model of net theory. Elementary transition systems were identified by imposing some axioms on ordinary transition systems, with the axioms being predominantly formulated in terms of a structural notion called regions. In this paper, we have shown that elementary transition systems can also be used to characterize, at the level of transition systems, yet another basic model of concurrency, namely, prime event structures. More precisely, we have shown that by smoothly strengthening the axioms for elementary transition systems one obtains a subclass called occurrence transition systems. This subclass turns to be *the* transition system version of prime event structures. We have chosen to phrase this result in the language of category theory. Thus our first main result (Theorem 4.2) states that the categories OTS and PES are equivalent categories.

Our second main result concerns the unfolding of elementary transition systems into occurence transition systems. The fact that one can do so follows from our first main result (Theorem 4.2), the main result of Nielsen *et al.* (1992)—stated as Theorem 1.6 in this paper—and the results of Nielsen *et al.* (1981) adapted for elementary net systems as in Nielsen *et al.* (1990). This is informally captured by the conjecture following Definition 5.4. However, the question arises as to why this unfolding operation is to be preferred over the multitude of other unfolding operations one can think of. (Here is an easy alternative: Unfold *every* elementary transition system into the occurrence transition system consisting of just one state, no events and no transitions!). Here, as demonstrated by Winskel (1987), category theory—modulo the chosen notion of behaviour preserving morphisms—can be used to provide convincing evidence that the chosen unfolding operation is the "correct" one. In this spirit, our second main result (Theorem 5.15) states that our unfolding operation extends uniquely to the right adjoint of the inclusion functor from OTS to ETS. What this means is that for every ETS object *TS*, the pair $(Uf(TS), fold_{TS})$ consisting of the unfolding of *TS* and the associated folding morphism from $Uf(TS)$ to *TS* the "best" pair possible; every other pair (OTS, f) where *OTS* is an occurrence transition system and f is a morphism from *OTS* to *TS* must uniquely "factor" through the pair $(Uf(TS), fold_{TS})$ as described in Theorem 5.14. This, in our opinion, is one of the main justifications for using the language of category theory to describe relationships between models of concurrency. Through this language, one can hope to show that constructions that transform one type of model into another are not just *ad hoc* translations but instead enjoy certain universal properties. We refer the reader to Winskel and Nielsen (1995) for a variety of such results and further justifications for using the language of category theory to study models of concurrency.

Admittedly, we have not shown here that the chosen notion of unfolding indeed corresponds to the well-known unfolding operation at the net level. As pointed out in Section 5, we have not established this mainly to avoid a considerable technical expansion of the paper and construction of arguments that are more or less available in the literature. We wish to point out, however, that this correspondence between the unfolding operations in the two worlds does *not* directly lead to a proof of Theorem 5.15. For one thing, as pointed out in Section 5, Winskel uses—and must use—a weaker notion of net morphisms (compared to the one used in Nielsen *et al.*, 1992) to establish his co-reflection between occurrence nets and 1-safe Petri nets.

It would be interesting to explore the structure of the nets that arise out of occurrence transition systems. A characterization of this class of nets might then lead to a translation of our unfolding operation in the language of nets.

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