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Punctured Distributions in the Rational Function Fields

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Let M be a nonconstant polynomial in the polynomial ring $R_T = \mathbb{F}_q[T]$ over the finite field $\mathbb{F}_q$. We show that the universal ordinary punctured distribution on $\frac{1}{M} R_T / R_T$ is a free abelian group and determine its rank. We also compute the torsion subgroups of the universal ordinary punctured even and odd distributions. © 2001 Academic Press

INTRODUCTION

Distributions in the rational function fields were studied by S. Galovich and M. Rosen in studying the arithmetic of cyclotomic function fields. However, punctured distributions were not fully discussed in their papers. In this article we introduce an ordinary punctured distribution other than those appeared in Galovich and Rosen [3], but used by Ennola [1] and Schmidt [7, 8] without the notion of distributions. Using this we prove that the universal ordinary punctured distribution is a free abelian group of rank $\phi(M) + \pi(M) - 1$ (see below.). Then we show that up to the torsion and the roots of unity all the relations between cyclotomic units in function fields are consequences of the distribution relation of the torsion points of the Carlitz module. Finally we compute the torsion subgroups of the universal punctured even and odd distributions in the rational function fields. Recently L. Yin extended the concept of distributions to the case of global fields [10]. We fix the following notations:

$R_T = \mathbb{F}_q[T]$

$k = \mathbb{F}_q(T)$

$R_T(M) = \frac{1}{M} R_T / R_T$, for a monic polynomial M of R_T

$\lambda_M =$ a primitive Mth root of the Carlitz module ρ

$\phi(M) =$ the Euler ϕ function

$\pi(M) =$ the number of monic irreducibles dividing M

$v_P =$ the P-adic valuation of k, where P is an irreducible polynomial in R_T.

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The term "$\sum_{A \bmod B}$" always implies that the sum is over the polynomials A with $\deg A < \deg B$. The term "$\sum_{A \bmod {}^*B}$" implies that the sum is over the polynomials A relatively prime to B with $\deg A < \deg B$.

1. DISTRIBUTIONS

A *distribution* is a family of functions

$$\{f_N\colon R_T(N) \to V \mid N \in R_T,\ \text{monic}\},$$

V an abelian group, such that for $N \mid M$ and $\deg A < \deg N$ we have

$$f_N\left(\frac{A}{N}\right) = \sum_{B \equiv A \bmod N} f_M\left(\frac{B}{M}\right). \tag{$*$}$$

A *distribution of level* M is a family of functions

$$\{f_N\colon R_T(N) \to V \mid N \text{ divides } M,\ \text{monic}\},$$

which satisfies $(*)$. A *punctured distribution* is such a family which is not defined at 0 but satisfies the relation for $A \neq 0$. A distribution is said to be *ordinary* if $f_M(\frac{A}{M}) = f_N(\frac{B}{N})$ when $\frac{A}{M} = \frac{B}{N}$. In this case the family $\{f_N\}$ can be replaced by a single function $f\colon k/R_T \to V$ ($R_T(M) \to V$ in the case of level M). In this article we are interested mainly in the ordinary punctured distributions. A distribution is said to be *even* (*real* in the terminology of [2]) if $f_N(\alpha x) = f_N(x)$ for every $\alpha \in \mathbb{F}_q^*$ and every monic N, and *odd* if $\sum_{\alpha \in \mathbb{F}_q^*} f_N(\alpha x) = 0$. Note that for a distribution $\{f_N\}$, $\{h_N(x) = \sum_{\alpha \in \mathbb{F}_q^*} f_N(\alpha x)\}$ is an even distribution and $\{g_N = (q-1) f_N - h_N\}$ is an odd distribution. In the following we denote $\rho_A(x)$ by x^A for simplicity.

EXAMPLE 1. It is shown in [2, Proof of Lemma 5], that the function $\frac{A}{M} \mapsto \lambda_M^A$ is an even ordinary punctured multiplicative distribution. Thus for each place $\wp$ of k the map $\varphi^P(\frac{A}{M}) = v_\wp(\lambda_M^A)$, where $v_\wp$ is a fixed extension of v_P to $k(\lambda_M)$, is an even ordinary punctured distribution. If P divides M and $\wp_1$ and $\wp_2$ are two places over P, then $\varphi^{\wp_1}(\frac{A}{M}) = \varphi^{\wp_2}(\frac{A}{M})$. In this case we write φ^P instead of $\varphi^\wp$ for any place $\wp$ over P.

The Galois group G of $k(\lambda_M)$ over k is isomorphic to $(R_T/M)^*$. We fix an isomorphism $A \mapsto \sigma_A$ so that $\sigma_A(\lambda_M) = \rho_A(\lambda_M)$.

EXAMPLE 2. Let $\chi: \mathbb{F}_q^* \to \mathbb{C}^*$ be a nontrivial character. Let

$$\Theta^\chi\left(\frac{A}{M}\right) = \Theta^\chi_M\left(\frac{A}{M}\right) = \begin{cases} \sum_{X \bmod {}^*M} \tilde{\chi}_M(AX)\,\sigma_X^{-1}, & \text{if } M \nmid A \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\chi}_M(B)$ is the value of χ at the leading coefficient of C which is congruent to B modulo M and $\deg C < \deg M$. Then it is an ordinary χ-distribution, that is, $\Theta^\chi(\alpha x) = \chi(\alpha)\,\Theta^\chi(x)$ for $\alpha \in \mathbb{F}_q^*$. In particular Θ^χ is an odd distribution. Again Θ^χ is the distribution $\mathrm{St}(\phi^\chi)$ in [3].

Let S be the set of all nonzero polynomials of degree less than $\deg M$. Let $\mathscr{V}$ be a free abelian group of rank $q^{\deg M} - 1$ with basis $\{\mathfrak{e}_A; A \in S\}$. For a nonconstant polynomial D dividing M and a nonzero polynomial X of degree less than $\deg M - \deg D$, we define

$$\mathfrak{a}(D, X) = \mathfrak{e}_{XD} - \sum_{Y \bmod D} \mathfrak{e}_{X + Y(M/D)}.$$

For a polynomial X in S we define

$$\mathfrak{n}(X) = \sum_{\alpha \in \mathbb{F}_q^*} \mathfrak{e}_{\alpha X},$$

and

$$\mathfrak{n}'(X) = \mathfrak{e}_X - \mathfrak{e}_{\gamma X},$$

where γ is a fixed generator of $\mathbb{F}_q^*$. Let $\mathscr{U}$ be the subgroup of $\mathscr{V}$ generated by $\mathfrak{a}(D, X)$'s. Let $\mathscr{R}_1$ be the subgroup of $\mathscr{V}$ generated by $\mathscr{U}$ and $\mathfrak{n}(X)$, and $\mathscr{R}_1'$ the subgroup generated by $\mathscr{U}$ and $\mathfrak{n}'(X)$. An ordinary distribution is called *universal* if it is an initial object in the category of ordinary distributions. The universal ordinary punctured (even or odd) distribution (of level M) is defined in a similar way. Then $\mathscr{A} = \mathscr{V}/\mathscr{U}$ is isomorphic to the subgroup generated by the images of $R_T(M)$ under a universal ordinary punctured distribution of level M, $\mathscr{V}/\mathscr{R}_1'$ is isomorphic to the subgroup generated by the images of $R_T(M)$ under a universal even ordinary punctured distribution of level M, and $\mathscr{V}/\mathscr{R}_1$ is isomorphic to the subgroup generated by the images of $R_T(M)$ under a universal odd ordinary punctured distribution of level M.

Let ∞_i, $i = 1, ..., \frac{\phi(M)}{q-1}$, be the places of $k(\lambda_M)$ over ∞. Define

$$\Phi_M: R_T(M) \to \mathbb{Q}^{\phi(M)/(q-1) + \pi(M)}$$

by

$$\Phi_M\left(\frac{A}{M}\right)=\left(\varphi^{\infty_i}\left(\frac{A}{M}\right),\ \varphi^P\left(\frac{A}{M}\right) : i=1, ..., \frac{\phi(M)}{q-1},\ P \mid M\right).$$

LEMMA 1.1. *The abelian group generated by $\Phi_M(R_T(M))$ has rank at least $\frac{\phi(M)}{q-1}+\pi(M)-1$.*

Proof. It is clear, by definition, that $\varphi^P(\frac{1}{P})=\frac{1}{\phi(P)}$ and $\varphi^Q(\frac{1}{P})=0$ for $P\neq Q$. We know that $\sum_{A \bmod M} c_A \varphi^P(\frac{A}{M})=0$ for every P dividing M if and only if $\prod_{A \bmod M} (\lambda_{\frac{A}{M}})^{c_A}$ is a unit, and that the group of cyclotomic units has rank $\frac{\phi(M)}{q-1}-1$. Then various combinations of the vectors $\Phi_M(\frac{A}{M})$ give the valuations of the $\frac{\phi(M)}{q-1}-1$ independent units and the P-components of these vectors are 0. Thus these vectors and $\Phi_M(\frac{1}{P})$'s, $P \mid M$ a prime, are linearly independent.

The following proposition is the analog of Proposition 12.11 of [9], and the proof is exactly the same.

PROPOSITION 1.2. *Let f be the universal punctured ordinary distribution. Then the subgroup generated by $f(R_T(M))$ requires at most $\phi(M)+\pi(M)-1$ generators.*

Suppose that h^+ and h^- are any two punctured distributions with h^+ even and h^- odd. Let H^+ (resp. H^-) be the group generated by $h^+(R_T(M))$(resp. $h^-(R_T(M))$). Define

$$h\left(\frac{A}{M}\right)=\left(h^+\left(\frac{A}{M}\right),\ h^-\left(\frac{A}{M}\right)\right)\in H^+\oplus H^-.$$

Let H be the subgroup generated by $h(R_T(M))$. Then

$$\sum_{\alpha\in\mathbb{F}_q^*} h\left(\frac{\alpha A}{M}\right)=\left((q-1)\,h^+\left(\frac{A}{M}\right),0\right)\in H$$

and

$$(q-1)\,h\left(\frac{A}{M}\right)-\sum_{\alpha\in\mathbb{F}_q^*} h\left(\frac{\alpha A}{M}\right)=\left(0,(q-1)\,h^-\left(\frac{A}{M}\right)\right)\in H.$$

Hence

$$(q-1)\,H^+\oplus(q-1)\,H^-\subset H\subset H^+\oplus H^-,$$

so $\operatorname{rank}(H)=\operatorname{rank}(H^+)+\operatorname{rank}(H^-)$. Then Lemma 1.1, Proposition 1.2, and Proposition 3.4 of [3] give the following theorem.

THEOREM 1.3. *Let f be a universal ordinary punctured distribution. Then the subgroup generated by $f(R_T(M))$ is a free abelian group of rank $\phi(M)+\pi(M)-1$.*

For an ordinary distribution f of level M we can define a homomorphism $\tilde{f}$ on $\mathscr{V}$ by $\tilde{f}(\sum n_X \mathfrak{e}_X)=\sum n_X f(\frac{X}{M})$. From now on we will not distinguish distributions and homomorphisms on $\mathscr{V}$ induced by the distributions and drop the tilde from the notation.

2. RELATIONS BETWEEN CYCLOTOMIC UNITS

From now on we assume that M is a monic polynomial. We say that a multiplicative homomorphism

$$\Psi\colon (R_T/M)^* \to \mathbb{C}^*$$

is a *polynomial-Dirichlet character*, or simply, a *character*. Then as in the classical case the conductor of a character is defined. As usual Ψ_1 denotes the trivial character. We say that Ψ is *even* if $\Psi(\alpha A)=\Psi(A)$ for every $\alpha \in \mathbb{F}_q^*$, and *odd* otherwise.

Let

$$\mathscr{R}=\bigcap_{\chi \neq \chi_1} \ker \Theta_M^{\chi},$$

and

$$\mathscr{R}'=\ker \Phi_M .$$

Then it is clear that $\mathscr{R}_1 \subset \mathscr{R}$ and $\mathscr{R}'_1 \subset \mathscr{R}'$.

Remark. $\mathscr{R}'$ is the analog of ker (Θ) in [1]. In [1] the distribution φ^P did not appear, but we have to use φ^P to define $\mathscr{R}'$. The reason for this is that in the classical case log p's for different prime numbers are algebraically independent, but in the function field case $\log_q |P|_\infty$'s are all integers.

Let $\mathfrak{r}=\sum_{A\in S} c_A \mathfrak{e}_A$ be an element of $\mathscr{V}$, and Ψ a character with conductor F. For a nonconstant monic polynomial D dividing M and $F \mid D$, we define

$$T(\Psi, D, \mathfrak{r})=\sum_{A \bmod {}^*D} \Psi(A)\, c_{(M/D)A} .$$

In case $\deg F \geqslant 1$, let

$$Y(\Psi, \mathfrak{r}) = \sum_{D,\, F \mid D \mid M} \frac{1}{\phi(D)} \prod_{P \mid D} (1 - \bar{\Psi}(P))\, T(\Psi, D, \mathfrak{r}).$$

Remarks. (1) It is clear from the definition that $Y(\Psi, \mathfrak{r})$ is the analog of $Y(\chi, R)$ and $\varphi^P(\mathfrak{r})$ is the analog of $Y_p(R)/\phi(p^{\gamma_p})$ of [1].

(2) From the definition of Φ_M we may view $\mathscr{R}'$ as the relation subgroup of cyclotomic units.

In this section we are going to prove the following three theorems.

THEOREM 2.1. *Let $\mathfrak{r} \in \mathscr{V}$. Then $\mathfrak{r} \in \mathscr{R}'$ if and only if*

$$Y(\Psi, \mathfrak{r}) = 0$$

for every even character $\Psi \neq \Psi_1$ with conductor dividing M, and

$$\varphi^P(\mathfrak{r}) = 0,$$

for every prime P dividing M.

THEOREM 2.2. $\mathfrak{r} \in \mathscr{U}$ *if and only if*

$$Y(\Psi, \mathfrak{r}) = 0,$$

for every nontrivial character Ψ with conductor dividing M, and

$$\varphi^P(\mathfrak{r}) = 0,$$

for every prime P dividing M.

THEOREM 2.3. *If $\mathfrak{r} \in \mathscr{R}'$, then $(q-1)\,\mathfrak{r} \in \mathscr{R}'_1$.*

Proof of Theorem 2.1. Assume that Ψ is even. Choose $\lambda_N = \xi(N)\, e_{(N)}(1)$, where $\xi(N)\, NR_T$ is the lattice associated with the Carlitz module ρ and $e_{(N)}$ the lattice function corresponding to the ideal (N). Then for $M = ND$, $\lambda_D = \rho_N(\lambda_M)$. Let F_Ψ be the conductor of Ψ. For a non-constant polynomial D such that $D \mid M$, $F_\Psi \mid D$, define

$$S(\Psi, D) = \ln q \sum_{X \bmod {}^*D,\ \text{monic}} \Psi(X) \log_q |\lambda_D^X|_\infty$$

and

$$S(\Psi) = S(\Psi, F_\Psi).$$

If $\Psi \neq \Psi_1$, then from Proposition 7.15 of [5]

$$S(\Psi) = -(q-1)\, L'(0, \Psi) \neq 0,$$

where $L(s, \Psi)$ is the L-function of R_T, and from Lemma 6 of [2]

$$S(\Psi, D) = \prod_{P \mid D,\ \text{monic}} (1 - \Psi(P))\, S(\Psi).$$

As in the classical case

$$S(\Psi_1, D) = \ln q \sum \log_q |\lambda_D^X|_\infty = \ln q \cdot \log_q \left|\prod \lambda_D^X\right|_\infty$$

$$= \begin{cases} \dfrac{\ln q}{q-1} \deg P, & \text{if } D \text{ is a power of a prime } P \\ 0, & \text{otherwise.} \end{cases}$$

Then from the definition of $S(\Psi, D)$ we have

$$\ln q \cdot \log_q |\lambda_D^X|_\infty = \frac{q-1}{\phi(D)} \sum_{\Psi\ \text{even},\ F_\Psi \mid D} \bar{\Psi}(X)\, S(\Psi, D),$$

for $D \mid M$, D nonconstant, $(X, D) = 1$.

For $\mathfrak{r} = \sum_{X \bmod M} c_X \mathfrak{e}_X$, define

$$\mathfrak{r}' = \sum_{X \bmod M,\ \text{monic}} c'_X \mathfrak{e}_X,$$

where $c'_X = \sum_{\alpha \in \mathbb{F}_q^*} c_{\alpha X}$. Then it is clear that $\mathfrak{r} \in \mathscr{R}'$ if and only if $\mathfrak{r}' \in \mathscr{R}'$, because $v(\mathbb{F}_q^*) = 0$ for any valuation v of k. Thus we may assume that $c_X = 0$ if X is not monic. Then as in [1] we have

$$(q-1) \sum_{\Psi \neq \Psi_1,\ \text{even},\ F_\Psi \mid M} \bar{\Psi}(K)\, S(\Psi)\, Y(\bar{\Psi}, \mathfrak{r}) + \ln q \sum_{P \mid M} \varphi^P(\mathfrak{r}) \deg P = 0,$$

for every K prime to M if and only if $\mathfrak{r} \in \mathscr{R}'$. Since $\varphi^P(\mathfrak{r}) = 0$ for $\mathfrak{r} \in \mathscr{R}'$ by the definition of Φ_M,we get the result on multiplying by $\Psi(K)$ and summing over K.

Proof of Theorem 2.2. The necessity is exactly the same as in the classical case. The lemmas in [1] hold with only minor changes, so we do not state them explicitly. We are now going to prove the sufficiency.

Let $\mathfrak{r}$ be any element of $\mathscr{V}$ such that $Y(\Psi, \mathfrak{r}) = 0$ for every $\Psi \neq \Psi_1$ and $\varphi^P(\mathfrak{r}) = 0$ for every prime P dividing M. Assume that $\Delta(\mathfrak{r}) = 1$, where Δ is defined as in [1, Sect. 5]. Define $\mathfrak{r}_P$ and $\bar{\mathfrak{r}}_P$ as in [1]. Then we get

$$\mathfrak{r}_P - \bar{\mathfrak{r}}_P = c^{(P)} \sum \mathfrak{e}_{MX/P^{v_P(M)}},$$

where the sum runs over the X's of degree less than $v_P(M) \deg P$ and prime to P. Since $\varphi^P(\mathfrak{r}_P - \bar{\mathfrak{r}}_P) = 0$, $c^{(P)} = 0$. Therefore $\mathfrak{r}_P \in \mathscr{U}$. The rest of the proof is almost the same as in [1].

Proof of Theorem 2.3. Let $\mathfrak{r} = \sum_{X \bmod M} c_X \mathfrak{e}_X$ belong to $\mathscr{R}'$. Let

$$\mathfrak{r}' = \frac{1}{2} \sum_{X \bmod M,\ \text{monic}} \left(\sum_{\alpha, \beta \in \mathbb{F}_q^*} (c_{\alpha X} - c_{\beta X})(\mathfrak{e}_{\alpha X} - \mathfrak{e}_{\beta X}) \right).$$

Then it can be easily verified that $\mathfrak{r}'$ is in $\mathscr{R}'_1$ and

$$\mathfrak{r}'' = (q-1)\,\mathfrak{r} - \mathfrak{r}' = \sum_{X \bmod M} \left(\sum_{\alpha \in \mathbb{F}_q^*} c_{\alpha X} \right) \mathfrak{e}_X.$$

Thus the coefficients of $\mathfrak{e}_{\alpha X}$ in $\mathfrak{r}''$ are the same for any $\alpha \in \mathbb{F}_q^*$. Then $Y(\Psi, \mathfrak{r}'') = 0$ for any odd character Ψ. Thus $\mathfrak{r}''$ lies in $\mathscr{U}$ by Theorem 2.1 and Theorem 2.2, and we get the result.

3. TORSION OF PUNCTURED DISTRIBUTIONS

It is easy to see that $\mathscr{V}$ is a G-module via $\sigma_A(\mathfrak{e}_X) = \mathfrak{e}_{AX}$. Let J be the subgroup of G consisting of σ_α with $\alpha \in \mathbb{F}_q^*$. Let

$$N_J = \sum_{\alpha \in \mathbb{F}_q^*} \sigma_\alpha \qquad \text{and} \qquad I_J = 1 - \sigma_\gamma,$$

where γ is a generator of $\mathbb{F}_q^*$. In this section we are going to compute the torsion subgroups of the modules $\mathscr{V}/(\mathscr{U} + I_J\mathscr{V})$ and $\mathscr{V}/(\mathscr{U} + N_J\mathscr{V})$. Note that $\mathscr{U} + I_J\mathscr{V}$ is isomorphic to $\mathscr{R}_1$ and $\mathscr{U} + N_J\mathscr{V}$ is isomorphic to $\mathscr{R}'_1$. In the following $T(\mathscr{B})$ denotes the torsion subgroup of a group $\mathscr{B}$.

LEMMA 3.1. *We have*

(a) $H^0(J, \mathscr{A}) = T(\mathscr{A}/N_J\mathscr{A}) = T(\mathscr{V}/(\mathscr{U} + N_J\mathscr{V}))$.

(b) $H^{-1}(J, \mathscr{A}) = T(\mathscr{A}/I_J\mathscr{A}) = T(\mathscr{V}/(\mathscr{U} + I_J\mathscr{V}))$.

Proof. Once we have Theorem 2.2, the proof is exactly the same as the classical case (cf. [8, Lemma 2.1]).

LEMMA 3.2. *$\mathfrak{r} \in \mathcal{R}$ if and only if for each odd character Ψ of $(R_T/M)^*$ we have $Y(\Psi, \mathfrak{r}) = 0$.*

Proof. Under the fixed isomorphism between $(R_T/M)^*$ and $G = \mathrm{Gal}(k(\lambda_M)/k)$ we can view a character Ψ of $(R_T/M)^*$ as a function on G. Suppose that, for a nontrivial character χ of $\mathbb{F}_q^*$ and a nontrivial character Ψ of $(R_T/M)^*$ with conductor $F_\Psi | M$, we have

$$\Psi\left(\Theta_M^\chi\left(\frac{1}{M}\right)\right) = \Psi\left(\Theta_{F_\Psi}^\chi\left(\frac{1}{F_\Psi}\right)\right) \prod_{P \mid M} (1 - \bar{\Psi}(P)), \tag{1}$$

and

$$\Psi\left(\Theta_M^\chi\left(\frac{X}{M}\right)\right) = \begin{cases} \Psi(X_0)\, \Psi\left(\Theta_{M_0}^\chi\left(\frac{1}{M_0}\right)\right) \dfrac{\phi(M)}{\phi(M_0)}, & \text{if } \; F_\Psi \mid M_0 \\ 0, & \text{otherwise,} \end{cases}$$

where $X = (X, M)\, X_0$ and $M = (X, M)\, M_0$. It is shown in [3], Proposition 3.4, that $\Psi(\Theta_{F_\Psi}^\chi(1/F_\Psi)) \neq 0$ if and only if $\Psi = \chi$ on $\mathbb{F}_q^*$. Then the result follows from the same argument as in [7, Satz 1], if we prove (1) and (2). But we should note that we do not have the condition that $\sum_{0 \neq X \bmod M} c_X = 0$ because $\Psi_1(\Theta_M^\chi(X/M)) = 0$ for each nontrivial character χ of $\mathbb{F}_q^*$. (1) is just a special case of Proposition 3.1 of [10]. For (2) we first note that if $B \equiv B' \bmod M_0$, then $\tilde{\chi}_M(XB) = \tilde{\chi}_M(XB')$. Also it is not hard to show that $\tilde{\chi}_M(XB) = \tilde{\chi}_{M_0}(X_0 B)$. Then we have

$$\Theta_M^\chi\left(\frac{X}{M}\right) = \Theta_{M_0}^\chi\left(\frac{X_0}{M_0}\right) \sum_{A \in S_0} \sigma_A^{-1},$$

where $S_0 = \{A \in R_T : \deg A < \deg M,\ (A, M) = 1,\ A \equiv 1 \bmod M_0\}$. Then one can follow the proof of Lemma 2 of [7] to get (2).

LEMMA 3.3. *We have*

(a) $T(\mathcal{V}/(\mathcal{U} + N_J\mathcal{V})) = \mathcal{R}/(\mathcal{U} + N_J\mathcal{V}) = \mathcal{R}/\mathcal{R}_1$.

(b) $T(\mathcal{V}/(\mathcal{U} + I_J\mathcal{V})) = \mathcal{R}'/(\mathcal{U} + I_J\mathcal{V}) = \mathcal{R}'/\mathcal{R}'_1$.

Proof. It is clear from the definitions that $\mathcal{R}_1 \subset \mathcal{R}$, $\mathcal{R}'_1 \subset \mathcal{R}'$, and both $\mathcal{R}$ and $\mathcal{R}'$ are torsion free. Thus for (a) it suffices to prove

$$(q-1)\, \mathcal{R} \subset \mathcal{U} + N_J\mathcal{V}.$$

For $\mathfrak{r}=\sum c_X \mathfrak{e}_X \in \mathscr{R}$ define

$$\begin{aligned}\mathfrak{r}' &= \tfrac{1}{2} \sum_{X \bmod M,\ \text{monic}} \left(\sum_{\alpha,\, \beta \in \mathbb{F}_q^*} (c_{\alpha X} - c_{\beta X})(\mathfrak{e}_{\alpha X} - \mathfrak{e}_{\beta X}) \right) \\ &= (q-1)\,\mathfrak{r} - \sum_{X \bmod M,\ \text{monic}} \left(\sum_{\alpha \in \mathbb{F}_q^*} c_{\alpha X} \right) \mathfrak{n}(X)\end{aligned}$$

as in the proof of Theorem 2.3. Then $\varphi^P(\mathfrak{r}')=0$ and $Y(\Psi, \mathfrak{r}')=0$ for even character Ψ from the first expression of $\mathfrak{r}'$ above. Also $Y(\Psi, \mathfrak{r}')=0$ for odd character Ψ from the second expression of $\mathfrak{r}'$ and Lemma 3.2. Thus we see that $\mathfrak{r}' \in \mathscr{U}$ using Theorem 2.2, and $(q-1)\,\mathfrak{r} - \mathfrak{r}' = \sum_{X \bmod M,\ \text{monic}} (\sum_{\alpha \in \mathbb{F}_q^*} c_{\alpha X})\, \mathfrak{n}(X) \in N_J \mathscr{V}$.

For (b) it suffices to prove that $(q-1)\,\mathscr{R}' \subset \mathscr{R}'_1$ which is the content of Theorem 2.3.

COROLLARY 3.4. *We have*

(a) $\mathscr{R}/\mathscr{R}_1 \simeq H^0(J, \mathscr{A})$.

(b) $\mathscr{R}'/\mathscr{R}'_1 \simeq H^{-1}(J, \mathscr{A})$.

For each monic prime P dividing M let $\mathscr{T}_P$ be the inertia group at P in G and T_P be defined modulo M by

$$T_P \equiv 1 \ (\text{mod } P^{v_P(M)}), \qquad T_P \equiv P \ (\text{mod } M/P^{v_P(M)}).$$

For a monic divisor D of M let

$$\mathscr{H}_D = \{\sigma_A;\ A \equiv 1(D)\}.$$

Let $\mathfrak{A}$ be the $\mathbb{Z}[G]$-submodule of $\mathbb{Q}[G]$ generated by the elements

$$\alpha_D = \sum_{\sigma \in \mathscr{H}_D} \sigma \prod_{P \mid D} \left(1 - \sigma_{T_P}^{-1} \frac{1}{|\mathscr{T}_P|} \left(\sum_{\tau \in \mathscr{T}_P} \tau \right) \right).$$

Let

$$\omega = \sum_{\Psi \neq \Psi_0} \phi(F_\Psi)\, e_\Psi \mathfrak{e}_{M/F_\Psi},$$

where $e_\Psi = 1/|G| \sum_{\sigma \in G} \Psi(\sigma^{-1})\, \sigma$. $\mathfrak{A}$ and ω are first defined by Galovich and Rosen in [4].

LEMMA 3.5. *For each nonconstant monic divisor D of M both $(|G| - \sum_{\sigma \in G} \sigma)\, \mathscr{V}$ and $\alpha_D \omega$ lie in $(\varphi^P)^{-1}(0)$ for every prime P dividing M.*

Proof. For each σ in G, $\sigma \mathfrak{e}_X = \mathfrak{e}_{XY}$ for some polynomial Y prime to M. Hence $\varphi^P(\mathfrak{e}_X - \sigma \mathfrak{e}_X) = 0$ for every P dividing M. Since the unit group of $\mathcal{O}_M$, the ring of integers in $k(\lambda_M)$, is stable under the action of G, we get the result.

Now one can follow the arguments in [8] to show that the G-modules $(|G| - \sum_{\sigma \in G} \sigma) \mathscr{A}$ and $(1 - e_{\Psi_0}) \mathfrak{A}$ are isomorphic. It is shown in [4, Proposition 5.3] that $H^n(J, \mathfrak{A}) \simeq (\mathbb{Z}/(q-1))^{2^{\pi(M)-1}}$. Almost the same argument as in [8, Lemma 4.1 and Lemma 5.1] gives

LEMMA 3.6. $H^{2n}(J, (1 - e_{\Psi_0}) \mathfrak{A}) \simeq (\mathbb{Z}/(q-1))^{2^{\pi(M)-1}-1}$, *and*

$$H^{2n-1}(J, (1 - e_{\Psi_0}) \mathfrak{A}) \simeq (\mathbb{Z}/(q-1))^{2^{\pi(M)-1}}.$$

LEMMA 3.7. (a) *$\mathscr{A}^G$ is a free abelian group of rank $\pi(M)$ with basis*

$$\mathfrak{B} = \left\{ \mathfrak{b}_P = \sum_{V \bmod {}^*P} \bar{\mathfrak{e}}_{V(M/P)};\ P \mid M \right\}.$$

Here $\bar{\mathfrak{e}}$ means the image of $\mathfrak{e}$ in $\mathscr{A}$.

(b) $\mathscr{A}^G / (N_J \mathscr{A})^G = 0$.

Combining all the results in this section we get

THEOREM 3.8. *We have*

$$H^{2n}(J, \mathscr{A}) \simeq (\mathbb{Z}/(q-1))^{2^{\pi(M)-1}-1},$$

and

$$H^{2n-1}(J, \mathscr{A}) \simeq (\mathbb{Z}/(q-1))^{2^{\pi(M)-1}-\pi(M)}.$$

REFERENCES

1. V. Ennola, On relations between cyclotomic units, *J. Number Theory* **4** (1972), 236–247.
2. K. Feng and L. Yin, Maximal independent system of units in cyclotomic function fields, *Science Sinica* **34** (1991), 908–919.
3. S. Galovich and M. Rosen, Distributions in rational function fields, *Math. Ann.* **256** (1981), 549–560.
4. S. Galovich and M. Rosen, Units and class groups in cyclotomic function fields, *J. Number Theory* **14** (1982), 156–184.
5. B. Gross and M. Rosen, Fourier series and the special values of L-functions, *Adv. Math.* **69** (1988), 1–31.
6. D. Hayes, Explicit class field theory in rational function fields, *Trans. Amer. Math. Soc.* **189** (1974), 163–191.

7. C. G. Schmidt, Relationen von Gaußschen Summen und Kreiseinheiten, *Arch. Math.* **31** (1978), 457–463.
8. C. G. Schmidt, Die Relationenfaktorgruppen von Stickelberger-Elementen und Kreiszahlen, *J. Reine Angew. Math* **315** (1980), 60–72.
9. L. Washington, "Introduction to Cyclotomic Fields," Springer-Verlag, Berlin/Heidelberg, 1980.
10. L. Yin, Distributions on a global field, *J. Number Theory* **80** (2000), 154–167.