JOURNAL OF ALGEBRA **208,** 185–215 (1998)
ARTICLE NO. JA987503

Crystals for Demazure Modules of Classical Affine Lie Algebras

Atsuo Kuniba

Institute of Physics, University of Tokyo, Komaba, Tokyo 153, Japan

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205

Masato Okado

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan

Taichiro Takagi

Department of Mathematics and Physics, National Defense Academy, Okosuka 239, Japan

and

Jun Uchiyama

Department of Physics, Rikkyo University, Nishi-Ikebukuro, Tokyo 171, Japan

Communicated by Georgia Benkart

Received July 21, 1997

We study, in the path realization, crystals for Demazure modules of affine Lie algebras of types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$. We find a special sequence of affine Weyl group elements for the selected perfect crystal, and show that if the highest weight is $l\Lambda_0$, the Demazure crystal has a remarkably simple structure.

0021-8693/98 $25.00

0. INTRODUCTION

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Let $U_q(\mathfrak{g})$ be its quantized universal enveloping algebra, and let $V(\lambda)$ be the integrable $U_q(\mathfrak{g})$-module with dominant integral highest weight λ. Let W be the Weyl group of $\mathfrak{g}$. Fixing an element w of W, the Demazure module $V_w(\lambda)$ is defined as a finite-dimensional subspace of $V(\lambda)$ generated from the extremal weight space $V(\lambda)_{w\lambda}$ by the e_i generators of $U_q(\mathfrak{g})$. Although the Demazure module itself can be defined in the same way for the classical case $q = 1$, we stay in the quantum case. The reason is the existence of a "good" basis. Let $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ be the crystal basis of $V(\lambda)$ [K1]. In [K2], Kashiwara showed there exists a subset $\mathscr{B}_w(\lambda)$ of $\mathscr{B}(\lambda)$ such that

$$V_w(\lambda) = \bigoplus_{b \in \mathscr{B}_w(\lambda)} \mathbf{Q}(q) G_\lambda(b).$$

Here $G_\lambda(b)$ is the lower global base [K3]. Moreover, $\mathscr{B}_w(\lambda)$ has a quite remarkable simple recursive property,

$$\text{If } r_i w \succ w, \qquad \text{then } \mathscr{B}_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n \mathscr{B}_w(\lambda) \setminus \{0\}.$$

Here r_i is a simple reflection of W, and $\succ$ denotes the Bruhat order. In fact, using this property Kashiwara gave a new proof of Demazure's character formula for an arbitrary symmetrizable Kac–Moody algebra [K2].

Let us now focus on the quantum affine algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is of affine type. In this case, we have a description of $\mathscr{B}(\lambda)$ in terms of paths [KMN1, KMN2]. Roughly speaking, the set of paths is a suitable subset of the half infinite tensor product of a "perfect" crystal B, which is a crystal of a finite-dimensional $U_q'(\mathfrak{g})$-module having some nice properties [KMN1]. On this set, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are explicitly given. In [KMOU], we gave a criterion for $\mathscr{B}_w(\lambda)$ to have a tensor product structure. To describe the general situation, the mixing index κ was introduced. Taking $\kappa = 1$ for simplicity, the result in [KMOU] is stated as follows. Consider an increasing sequence $\{w^{(k)}\}_{k \geq 0}$ of W with respect to the Bruhat order. If a perfect crystal B and $\{w^{(k)}\}$ satisfy several assumptions, then $\mathscr{B}_{w^{(k)}}(\lambda)$ is given by

$$\begin{array}{rcl} \mathscr{B}(\lambda) & \subset & \cdots \otimes B \otimes B \otimes B \otimes B \otimes \cdots \otimes B \\ & \cup & \\ \mathscr{B}_{w^{(k)}}(\lambda) & = & \cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1} \otimes B_a^{(j)} \otimes B \otimes \cdots \otimes B. \end{array} \tag{0.1}$$

Here j, a are determined from k, $B_a^{(j)}$ is a subset of B, and $\bar{b} = \cdots \otimes \bar{b}_j \otimes \cdots \otimes \bar{b}_1$ is the ground state path corresponding to the highest weight vector in $\mathscr{B}(\lambda)$.

The purpose of this article is to show that if $\lambda = l\Lambda_0$ (we also discuss some other similar cases), we can find the sequence $\{w^{(k)}\}$ satisfying (0.1) for $\mathfrak{g}$ of classical types (i.e., $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, C_n^{(1)}$). We choose the perfect crystal B from the list in [KMN2], except for the $C_n^{(1)}$ case. To illustrate, we take an example of $\mathfrak{g} = A_3^{(1)}$, $B = B(\Lambda_2)$. The crystal graph of B is given in Fig. 1. Since B is a level 1 perfect crystal, let us take $\lambda = \Lambda_0$. Then the path $\bar{b}$ is given by

$$\bar{b} = \cdots \otimes \boxed{\begin{matrix}1\\2\end{matrix}} \otimes \boxed{\begin{matrix}3\\4\end{matrix}} \otimes \boxed{\begin{matrix}1\\2\end{matrix}} \otimes \boxed{\begin{matrix}3\\4\end{matrix}}.$$

In this case, the sequence of Weyl group elements $\{w^{(k)}\}_{k \geq 0}$ is given as follows:

$$w^{(0)} = 1, \qquad w^{(k+1)} = r_i w^{(k)} \qquad (k \geq 0),$$

where $i = 0$ $(k \equiv 0, 3)$, $i = 3$ $(k \equiv 1, 6)$, $i = 1$ $(k \equiv 2, 5)$, $i = 2$ $(k \equiv 4, 7)$. Here $\equiv$ denotes "congruence modulo 8." The integers j, a in (0.1) are determined from k by $k = 4(j-1) + a$, $j \geq 1$, $0 \leq a < 4$; and $B_a^{(j)}$ are given as follows:

If j is odd,

$$B_0^{(j)} = \left\{\boxed{\begin{matrix}3\\4\end{matrix}}\right\}, \qquad B_1^{(j)} = B_0^{(j)} \cup \left\{\boxed{\begin{matrix}1\\3\end{matrix}}\right\},$$

$$B_2^{(j)} = B_1^{(j)} \cup \left\{\boxed{\begin{matrix}1\\4\end{matrix}}\right\}, \qquad B_3^{(j)} = B_2^{(j)} \cup \left\{\boxed{\begin{matrix}2\\3\end{matrix}}, \boxed{\begin{matrix}2\\4\end{matrix}}\right\}.$$

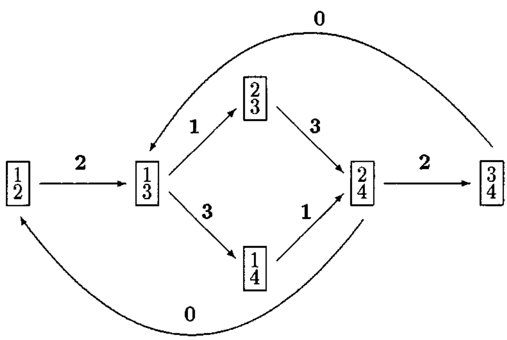

FIG. 1. Level 1 perfect crystal $B(\Lambda_2)$ for $A_3^{(1)}$.

If j is even,

$$B_0^{(j)} = \left\{ \boxed{\begin{matrix}1\\2\end{matrix}} \right\}, \qquad B_1^{(j)} = B_0^{(j)} \cup \left\{ \boxed{\begin{matrix}1\\3\end{matrix}} \right\},$$

$$B_2^{(j)} = B_1^{(j)} \cup \left\{ \boxed{\begin{matrix}2\\3\end{matrix}} \right\}, \qquad B_3^{(j)} = B_2^{(j)} \cup \left\{ \boxed{\begin{matrix}1\\4\end{matrix}}, \boxed{\begin{matrix}2\\4\end{matrix}} \right\}.$$

This paper is organized as follows. In the next section, we review perfect crystals, Demazure crystals, and the criterion developed in [KMOU]. In Sections 2 to 8, we list the sequence of Weyl group elements $\{w^{(k)}\}$, the subset $B_a^{(j)} \subset B$, and other important data for Demazure crystals. In the last section, some observations and conjectures are given.

1. CRYSTALS FOR DEMAZURE MODULES

In [KMOU], a criterion for Demazure crystals is presented. It clarifies their tensor product structure, and involves a parameter κ that measures the degree of *mixing*. Here we summarize the criterion for $\kappa = 1$.

1.1. *Perfect Crystal*

We follow the notations of the quantized universal enveloping algebra and the crystal basis in [KMN1], except that we shall use a different font for the crystal basis $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ of the irreducible highest weight module $V(\lambda)$, to avoid the confusion with the crystal basis (L, B) of V in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$, which may also have an argument. We review necessary properties of perfect crystals. Our main reference is [KMN1].

Let B be a perfect crystal of level l. Then for any $\lambda \in (P_{cl}^+)_l$, there exists a unique element $b(\lambda) \in B$ such that $\varphi(b(\lambda)) = \lambda$. Let σ be the automorphism of $(P_{cl}^+)_l$ given by $\sigma\lambda = \varepsilon(b(\lambda))$. We set $\bar{b}_k = b(\sigma^{k-1}\lambda)$ and $\lambda_k = \sigma^k\lambda$. Then perfectness ensures the following isomorphism of crystals:

$$\mathscr{B}(\lambda_{k-1}) \simeq \mathscr{B}(\lambda_k) \otimes B. \tag{1.1}$$

Define the set of paths $\mathscr{P}(\lambda, B)$ by

$$\mathscr{P}(\lambda, B) = \left\{ p = \cdots \otimes p(2) \otimes p(1) \mid p(j) \in B,\ p(k) = \bar{b}_k \text{ for } k \gg 1 \right\}.$$

Iterating the isomorphism (1.1), we see $\mathscr{B}(\lambda)$ is isomorphic to $\mathscr{P}(\lambda, B)$. Under this isomorphism, the highest weight vector u_λ in $\mathscr{B}(\lambda)$ corresponds to the path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_2 \otimes \bar{b}_1$, which we call the *ground state* path. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathscr{P}(\lambda, B)$ are determined by *signature rule*, which we explain in the next subsection.

1.2. *Signature Rule*

We need to know the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on the set of paths $\mathscr{P}(\lambda, B)$. To see this, we consider the following isomorphism:

$$\mathscr{P}(\lambda, B) \simeq \mathscr{B}(\lambda_k) \otimes B^{\otimes k}. \tag{1.2}$$

For each $p \in \mathscr{P}(\lambda, B)$, if we take k sufficiently large, we can assume that p corresponds to $u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$, where u_{λ_k} is the highest weight vector of $\mathscr{B}(\lambda_k)$ and $p(j) \in B$ $(j = 1, \dots, k)$. Then we apply Proposition 2.1.1 in [KN] to see which component $\tilde{e}_i$ or $\tilde{f}_i$ acts on. Let us suppose that $\tilde{e}_i$ or $\tilde{f}_i$ acts on the jth component from the right end. If $j < k + 1$, we have

$$\tilde{e}_i p = \cdots \otimes \tilde{e}_i p(j) \otimes \cdots \otimes p(2) \otimes p(1) \tag{1.3}$$

or

$$\tilde{f}_i p = \cdots \otimes \tilde{f}_i p(j) \otimes \cdots \otimes p(2) \otimes p(1). \tag{1.4}$$

If $j = k + 1$, we see we should have taken k larger. This happens only for $\tilde{f}_i$.

This determination of the component can be rephrased using the notion of *signature*. Let $p \in \mathscr{P}(\lambda, B)$ correspond to $u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$ under (1.2) as above. With $p(j)$ $(1 \le j \le k)$, we associate

$$\begin{aligned} \epsilon^{(j)} &= \left(\epsilon_1^{(j)}, \dots, \epsilon_m^{(j)}\right), \\ m &= \varepsilon_i(p(j)) + \varphi_i(p(j)), \\ \epsilon_a^{(j)} &= -, \text{ if } 1 \le a \le \varepsilon_i(p(j)), \qquad +, \text{ if } \varepsilon_i(p(j)) < a \le m. \end{aligned}$$

For the highest weight vector u_{λ_k}, we take

$$\epsilon^{(k+1)} = (\underbrace{+, \dots, +}_{\langle \lambda_k, h_i \rangle}).$$

We then append these $\epsilon^{(j)}$'s so that we have

$$\epsilon = (\epsilon^{(k+1)}, \epsilon^{(k)}, \dots, \epsilon^{(1)}).$$

We call it the (i-)signature of p truncated at the kth position.

Next we consider a sequence of signatures,

$$\epsilon = \eta_0, \eta_1, \dots, \eta_{\max}.$$

Here η_{j+1} is obtained from η_j by deleting the leftmost adjacent $(+,-)$ pair of η_j. Eventually, we arrive at the following signature:

$$\eta_{\max} = (\underbrace{-,\ldots,-}_{n_-},\underbrace{+,\ldots,+}_{n_+}),$$

with $n_\pm \geq 0$. We call it the *reduced signature* and denote it by $\bar{\epsilon}$.

The component on which $\tilde{e}_i$ or $\tilde{f}_i$ acts in (1.3) or (1.4) reads as follows. If $n_- = 0$ (resp., $n_+ = 0$), we set $\tilde{e}_i p = 0$ (resp., $\tilde{f}_i p = 0$). Otherwise, take the rightmost $-$ (resp., leftmost $+$) and find the component $\epsilon^{(j)}$ to which it belonged. Then, this j is the position in (1.3) (resp., (1.4)) we looked for. Note that if k is large enough, the position j does not depend on the choice of k.

Remark 1.1. Of course, this signature rule can be applied to the tensor product of crystals $B_1 \otimes \cdots \otimes B_l$.

EXAMPLE 1.1. Let $\mathfrak{g} = A_1^{(1)}$, and let B be the classical crystal of the three-dimensional irreducible representation. Its crystal graph is described as follows:

$$B \quad 00 \underset{0}{\overset{1}{\rightleftharpoons}} 01 \underset{0}{\overset{1}{\rightleftharpoons}} 11$$

it is known that B is perfect of level 2. We have isomorphisms $B(\lambda) \simeq \mathscr{P}(\lambda, B)$ for $\lambda = 2\Lambda_0, \Lambda_0 + \Lambda_1, 2\Lambda_1$. Let $\lambda = 2\Lambda_0$. We see $\lambda_k = 2\Lambda_0$ (k: even), $2\Lambda_1$ (k: odd). The ground-state path is given by

$$\bar{p} = \cdots \otimes 11 \otimes 00 \otimes 11 \otimes 00 \otimes 11.$$

Consider a path

$$p = \cdots \otimes 11 \otimes 01 \otimes 01 \otimes 01 \otimes 00.$$

The dotted part of p is the same as that of $\bar{p}$. Then the 1-signature of p truncated at the fifth position and its reduced signature read as follows:

$$\epsilon = (++, --, -+, -+, -+, ++),$$

$$\bar{\epsilon} = \left(\overset{4}{-} \ \overset{2}{+} \ \overset{1}{+} \ \overset{1}{+} \right).$$

Here the number above each sign shows the component to which it belonged. Consequently, we have

$$\tilde{e}_1 p = \cdots \otimes 11 \otimes 00 \otimes 01 \otimes 01 \otimes 00,$$

$$\tilde{f}_1 p = \cdots \otimes 11 \otimes 01 \otimes 01 \otimes 11 \otimes 00.$$

1.3. *Demazure Crystal*

Let $\{r_i\}_{i \in I}$ be the set of simple reflections, and let W be the Weyl group. For $w \in W$, let $l(w)$ denote the length of w, and let $\prec$ denote the Bruhat order on W. Let $U_q^+(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i's. We consider the irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ ($\lambda \in P_{cl}^+$). Let $V_w(\lambda)$ denote the $U_q^+(\mathfrak{g})$-module generated by the extremal weight space $V(\lambda)_{w\lambda}$. These modules $V_w(\lambda)$ ($w \in W$) are called the Demazure modules. Let $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ be the crystal basis of $V(\lambda)$. In [K2] Kashiwara showed that for each $w \in W$, there exists a subset $\mathscr{B}_w(\lambda)$ of $\mathscr{B}(\lambda)$ such that

$$\frac{V_w(\lambda) \cap \mathscr{L}(\lambda)}{V_w(\lambda) \cap q\mathscr{L}(\lambda)} = \bigoplus_{b \in \mathscr{B}_w(\lambda)} \mathbf{Q}b.$$

Furthermore, $\mathscr{B}_w(\lambda)$ has the following recursive property:

$$\text{If } r_i w \succ w, \text{ then} \qquad \mathscr{B}_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n \mathscr{B}_w(\lambda) \setminus \{0\}. \tag{1.5}$$

We call $\mathscr{B}_w(\lambda)$ the *Demazure crystal* associated with the Demazure module $V_w(\lambda)$.

1.4. *Criterion*

In this subsection, we review the result of [KMOU] when $\kappa = 1$. (For the definition of the *mixing index* κ, see Section 2.3 of [KMOU].)

Let λ be an element of $(P_{cl}^+)_l$, and let B be a classical crystal. To state the result, we need to assume four conditions:

(I) B is perfect of level l. Thus, we can assume an isomorphism between $\mathscr{B}(\lambda)$ and the set of paths $\mathscr{P}(\lambda, B)$. Let $\bar{p} = \cdots \otimes \bar{b}_2 \otimes \bar{b}_1$ denote the ground state path. Fix a positive integer d. For a set of elements $i_a^{(j)}$ ($j \geq 1$, $1 \leq a \leq d$) in I, we define $B_a^{(j)}$ ($j \geq 1$, $0 \leq a \leq d$) by

$$B_0^{(j)} = \left\{\bar{b}_j\right\}, \qquad B_a^{(j)} = \bigcup_{n \geq 0} \tilde{f}_{i_a^{(j)}}^n B_{a-1}^{(j)} \setminus \{0\} \qquad (a = 1, \dots, d).$$

(II) For any $j \geq 1$, $B_d^{(j)} = B$.

(III) For any $j \geq 1$ and $1 \leq a \leq d$, $\langle \lambda_j, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$.

We now define an element $w^{(k)}$ of the Weyl group W by

$$w^{(0)} = 1, \qquad w^{(k)} = r_{i_a^{(j)}} w^{(k-1)} \qquad \text{for } k > 0.$$

Here, if $k = 0$, we set $j = 1$, $a = 0$, and otherwise, j and a are fixed from k by $k = (j-1)d + a$, $j \geq 1$, $1 \leq a \leq d$.

(IV) $w^{(0)} \prec w^{(1)} \prec \cdots \prec w^{(k)} \prec \cdots$.

We shall discuss how to check this condition in the next subsection.

Then the statement of [KMOU] reads as follows.

THEOREM 1.1 ([KMOU]). *Under the assumptions* (I–IV), *we have*

$$\mathscr{B}_{w^{(k)}}(\lambda) \simeq u_{\lambda_j} \otimes B_a^{(j)} \otimes B^{\otimes(j-1)}.$$

1.5. *Extremal Vectors*

We present a proposition that is convenient for determining the extremal vectors. For an element b in B or $\mathscr{B}(\lambda)$, let $\tilde{f}_i^{\max} b$ stand for $\tilde{f}_i^{\varphi_i(b)} b$. Assume B is perfect of level l. Therefore, for any $\lambda \in (P_{cl}^+)_l$ we have the isomorphism

$$\mathscr{B}(\lambda) \simeq \mathscr{P}(\lambda, B). \tag{1.6}$$

For $w \in W$, let us denote by $u_{w\lambda}$ the vector in the lower global crystal basis of weight $w\lambda$. Then we have

$$\begin{aligned} u_{w\lambda} &= u_\lambda \qquad \text{if } w = 1, \\ u_{r_i w\lambda} &= f_i^{(m)} u_{w\lambda} \qquad \text{if } m = \langle w\lambda, h_i \rangle \geq 0. \end{aligned} \tag{1.7}$$

(see Section 3.2 of [K2] for this, along with the definition of $f_i^{(m)}$). Note that we can regard $u_{w\lambda}$ as an element of $\mathscr{B}(\lambda)$, since the $w\lambda$ weight space is one-dimensional. Again, since $r_i w\lambda$ weight space is one-dimensional, in $\mathscr{B}(\lambda)$, using (1.7), we can and do identify $u_{r_i, w\lambda}$ with $\tilde{f}_i^m u_{w\lambda}$. These $u_{w\lambda}$ are called *extremal vectors*. We want to find out the extremal vector $u_{w^{(k)}\lambda}$ in the right-hand side of (1.6). For any $j \geq 1$, we set

$$b_0^{(j)} = \bar{b}_j, \qquad b_a^{(j)} = \tilde{f}_{i_a^{(j)}}^{\max} b_{a-1}^{(j)} \qquad (a = 1, \dots, d).$$

PROPOSITION 1.1. *Assume that condition* (III) *holds. If* $\varepsilon_{i_{a+1}^{(j)}}(b_a^{(j)}) = 0$, $\varphi_{i_{a+1}^{(j)}}(b_a^{(j)}) > 0$ $(a = 1, \dots, d)$, $b_0^{(j+1)} = \tilde{f}_{i_1^{(j+1)}}^{m^{(j+1)}} b_d^{(j)}$, *with* $m^{(j+1)} = \langle \lambda_{j+1}, h_{i_1^{(j+1)}} \rangle$ *for any* $j \geq 1$, *then we have*

$$u_{w^{(k)}\lambda} = \cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1} \otimes \left(b_a^{(j)}\right)^{\otimes j}.$$

Here, if $k = 0$, *we set* $j = 1$, $a = 0$, *and otherwise,* j *and* a *are fixed from* k *by* $k = (j-1)d + a$, $j \geq 1$, $1 \leq a \leq d$. *Furthermore,* $i_{d+1}^{(j)}$ *should be understood as* $i_1^{(j+1)}$.

Proof. We prove by induction on k. If $k = 0$, the statement should be understood as

$$u_{w^{(0)}\lambda} = \cdots \otimes \bar{b}_3 \otimes \bar{b}_2 \otimes \bar{b}_1.$$

Thus it is trivial for $w^{(0)} = 1$.

Now assume $k > 0$. First, we assume $a \neq 1$. From the induction hypothesis, we have

$$\begin{aligned} u_{w^{(k-1)}\lambda} &= \cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1} \otimes \left(b^{(j)}_{a-1}\right)^{\otimes j} \\ &\simeq u_{\lambda_j} \otimes \left(b^{(j)}_{a-1}\right)^{\otimes j}. \end{aligned}$$

From the assumption of the proposition and condition (III), we have

$$\varepsilon_{i^{(j)}_a}\left(b^{(j)}_{a-1}\right) = 0, \qquad \varphi_{i^{(j)}_a}\left(b^{(j)}_{a-1}\right) > 0, \qquad \langle \lambda_j, h_{i^{(j)}_a}\rangle = 0.$$

Setting $\nu = \varphi_{i^{(j)}_a}(b^{(j)}_{a-1})$, the $i^{(j)}_a$-signature of $u_{w^{(k-1)}\lambda}$ reads as

$$\Big(\underbrace{+^{\nu}, \dots, +^{\nu}}_{j}\Big).$$

Since $\langle w^{(k-1)}\lambda, h_{i^{(j)}_a}\rangle = j(\varphi_{i^{(j)}_a}(b^{(j)}_{a-1}) - \varepsilon_{i^{(j)}_a}(b^{(j)}_{a-1})) = j\nu$, it is now clear that

$$\begin{aligned} u_{w^{(k)}\lambda} &= u_{r_{i^{(j)}_a} w^{(k-1)}\lambda} = \tilde{f}^{j\nu}_{i^{(j)}_a} u_{w^{(k-1)}\lambda} \\ &= \cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1} \otimes \left(\tilde{f}^{\nu}_{i^{(j)}_a} b^{(j)}_{a-1}\right)^{\otimes j} \\ &= \cdots \otimes \bar{b}_{j+2} \otimes \bar{b}_{j+1} \otimes \left(b^{(j)}_a\right)^{\otimes j}. \end{aligned}$$

Taking the assumption $b^{(j+1)}_0 = \tilde{f}^{m^{(j+1)}}_{i^{(j+1)}_1} b^{(j)}_d$ into account, the proof in the case of $a = 1$ can be done similarly. This completes the proof. ∎

This proposition can be used to check condition (IV). In fact, assuming the above proposition, we can show (IV). Let us consider the situation where condition (IV) is satisfied up to $w^{(k)}$. Then we easily have $\varepsilon_{i^{(j)}_{a+1}}(u_{w^{(k)}\lambda}) = 0$, $\langle w^{(k)}\lambda, h_{i^{(j)}_{a+1}}\rangle = \varphi_{i^{(j)}_{a+1}}(u_{w^{(k)}\lambda}) > 0$. Noting Proposition 2 in [KMOU], we get $w^{(k+1)} = r_{i^{(j)}_{a+1}} w^{(k)} \succ w^{(k)}$, which proves (IV) for $k + 1$. In the cases we shall deal with in this paper, we choose suitable known perfect crystals; hence condition (I) already holds. Thus, we are going to check (II), (III), and condition (IV′):

(IV′) For any $j \geq 1$, $\varepsilon_{i^{(j)}_{a+1}}(b^{(j)}_a) = 0$, $\varphi_{i^{(j)}_{a+1}}(b^{(j)}_a) > 0$ $(a = 1, \dots, d)$ and $b^{(j+1)}_0 = \tilde{f}^{m^{(j+1)}}_{i^{(j+1)}_1} b^{(j)}_d$, with $m^{(j+1)} = \langle \lambda_{j+1}, h_{i^{(j+1)}_1}\rangle$ ($i^{(j)}_{d+1}$ should be understood as $i^{(j+1)}_1$), which implies condition (IV).

2. $A_n^{(1)}$ CASE

In this section we give explicit descriptions of the $A_n^{(1)}$ Demazure crystals $\mathscr{B}_w(l\Lambda_0)$, $l \geq 1$ for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \geq 0\}$. (Note that because of the $\mathbf{Z}_{n+1}$ symmetry, we can obtain $\mathscr{B}_w(l\Lambda_i)$ for all i.) Our starting point is the perfect crystal B of level l, which is isomorphic to $B(l\overline{\Lambda}_k)$ as crystals for $U_q(A_n)$ [KMN2]. As can be seen, the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $A_n^{(1)}$ $(n \geq 2)$ is shown in Fig. 2 . The labels are the levels of the fundamental weights corresponding to the vertices.

2.1. *Description of the Perfect Crystal*

Let $I = \mathbf{Z}/(n+1)\mathbf{Z}$ be the index set of the simple roots for $U_q(A_n^{(1)})$, and let $J = \{1, \dots, n\}$ be that for $U_q(A_n)$. For $i \in I$ we define $\iota^{(i)}: J \to I$ by $\iota^{(i)}(j) = i + j \bmod n+1$, $\bar{\iota}^{(i)}: J \to I$ by $\bar{\iota}^{(i)}(j) = i - j \bmod n+1$. Let $B(l\overline{\Lambda}_k)$ $(1 \leq k \leq n)$ be the crystal of type A_n as described in [KN]. Then for any integers k, l such that $1 \leq k \leq n$, $l \geq 1$, there exists a unique crystal $B = B^{k,l}$ of type $A_n^{(1)}$ such that $\iota^{(i)*}(B^{k,l}) = B(l\overline{\Lambda}_k)$ and $\bar{\iota}^{(i)*}(B^{k,l}) = B(l\overline{\Lambda}_{k'})$ for all i, where $k' = n + 1 - k$. This $B^{k,l}$ is perfect of level l.

We review the crystal $B(l\overline{\Lambda}_k)$ here. Set $K = \{1, 2, \dots, n, n+1\}$. With each $b \in B(l\overline{\Lambda}_k)$, we associate a table $(m_{i,i'})_{\{1 \leq i \leq k,\ 1 \leq i' \leq l\}} = m(b)$, where $m_{i,i'} \in K$, $m_{i,i'} \leq m_{i,i'+1}$ and $m_{i,i'} < m_{i+1,i'}$. To see the actions of $\tilde{e}_j$ and $\tilde{f}_j$ $(j \neq 0)$, we begin with the case of $l = 1$; each element in $B(\overline{\Lambda}_k)$ corresponds to a column $(m_i)_{1 \leq i \leq k}$ such that $m_i \in K$, $m_i < m_{i+1}$. $\tilde{e}_j(m_i)$ (resp., $\tilde{f}_j(m_i)$) is obtained by changing $j+1$ in the column to j (resp., j to $j+1$). If there is no $j+1$ (resp., j) in the column, or the result breaks the requirements $m_i < m_{i+1}$ $(1 \leq i < k)$, the action should be set to 0. To deal with the general case, we define the following injection:

$$\begin{aligned} B\left(l\overline{\Lambda}_k\right) &\hookrightarrow B\left(\overline{\Lambda}_k\right) \otimes B\left(\overline{\Lambda}_k\right) \otimes \cdots \otimes B\left(\overline{\Lambda}_k\right) \\ \left(m_{i,i'}\right) &\mapsto \left(m_i^{(l)}\right) \otimes \left(m_i^{(l-1)}\right) \otimes \cdots \otimes \left(m_i^{(1)}\right), \end{aligned} \tag{2.1}$$

where $(m_i^{(a)})$ denotes the ath column in the table $(m_{i,i'})$. Then, the actions on $B(l\overline{\Lambda}_k)$ are defined so that they commute with this injection. The actions on the right-hand side of (2.1) can be calculated using the signature rule in Section 1.2.

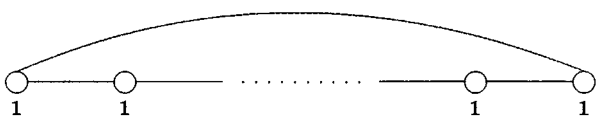

FIG. 2. Dynkin diagram $A_n^{(1)}$ $(n \geq 2)$.

The description of the bijection σ is as follows. Let us use the notation $\lambda = (m_0, m_1, \ldots, m_n)$ for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$. Then,

$$\sigma : (m_0, m_1, \ldots, m_{k'-1}, m_{k'}, \ldots, m_n) \mapsto (m_k, m_{k+1}, \ldots, m_n, m_0, \ldots, m_{k-1}).$$

Next we describe $\bar{p}$. Fixing j, we consider the following sequence of integers:

$$n + 2 - jk, n + 3 - jk, \ldots, n + 1 - (j-1)k.$$

If there is no integer congruent to 0 mod $n+1$ in the above sequence, then $m(\bar{b}_j)$ is given by $m_{i,i'} \equiv i + n + 1 - jk \bmod n+1$. If it is not the case, let α be the position of the integer congruent to $0 \bmod n+1$. Then, $m(\bar{b}_j)$ is given by $m_{i,i'} = i$ $(1 \le i \le k - \alpha)$, $= i + k'$ $(k - \alpha < i \le k)$. Here and in what follows in this section, we set $k' = n + 1 - k$. In either case, $m_{i,i'}$ does not depend on i'.

2.2. *Description of Demazure Crystals*

We set $d = kk'$. We wish to present $i_a^{(j)}$. For this purpose, it is convenient to associate (g, r) with a. Fixing a $(1 \le a \le d)$, we set

$$g = \left[\frac{a-1}{k'}\right], \qquad r = a - 1 - k'g,$$

where $[\gamma]$ denotes the largest integer not exceeding γ. Note that $0 \le g \le k-1$, $0 \le r \le k'-1$. Then, $i_a^{(j)}$ is given by

$$i_a^{(j)} = k(1-j) - g + r.$$

Now we have

THEOREM 2.1. *With the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III) *and* (IV′).

Thanks to the symmetry under $\iota^{(i)*}$, we can reduce checking the conditions to that for a particular j. We choose it to be $n+1$. Then we have $i_a^{(n+1)} = k - g + r$. Considering the ranges of g and r, we know $i_a^{(n+1)} \not\equiv 0$ for any a $(1 \le a \le d)$. This reduces the problem for B to that for $B(l\overline{\Lambda}_k)$. We need the following data: $m(\bar{b}_{n+1})$ is given by $m_{i,i'} = i$, and $m(\bar{b}_{n+2})$ by $m_{i,i'} = i + k'$.

PROPOSITION 2.1. *For a, take g, r as above. Then,*

(1) $B_a^{(n+1)} = \{m_{i,i'} \mid m_{i,i'} = i \ (i < k - g),\ m_{k-g,i'} \le k - g + r + 1\}$.

(2) $m(b_a^{(n+1)})$ *is given by* $m_{i,i'} = i$ $(i < k-g)$, $= k - g + r + 1$ $(i = k - g)$, $= i + k'$ $(i > k - g)$.

Proof. We prove by induction on a. First, look at (1). If $a = 1$, we have $g = r = 0$, $i_1^{(n+1)} = k$. The assertion is clear from the application rule through (2.1). Now set $\overline{B}_a$ to be the r.h.s. of (1). For any a $(1 < a \le d)$, we are to show

$$\overline{B}_a = \bigcup_{m \ge 0} \tilde{f}^m_{i_a^{(n+1)}} \overline{B}_{a-1} \setminus \{0\},$$

which is equivalent to showing both

$$\overline{B}_a \supset \bigcup_{m \ge 0} \tilde{f}^m_{i_a^{(n+1)}} \overline{B}_{a-1} \setminus \{0\} \qquad \text{and} \qquad \overline{B}_{a-1} \supset \bigcup_{m \ge 0} \tilde{e}^m_{i_a^{(n+1)}} \overline{B}_a \setminus \{0\}.$$

These are also clear from the application rule.

The proof of (2) is similar. ∎

3. $B_n^{(1)}$ CASE

In this section we give explicit descriptions of the $B_n^{(1)}$ Demazure crystals $\mathscr{B}_w(l\Lambda)$, $l \ge 1$, $\Lambda = \Lambda_0, \Lambda_1, \Lambda_n$ (level one dominant weights) for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \ge 0\}$. Our starting point is the perfect crystal B of level l, which is isomorphic to $B(l\overline{\Lambda}_1)$ as crystals for $U_q(B_n)$ [KMN2]. As can be seen, the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $B_n^{(1)}$ $(n \ge 3)$ is shown in Fig. 3. The labels are the levels of the fundamental weights corresponding to the vertices.

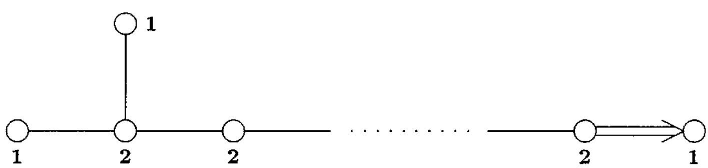

FIG. 3. Dynkin diagram $B_n^{(1)}$ $(n \ge 3)$.

3.1. *Description of the Perfect Crystal*

For any integer $l \geq 1$, we recall the perfect crystal B from [KMN2]. As a set,

$$B = \left\{ (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in \mathbf{Z}^{2n} \times \{0,1\} \,\middle|\, \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \geq 0, \\ x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \end{array} \right\}. \tag{3.1}$$

The action of $\tilde{f}_i$ is defined as follows. For $b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2 + 1, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

for $i = 1, 2, \dots, n-1$, and

$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, x_0 + 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n, x_0 - 1, \bar{x}_n + 1, \dots, \bar{x}_1) & \text{if } x_0 = 1. \end{cases}$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+, \qquad \varphi_n(b) = 2x_n + x_0,$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+,$$

For $i = 1, 2, \dots, n-1$, and

$$\varepsilon_0(b) = x_1 + (x_2 - \bar{x}_2)_+, \qquad \varepsilon_n(b) = 2\bar{x}_n + x_0,$$

where $(x)_+ = \max(x, 0)$. Furthermore, $wt(b) = \sum_{i=0}^{n} (\varphi_i(b) - \varepsilon_i(b))\Lambda_i$.

Furthermore, in this case the automorphism σ is given by

$$\sigma : (m_0, m_1, m_2, \dots, m_n) \to (m_1, m_0, m_2, \dots, m_n)$$

for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$. Choose $b(l\Lambda_i) \in B$ as follows:

$$b(i) = b(l\Lambda_i) = \begin{cases} (0, 0, \dots, l) & i = 0, \\ (l, 0, \dots, 0) & i = 1, \\ (0, \dots, 0, m, x_0, m, 0, \dots, 0) & i = n, \end{cases}$$

where $x_0 = \epsilon(l)$ and $m = \frac{1}{2}(l - x_0)$. Here and in what follows, we use the function

$$\epsilon(i) = \begin{cases} 0 & i: \text{even}, \\ 1 & i: \text{odd}. \end{cases} \tag{3.2}$$

Then note that $\varphi(b(i)) = l\Lambda_i$, $i = 0, 1, n$. So the highest weight vector u_λ with $\lambda = l\Lambda_i$ corresponds to the ground state path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_2 \otimes \bar{b}_1$ where

$$\bar{b}_k = \begin{cases} b(\epsilon(k+i+1)) & i = 0, 1, \\ b(n) & i = n. \end{cases}$$

Thus, we have, for $\lambda = l\Lambda_i$, $i = 0, 1, n$,

$$\lambda_k = \begin{cases} l\Lambda_{\epsilon(k+i)} & i = 0 \text{ or } 1, \\ l\Lambda_n & i = n. \end{cases}$$

3.2. *Description of Demazure crystals*

For $\lambda = l\Lambda_0$, $l \geq 1$, set $d = 2n - 1$ and choose the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 2n-1\}$ defined as follows:

$$i_1^{(j)} = i_{2n-1}^{(j)} = \epsilon(j+1),$$

$$i_a^{(j)} = i_{2n-a}^{(j)} = a \qquad \text{for } 2 \leq a \leq n.$$

THEOREM 3.1. *For $\lambda = l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV$'$). *Furthermore, in this case,*

$$B_0^{(j)} = \begin{cases} \{(0,0,\ldots,l)\} & j \text{ odd}, \\ \{(l,0,\ldots,0)\} & j \text{ even}, \end{cases} \qquad B_{2n-1}^{(j)} = B,$$

and for $1 \leq a \leq n-1$, $B_a^{(j)}$ and $B_{n+a-1}^{(j)} \subseteq B$ are given as follows:

$$B_a^{(j)} = \begin{cases} \{(0, x_2, \ldots, x_{a+1}, 0, \ldots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \ldots, x_{a+1}, 0, \ldots, 0)\} & j \text{ even}, \end{cases}$$

$$B_{n+a-1}^{(j)} = \begin{cases} \{(0, x_2, \ldots, \bar{x}_{n-a+1}, 0, \ldots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \ldots, \bar{x}_{n-a+1}, 0, \ldots, 0)\} & j \text{ even}. \end{cases}$$

Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (3.1). *Furthermore, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)} = \begin{cases} \{(0,0,\ldots,l)\} & j \text{ odd}, \\ \{(l,0,\ldots,0)\} & j \text{ even}, \end{cases} \qquad b_{2n-1}^{(j)} = \begin{cases} \{(l,0,\ldots,0)\} & j \text{ odd}, \\ \{(0,0,\ldots,l)\} & j \text{ even}, \end{cases}$$

and for $1 \le a \le n-1$,

$$b_a^{(j)} = \Big(0, \dots, 0, \underset{a+1}{l}, 0, \dots, 0\Big), \qquad b_{n+a-1}^{(j)} = \Big(0, \dots, 0, \underset{\overline{n-a+1}}{l}, 0, \dots, 0\Big).$$

Proof. Since the j:odd case is similar, we only consider the j:even case, where $\lambda_j = l\Lambda_0$, $\overline{b}_j = (l, 0, \dots, 0)$, $i_a^{(j)} = i_{2n-a}^{(j)} = a$ $(1 \le a \le n)$.

Recalling the definitions in Section 1.4, the determination of the subset $B_a^{(j)}$ for $0 \le a \le n$ is easy. Let us consider $B_{n+1}^{(j)}$. By definition,

$$B_{n+1}^{(j)} = \bigcup_{m \ge 0} \tilde{f}_{n-1}^{m} \{ (x_1, \dots, x_n, x_0, \bar{x}_n, 0, \dots, 0) \} \setminus \{0\}.$$

In view of the rule of $\tilde{f}_{n-1}$, we see

$$\begin{aligned}&(x_1, \dots, x_n, x_0, \bar{x}_n, \bar{x}_{n-1}, 0, \dots, 0)\\ &\qquad = \tilde{f}_{n-1}^{\bar{x}_{n-1}+z_n} (x_1, \dots, x_{n-1} + z_n, x_n - z_n, x_0, \bar{x}_n + \bar{x}_{n-1}, 0, \dots, 0),\end{aligned}$$

where $z_n = (x_n - \bar{x}_n)_+$. This formula proves

$$B_{n+1}^{(j)} = \{ (x_1, \dots, x_n, x_0, \bar{x}_n, \bar{x}_{n-1}, 0, \dots, 0) \}.$$

The other cases are similar. Thus we have $B_{2n-1}^{(j)} = B$, and checked condition (II) with $d = 2n-1$. Since $\langle \lambda_j, h_{i_a^{(j)}} \rangle = \langle l\Lambda_0, h_{i_a^{(j)}} \rangle = 0$ for $1 \le a \le d$, condition (III) is trivial.

The calculation of $b_a^{(j)}$ is simpler. From the rules of ε_i and φ_i, we see $\varepsilon_{i_{a+1}^{(j)}}(b_a^{(j)}) = 0$, $\varphi_{i_{a+1}^{(j)}}(b_a^{(j)}) = l$ $(a \ne n-1)$, $2l$ $(a = n-1)$, $\langle \lambda_{j+1}, h_{i_1^{(j+1)}} \rangle = 0$, and $b_0^{(j+1)} = b_d^{(j)}$. Therefore, we have checked (IV′). ∎

Remark 3.1. In an analogous manner, it can be seen that Theorem 1.1 holds for $\lambda = l\Lambda_i$, $i = 1, n$ if we choose $i_a^{(i)}$ as follows:

$$\begin{aligned} i = 1: \qquad & i_1^{(j)} = i_{2n-1}^{(j)} = \epsilon(j), \\ & i_a^{(j)} = i_{2n-a}^{(j)} = a \quad \text{for } 2 \le a \le n. \\ i = n: \qquad & i_a^{(j)} = n - a + 1 \quad \text{for } 1 \le a \le n-1, \\ & \big(i_n^{(j)}, i_{n+1}^{(j)}\big) = (1, 0) \text{ or } (0, 1), \\ & i_{n+a}^{(j)} = a \quad \text{for } 2 \le a \le n-1. \end{aligned}$$

4. $D_n^{(1)}$ CASE

In this section we give explicit descriptions of the $D_n^{(1)}$ Demazure crystals $\mathscr{B}_w(l\Lambda)$, $l \geq 1$, $\Lambda = \Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n$ (level one dominant weights) for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \geq 0\}$. Our starting point is the perfect crystal B of level l, which is isomorphic to $B(l\overline{\Lambda}_1)$ as crystals for $U_q(D_n)$ [KMN2]. As can be seen, the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $D_n^{(1)}$ $(n \geq 4)$ is shown in Fig. 4 . The labels are the levels of the fundamental weights corresponding to the vertices.

4.1. *Description of the Perfect Crystal*

For any integer $l \geq 1$, we recall the perfect crystal B from [KMN2]. As a set

$$B = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbf{Z}^{2n} \,\middle|\, \begin{array}{l} x_n = 0 \text{ or } \bar{x}_n = 0,\ x_i, \bar{x}_i \geq 0, \\ \sum_{i=1}^n (x_i + \bar{x}_i) = l \end{array} \right\}. \tag{4.1}$$

The action of $\tilde{f}_i$ is defined as follows: For $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2 + 1, \ldots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \ldots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \ldots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

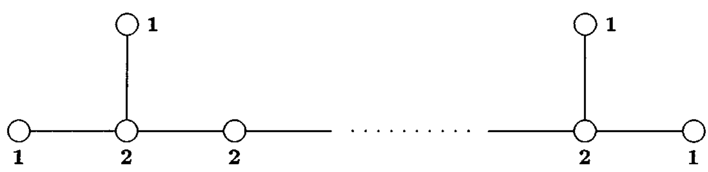

FIG. 4. Dynkin diagram $D_n^{(1)}$ $(n \geq 4)$.

for $i = 1, 2, \dots, n-2$, and

$$\tilde{f}_{n-1} b = \begin{cases} (x_1, \dots, x_{n-1} - 1, x_n + 1, \dots, \bar{x}_1) & \text{if } x_n \geq 0,\ \bar{x}_n = 0, \\ (x_1, \dots, x_n, \bar{x}_n - 1, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) & \text{if } x_n = 0,\ \bar{x}_n \geq 1, \end{cases}$$

$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, \bar{x}_n, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) & \text{if } x_n \geq 1,\ \bar{x}_n = 0, \\ (x_1, \dots, x_{n-1} - 1, x_n, \bar{x}_n + 1, \dots, \bar{x}_1) & \text{if } x_n = 0,\ \bar{x}_n \geq 0. \end{cases}$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+,$$

$$\varphi_{n-1}(b) = x_{n-1} + \bar{x}_n, \qquad \varphi_n(b) = x_{n-1} + x_n,$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+$$

for $i = 1, 2, \dots, n-2$, and

$$\varepsilon_0(b) = x_1 + (x_2 - \bar{x}_2)_+, \qquad \varepsilon_{n-1}(b) = \bar{x}_{n-1} + x_n, \qquad \varepsilon_n(b) = \bar{x}_{n-1} + \bar{x}_n,$$

where $(x)_+ = \max(x, 0)$. Moreover, $wt(b) = \sum_{i=0}^{n}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i$.

Furthermore, the automorphism σ is given by

$$\sigma : (m_0, m_1, m_2, \dots, m_{n-2}, m_{n-1}, m_n) \\ \to (m_1, m_0, m_2, \dots, m_{n-2}, m_n, m_{n-1})$$

for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$. Choose $b(l\Lambda_i) \in B$, $i = 0, 1, n-1, n$, as follows:

$$b(i) = b(l\Lambda_i) = \begin{cases} (0, 0, \dots, l) & i = 0, \\ (l, 0, \dots, 0) & i = 1, \\ \Big(0, \dots, 0, \underset{\bar{n}}{l}, 0, \dots, 0\Big) & i = n-1, \\ \Big(0, \dots, 0, \underset{n}{l}, 0, \dots, 0\Big) & i = n. \end{cases}$$

Note that $\varphi(b(i)) = l\Lambda_i$, $i = 0, 1, n-1, n$. The highest weight vector u_λ with $\lambda = l\Lambda_i$ corresponds to the ground state path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_1$, with

$$\bar{b}_k = \begin{cases} b(\epsilon(k + i + 1)) & i = 0, 1, \\ b(n - \epsilon(k + n - i + 1)) & i = n-1, n. \end{cases}$$

Here $\epsilon(i)$ is defined in (3.2). Thus we have, for $\lambda = l\Lambda_i$, $i = 0, 1, n-1, n$,

$$\lambda_k = \begin{cases} l\Lambda_{\epsilon(k+i)} & i = 0, 1, \\ l\Lambda_{n-\epsilon(k+n-i)} & i = n-1, n. \end{cases}$$

4.2. *Description of Demazure Crystals*

For $\lambda = l\Lambda_0$, $l \geq 1$, set $d = 2n-2$, and choose the sequence $\{i_a^{(j)} | j \geq 1, 1 \leq a \leq 2n-2\}$, defined as follows:

$$\begin{aligned} i_1^{(j)} = i_{2n-2}^{(j)} &= \epsilon(j+1), \\ \left(i_{n-1}^{(j)}, i_n^{(j)}\right) &= (n-1, n), \\ i_a^{(j)} = i_{2n-a-1}^{(j)} &= a \qquad \text{for } 2 \leq a \leq n-2. \end{aligned}$$

THEOREM 4.1. *For $\lambda = l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV′). *Furthermore, in this case,*

$$B_0^{(j)} = \begin{cases} \{(0, 0, \ldots, l)\} & j \text{ odd}, \\ \{(l, 0, \ldots, 0)\} & j \text{ even}, \end{cases} \qquad B_{2n-2}^{(j)} = B,$$

and for $1 \leq a \leq 2n-3$, $B_a^{(j)} \subseteq B$ are given as follows:

$$B_a^{(j)} = \begin{cases} \{(0, x_2, \ldots, x_{a+1}, 0, \ldots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \ldots, x_{a+1}, 0, \ldots, 0)\} & j \text{ even}, \end{cases}$$

$$B_{n-1}^{(j)} = \begin{cases} \{(0, x_2, \ldots, x_n, 0, \ldots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \ldots, x_n, 0, \ldots, 0)\} & j \text{ even}, \end{cases}$$

$$B_{n+a-1}^{(j)} = \begin{cases} \{(0, x_2, \ldots, \bar{x}_{n-a}, 0, \ldots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \ldots, \bar{x}_{n-a}, 0, \ldots, 0)\} & j \text{ even}, \end{cases}$$

where $1 \leq a \leq n-2$. Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (4.1). *Moreover, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)} = \begin{cases} \{(0, 0, \ldots, l)\} & j \text{ odd}, \\ \{(l, 0, \ldots, 0)\} & j \text{ even}, \end{cases} \qquad b_{2n-2}^{(j)} = \begin{cases} \{(l, 0, \ldots, 0)\} & j \text{ odd}, \\ \{(0, 0, \ldots, l)\} & j \text{ even}, \end{cases}$$

$$b_{n-1}^{(j)} = \Big(0, \ldots, 0, \underset{n}{l}, 0, \ldots, 0\Big),$$

and for $1 \le a \le n-2$,

$$b_a^{(j)} = \Bigl(0,\dots,0,\underset{a+1}{l},0,\dots,0\Bigr), \qquad b_{n+a-1}^{(j)} = \Bigl(0,\dots,0,\underset{\overline{n-a}}{l},0,\dots,0\Bigr).$$

Proof. The proof is similar to that of Theorem 3.1. ∎

Remark 4.1. We note that for $\lambda = l\Lambda_0$, choosing $(i_{n-1}^{(j)}, i_n^{(j)}) = (n, n-1)$ also works. Furthermore, for $\lambda = l\Lambda_i$, $i = 1, n-1, n$, it can be seen that Theorem 1.1 holds for the following choices of the sequence $\{i_a^{(j)}\}$:

$$i=1: \qquad \begin{cases} i_1^{(j)} = i_{2n-2}^{(j)} = \epsilon(j), \\ \bigl(i_{n-1}^{(j)}, i_n^{(j)}\bigr) = (n-1, n) \text{ or } (n, n-1), \\ i_a^{(j)} = i_{2n-a-1}^{(j)} = a \qquad \text{for } 2 \le a \le n-2, \end{cases}$$

$$i=n-1: \qquad \begin{cases} i_1^{(j)} = i_{2n-2}^{(j)} = n-\epsilon(j), \\ \bigl(i_{n-1}^{(j)}, i_n^{(j)}\bigr) = (1,0) \text{ or } (0,1), \\ i_a^{(j)} = i_{2n-a-1}^{(j)} = n-a \quad \text{for } 2 \le a \le n-2, \end{cases}$$

$$i=n: \qquad \begin{cases} i_1^{(j)} = i_{2n-2}^{(j)} = n-\epsilon(j+1), \\ \bigl(i_{n-1}^{(j)}, i_n^{(j)}\bigr) = (1,0) \text{ or } (0,1), \\ i_a^{(j)} = i_{2n-a-1}^{(j)} = n-a \quad \text{for } 2 \le a \le n-2. \end{cases}$$

5. $A_{2n-1}^{(2)}$ CASE

In this section we give explicit descriptions of the $A_{2n-1}^{(2)}$ Demazure crystals $\mathscr{B}_w(l\Lambda)$, $l \ge 1$, $\Lambda = \Lambda_0, \Lambda_1$ (level one dominant weights) for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \ge 0\}$. Our starting point is the perfect crystal B of level l, which is isomorphic to $B(l\overline{\Lambda}_1)$ as crystals for $U_q(C_n)$ [KMN2]. As it can be seen the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $A_{2n-1}^{(2)}$ $(n \ge 3)$ is shown in Fig. 5. The labels are the levels of the fundamental weights corresponding to the vertices.

5.1. *Description of the Perfect Crystal*

For any integer $l \ge 1$, we recall the perfect crystal B from [KMN2]. As a set,

$$B = \left\{ (x_1,\dots,x_n,\bar{x}_n,\dots,\bar{x}_1) \in \mathbf{Z}^{2n} \;\middle|\; x_i, \bar{x}_i \ge 0,\ \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}. \quad (5.1)$$

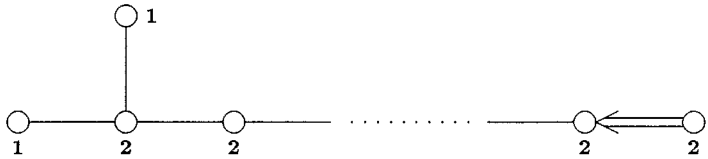

FIG. 5. Dynkin diagram $A^{(2)}_{2n-1}$ ($n \geq 3$).

The action of $\tilde{f}_i$ on B is defined as follows. For $b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2 + 1, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

for $i = 1, 2, \dots, n-1$, and

$$\tilde{f}_n b = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1).$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+, \qquad \varphi_n(b) = x_n,$$
$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+,$$

for $i = 1, 2, \dots, n-1$, and

$$\varepsilon_0(b) = x_1 + (x_2 - \bar{x}_2)_+, \qquad \varepsilon_n(b) = \bar{x}_n,$$

where $(x)_+ = \max(x, 0)$. Moreover, $wt(b) = \sum_{i=0}^n (\varphi_i(b) - \varepsilon_i(b))\Lambda_i$.

In this case the automorphism σ is given by

$$\sigma : (m_0, m_1, m_2, \dots, m_n) \to (m_1, m_0, m_2, \dots, m_n)$$

for $\lambda = \sum_{i=0}^n m_i \Lambda_i$. Choose $b(i) = b(l\Lambda_i) \in B$, $i = 0, 1$, as follows:

$$b(i) = \begin{cases} (0, 0, \dots, l) & i = 0, \\ (l, 0, \dots, 0) & i = 1. \end{cases}$$

Note that $\varphi(b(i)) = l\Lambda_i$, $i = 0, 1$. The highest weight vector u_λ with $\lambda = l\Lambda_i$ corresponds to the ground state path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_1$ with

$$\bar{b}_k = b(\epsilon(k + i + 1)).$$

Here $\epsilon(i)$ is defined in (3.2). We also have, for $\lambda = l\Lambda_i$, $i = 0, 1$,

$$\lambda_k = l\Lambda_{\epsilon(k+i)}.$$

5.2. *Description of Demazure Crystals*

For $\lambda = l\Lambda_0$, $l \geq 1$, set $d = 2n - 1$ and choose the sequence $\{i_a^{(j)} \mid j \geq 1,\ 1 \leq a \leq 2n-1\}$, defined as follows:

$$i_1^{(j)} = i_{2n-1}^{(j)} = \epsilon(j+1),$$
$$i_a^{(j)} = i_{2n-a}^{(j)} = a \qquad \text{for } 2 \leq a \leq n.$$

THEOREM 5.1. *For $\lambda = l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV$'$). *Furthermore, in this case,*

$$B_0^{(j)} = \begin{cases} \{(0,0,\dots,l)\} & j \text{ odd}, \\ \{(l,0,\dots,0)\} & j \text{ even}, \end{cases} \qquad B_{2n-1}^{(j)} = B,$$

and for $1 \leq a \leq 2n-2$, $B_a^{(j)} \subseteq B$ are given as follows:

$$B_a^{(j)} = \begin{cases} \{(0, x_2, \dots, x_{a+1}, 0, \dots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \dots, x_{a+1}, 0, \dots, 0)\} & j \text{ even}, \end{cases}$$

$$B_{n+a-1}^{(j)} = \begin{cases} \{(0, x_2, \dots, \bar{x}_{n-a+1}, 0, \dots, 0, \bar{x}_1)\} & j \text{ odd}, \\ \{(x_1, x_2, \dots, \bar{x}_{n-a+1}, 0, \dots, 0)\} & j \text{ even}, \end{cases}$$

where $1 \leq a \leq n-1$. Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (5.1). *Furthermore, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)} = \begin{cases} \{(0,0,\dots,l)\} & j \text{ odd}, \\ \{(l,0,\dots,0)\} & j \text{ even}, \end{cases} \qquad b_{2n-1}^{(j)} = \begin{cases} \{(l,0,\dots,0)\} & j \text{ odd}, \\ \{(0,0,\dots,l)\} & j \text{ even}, \end{cases}$$

and for $1 \leq a \leq n-1$,

$$b_a^{(j)} = \Big(0, \dots, 0, \underset{a+1}{l}, 0, \dots, 0\Big),$$

$$b_{n+a-1}^{(j)} = \Big(0, \dots, 0, \underset{\overline{n-a+1}}{l}, 0, \dots, 0\Big).$$

Proof. The proof is similar to that of Theorem 3.1. ∎

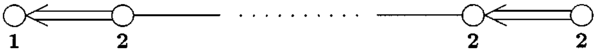

FIG. 6. Dynkin diagram $A_{2n}^{(2)}$ ($n \geq 2$).

Remark 5.1. For $\lambda = l\Lambda_1$ it can be seen, similarly, that Theorem 1.1 holds for the following choices of the sequence $\{i_a^{(j)}\}$:

$$i = 1: \qquad i_1^{(j)} = i_{2n-1}^{(j)} = \epsilon(j),$$
$$i_a^{(j)} = i_{2n-a}^{(j)} = a \quad \text{for } 2 \leq a \leq n.$$

6. $A_{2n}^{(2)}$ CASE

In this section we give explicit descriptions of the $A_{2n}^{(2)}$ Demazure crystals $\mathscr{B}_w(l\Lambda_0)$, $l \geq 1$ (Λ_0 being the level one dominant weight) for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \geq 0\}$. Our starting point is the perfect crystal B of level l, which is isomorphic to $B(0) \oplus B(\overline{\Lambda}_1) \oplus \cdots \oplus B(l\overline{\Lambda}_1)$ as crystals for $U_q(C_n)$ [KMN2]. It is shown that the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $A_{2n}^{(2)}$ ($n \geq 2$) is shown in Fig. 6 . The labels are the levels of the fundamental weights corresponding to the vertices.

6.1. *Description of the Perfect Crystal*

For any integer $l \geq 1$, we recall the perfect crystal B from [KMN2]. As a set,

$$B = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbf{Z}^{2n} \,\middle|\, x_i, \bar{x}_i \geq 0, \sum_{i=1}^{n} (x_i + \bar{x}_i) \leq l \right\}. \tag{6.1}$$

The action of $\tilde{f}_i$ on B is defined as follows. For $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, x_2, \ldots, \bar{x}_1) & \text{if } x_1 \geq \bar{x}_1, \\ (x_1, \ldots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_1 < \bar{x}_1, \end{cases}$$

$$\tilde{f}_i(b) = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \ldots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

for $i = 1, 2, \dots, n-1$, and

$$\tilde{f}_n b = \left(x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1 \right).$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = l - \sum_{i=1}^{n} (x_i + \bar{x}_i) + 2(\bar{x}_1 - x_1)_+, \qquad \varphi_n(b) = x_n,$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+,$$

for $i = 1, 2, \dots, n-1$, and

$$\varepsilon_0(b) = l - \sum_{i=1}^{n} (x_i + \bar{x}_i) + 2(x_1 - \bar{x}_1)_+, \qquad \varepsilon_n(b) = \bar{x}_n.$$

As before, $(x)_+ = \max(x, 0)$ and $wt(b) = \sum_{i=0}^{n}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i$. In this case the automorphism σ is given by

$$\sigma : (m_0, m_1, \dots, m_n) \mapsto (m_0, m_1, \dots, m_n),$$

for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$.

Choose $b(0) = (0, 0, \dots, 0) \in B$. Then $\varphi(b(0)) = l\Lambda_0$. The highest weight vector u_λ with $\lambda = l\Lambda_0$ corresponds to the ground state path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_1$, where $\bar{b}_k = b(0)$ for all k. We also have $\lambda_k = l\Lambda_0$ for all k.

6.2. *Description of Demazure Crystals*

For $\lambda = l\Lambda_0$, $l \geq 1$, set $d = 2n$ and choose the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 2n\}$, defined as follows:

$$i_a^{(j)} = \begin{cases} a - 1, & 1 \leq a \leq n+1, \\ 2n + 1 - a, & n + 2 \leq a \leq 2n. \end{cases}$$

THEOREM 6.1. *For $\lambda = l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV$'$). *Furthermore, in this case,*

$$B_0^{(j)} = \left\{ (0, \dots, 0) \right\}, \qquad B_{2n}^{(j)} = B, \qquad \textit{and}$$

$$B_a^{(j)} = \left\{ (x_1, \dots, x_a, 0, \dots, 0) \right\} \subseteq B, \qquad 1 \leq a \leq n,$$

$$B_{n+a}^{(j)} = \left\{ (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_{n-a+1}, 0, \dots, 0) \right\} \subseteq B, \qquad 1 \leq a \leq n-1.$$

Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (6.1). *Furthermore, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)} = (0,\dots,0),$$

and for $1 \le a \le n$,

$$b_a^{(j)} = \Big(0,\dots,0,\underset{a}{l},0,\dots,0\Big), \qquad b_{n+a}^{(j)} = \Big(0,\dots,0,\underset{n-a+1}{\underline{\;l\;}},0,\dots,0\Big).$$

Proof. For any j, we have $\lambda_j = l\Lambda_0$, $\bar{b}_j = (0,\dots,0)$, $i_a^{(j)} = a-1$, $i_{n+a}^{(j)} = n-a+1$ $(1 \le a \le n)$. From the rule of $\tilde{f}_i$, the determination of the subset $B_a^{(j)}$ for $0 \le a \le n+1$ is easy. With regard to $B_a^{(j)}$ for $n+2 \le a \le 2n$, the determination is just the same as for $B_n^{(1)}$ in Section 3. Therefore, the condition (II) is valid with $d = 2n$. We have $\langle \lambda_j, h_{i_a^{(j)}} \rangle = 0$ $(a \ne 1)$, $= l$ $(a = 1)$. Since $\varepsilon_0((0,\dots,0)) = l$, (III) is also valid.

The calculation of $b_a^{(j)}$ is simpler. From the rules of ε_i and φ_i, we see $\varepsilon_{i_{a+1}^{(j)}}(b_a^{(j)}) = 0, \varphi_{i_{a+1}^{(j)}}(b_a^{(j)}) = l$ $(a \ne d)$, $2l$ $(a = d)$, $\langle \lambda_{j+1}, h_{i_1^{(j+1)}} \rangle = l$, and $b_0^{(j+1)} = \tilde{f}_0^l b_d^{(j)}$. Therefore, we have checked (IV′). ∎

7. $D_{n+1}^{(2)}$ CASE

In this section we give explicit descriptions of the $D_{n+1}^{(2)}$ Demazure crystals $\mathscr{B}_w(l\Lambda)$, $l \ge 1$, $\Lambda = \Lambda_0, \Lambda_n$ (level one dominant weights) for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \ge 0\}$. Our starting point is the perfect crystal B of level l, which is isomorphic to $B(0) \oplus B(\overline{\Lambda}_1) \oplus \cdots \oplus B(l\overline{\Lambda}_1)$ as crystals for $U_q(B_n)$ [KMN2]. It is shown that the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $D_{n+1}^{(2)}$ $(n \ge 2)$ is shown in Fig. 7. The labels are the levels of the fundamental weights corresponding to the vertices.

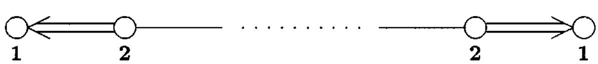

FIG. 7. Dynkin diagram $D_{n+1}^{(2)}$ $(n \ge 2)$.

7.1. *Description of the Perfect Crystal*

For any integer $l \geq 1$, we briefly recall the perfect crystal B from [KMN2]. As a set,

$$B = \left\{ (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in \mathbf{Z}^{2n} \times \{0,1\} \,\middle|\, \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \geq 0, \\ x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) \leq l \end{array} \right\}. \tag{7.1}$$

The action of $\tilde{f}_i$ on B is defined as follows. For $b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, x_2, \dots, \bar{x}_1) & \text{if } x_1 \geq \bar{x}_1, \\ (x_1, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_1 < \bar{x}_1, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

for $i = 1, 2, \dots, n-1$, and

$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, x_0 + 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n, x_0 - 1, \bar{x}_n + 1, \dots, \bar{x}_1) & \text{if } x_0 = 1. \end{cases}$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = l - x_0 - \sum_{i=1}^{n} (x_i + \bar{x}_i) + 2(\bar{x}_1 - x_1)_+, \qquad \varphi_n(b) = 2x_n + x_0,$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+,$$

for $i = 1, 2, \dots, n-1$, and

$$\varepsilon_0(b) = l - x_0 - \sum_{i=1}^{n} (x_i + \bar{x}_i) + 2(x_1 - \bar{x}_1)_+, \qquad \varepsilon_n(b) = 2\bar{x}_n + x_0.$$

As before $(x)_+ = \max(x, 0)$ and $wt(b) = \sum_{i=0}^{n} (\varphi_i(b) - \varepsilon_i(b))\Lambda_i$. In this case the automorphism σ is given by

$$\sigma\colon (m_0, m_1, \dots, m_n) \mapsto (m_0, m_1, \dots, m_n),$$

for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$.

Choose $b(0)=(0,\dots,0)\in B$, $b(n)=(0,\dots,0,m,x_0,m,0,\dots,0)\in B$, where $x_0=\epsilon(l)$ and $m=\frac{1}{2}(l-x_0)$. Here $\epsilon(i)$ is defined in (3.2). Then $\varphi(b(i))=l\Lambda_i$, $i=0,n$. The highest weight vector u_λ with $\lambda=l\Lambda_i$, $i=0,n$ corresponds to the ground state path $\bar{p}=\cdots\otimes\bar{b}_k\otimes\cdots\otimes\bar{b}_1$, where $\bar{b}_k=b(i)$, $i=0,n$ for all k. Furthermore, $\lambda_k=l\Lambda_i$, $i=0,n$ for all k.

7.2. *Description of Demazure Crystals*

For $\lambda=l\Lambda_0$, $l\geq 1$, set $d=2n$ and choose the sequence $\{i_a^{(j)}\mid j\geq 1,\ 1\leq a\leq 2n\}$, defined as follows:

$$i_a^{(j)}=\begin{cases}a-1, & 1\leq a\leq n+1,\\ 2n+1-a, & n+2\leq a\leq 2n.\end{cases}$$

THEOREM 7.1. *For $\lambda=l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV$'$). *Furthermore, in this case,*

$$B_0^{(j)}=\{(0,\dots,0)\}\subseteq B,\qquad B_{2n}^{(j)}=B,\qquad \textit{and}$$
$$B_a^{(j)}=\{(x_1,\dots,x_a,0,\dots,0)\}\subseteq B,\qquad 1\leq a\leq n,$$
$$B_{n+a}^{(j)}=\{(x_1,\dots,x_n,x_0,\bar{x}_n,\dots,\bar{x}_{n-a+1},0,\dots,0)\}\subseteq B,\qquad 1\leq a\leq n-1.$$

Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (7.1). *Furthermore, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)}=(0,\dots,0),$$

and for $1\leq a\leq n$,

$$b_a^{(j)}=\Big(0,\dots,0,\underset{a}{l},0,\dots,0\Big),\qquad b_{n+a}^{(j)}=\Big(0,\dots,0,\underset{n-a+1}{\underline{l}},0,\dots,0\Big).$$

Proof. The proof is similar to that of Theorem 6.1. ∎

Remark 7.1. For $\lambda=l\Lambda_n$, it can be seen, similarly, that Theorem 1.1 holds for the sequence $\{i_a^{(j)}\}$, given by

$$i_a^{(j)}=\begin{cases}n-a+1, & 1\leq a\leq n+1,\\ a-n-1, & n+2\leq a\leq 2n.\end{cases}$$

8. $C_n^{(1)}$ CASE

In this section we give explicit descriptions of the $C_n^{(1)}$ Demazure crystals $\mathscr{B}_w(l\Lambda_i)$, $i = 0, n$ for a suitably chosen linearly ordered chain of Weyl group elements $w \in \{w^{(k)} \mid k \geq 0\}$, using the perfect crystal B of level l (see [KKM]), which is isomorphic to $B(0) \oplus B(2\overline{\Lambda}_1) \oplus \cdots \oplus B(2l\overline{\Lambda}_1)$ as crystals for $U_q(C_n)$. It is shown that the mixing index $\kappa = 1$ in these cases. The Dynkin diagram $C_n^{(1)}$ $(n \geq 2)$ is shown in Fig. 8. The labels are the levels of the fundamental weights corresponding to the vertices.

8.1. *Description of the Perfect Crystal*

For $l \geq 1$, we briefly recall the perfect crystal B (see [KKM]). As a set,

$$B = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbf{Z}^{2n} \,\middle|\, \begin{array}{l} x_i, \bar{x}_i \geq 0, \\ 2l \geq \sum_{i=1}^n (x_i + \bar{x}_i) \in 2\mathbf{Z} \end{array} \right\}. \quad (8.1)$$

The action of $\tilde{f}_i$ on B is defined as follows. For $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B$,

$$\tilde{f}_0 b = \begin{cases} (x_1 + 2, x_2, \ldots, \bar{x}_2, \bar{x}_1) & \text{if } x_1 \geq \bar{x}_1, \\ (x_1 + 1, x_2, \ldots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_1 = \bar{x}_1 - 1, \\ (x_1, x_2, \ldots, \bar{x}_2, \bar{x}_1 - 2) & \text{if } x_1 \leq \bar{x}_1 - 2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \ldots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

for $i = 1, 2, \ldots, n-1$, and

$$\tilde{f}_n b = (x_1, \ldots, x_n - 1,\ \bar{x}_n + 1, \ldots, \bar{x}_1).$$

The action of $\tilde{e}_i$ is given by $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$. φ_i and ε_i are given by

$$\varphi_0(b) = l - \frac{1}{2} \sum_{i=1}^n (x_i + \bar{x}_i) + (\bar{x}_1 - x_1)_+, \qquad \varphi_n(b) = x_n,$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \qquad \varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+,$$

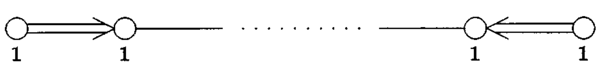

FIG. 8. Dynkin diagram $C_n^{(1)}$ $(n \geq 2)$.

for $i = 1, 2, \ldots, n-1$, and

$$\varepsilon_0(b) = l - \frac{1}{2}\sum_{i=1}^{n}(x_i + \bar{x}_i) + (x_1 - \bar{x}_1)_+, \qquad \varepsilon_n(b) = \bar{x}_n.$$

Recall $(x)_+ = \max(x, 0)$ and $wt(b) = \sum_{i=0}^{n}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i$. In this case the automorphism σ is given by

$$\sigma: (m_0, m_1, \ldots, m_n) \mapsto (m_0, m_1, \ldots, m_n),$$

for $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$.

Choose $b(0) = (0, \ldots, 0) \in B$ and

$$b(n) = \Big(0, \ldots, \underset{n}{l}, \underset{\bar{n}}{l}, \ldots, 0\Big) \in B.$$

Then $\varphi(b(i)) = l\Lambda_i$, $i = 0, n$. The highest weight vector u_λ with $\lambda = l\Lambda_i$, $i = 0, n$ corresponds to the ground state path $\bar{p} = \cdots \otimes \bar{b}_k \otimes \cdots \otimes \bar{b}_1$, where $\bar{b}_k = b(i)$, $i = 0, n$ for all k. Furthermore, for all k, $\lambda_k = l\Lambda_i$, $i = 0, n$.

8.2. *Description of Demazure Crystals*

For $\lambda = l\Lambda_0$, $l \geq 1$, set $d = 2n$ and choose the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 2n\}$, defined as follows:

$$i_a^{(j)} = \begin{cases} a - 1, & 1 \leq a \leq n+1, \\ 2n + 1 - a, & n + 2 \leq a \leq 2n. \end{cases}$$

THEOREM 8.1. *For $\lambda = l\Lambda_0$, with the above choice of $i_a^{(j)}$ and d, B satisfies* (II), (III), *and* (IV′). *Furthermore, in this case,*

$$B_0^{(j)} = \{(0, \ldots, 0)\} \subseteq B, \qquad B_{2n}^{(j)} = B, \qquad \textit{and}$$

$$B_a^{(j)} = \{(x_1, \ldots, x_a, 0, \ldots, 0)\} \subseteq B, \qquad 1 \leq a \leq n,$$

$$B_{n+a}^{(j)} = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_{n-a+1}, 0, \ldots, 0)\} \subseteq B, \qquad 1 \leq a \leq n-1.$$

Here on each set, x_i's and $\bar{x}_i$'s run over nonnegative integers satisfying the conditions in (8.1). *Furthermore, $b_a^{(j)}$ are given as follows*:

$$b_0^{(j)} = (0, \ldots, 0),$$

and for $1 \le a \le n$,

$$b_a^{(j)} = \Big(0, \dots, 0, \underset{a}{2l}, 0, \dots, 0\Big), \qquad b_{n+a}^{(j)} = \Big(0, \dots, 0, \underset{n-a+1}{\underline{2l}}, 0, \dots, 0\Big).$$

Proof. The proof is similar to that of Theorem 6.1. ∎

Remark 8.1. For $\lambda = l\Lambda_n$, it can be seen, similarly, that Theorem 1.1 holds for the sequence $\{i_a^{(j)}\}$ given by

$$i_a^{(j)} = \begin{cases} n - a + 1, & 1 \le a \le n + 1, \\ a - n - 1, & n + 2 \le a \le 2n. \end{cases}$$

9. DISCUSSION

In this section we discuss three topics: an interesting example from the $C_n^{(1)}$ case, $\kappa = 2$ conjecture, and the classical invariance of affine Demazure modules. For this purpose, we briefly review the situation of the mixing index $\kappa = 2$. Let us recall the definition of $B_a^{(j)}$ ($j \ge 1$, $0 \le a \le d$) in Section 1.3. We also define a subset $B_a^{(j+1,j)}$ ($j \ge 1$, $0 \le a \le d$) of $B \otimes B$ by

$$B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_d^{(j)},$$

$$B_a^{(j+1,j)} = \bigcup_{n \ge 0} \tilde{f}^n_{i_a^{(j+1)}} B_{a-1}^{(j+1,j)} \setminus \{0\} \qquad (a = 1, \dots, d).$$

Replace condition (II) by

(II′) For any $j \ge 1$, $B_d^{(j+1,j)} = B_d^{(j+1)} \otimes B$.

Note that $B_d^{(j)} \neq B$ in general. Then the corresponding theorem turns out to be

THEOREM 9.1 ([KMOU]). *Under the assumptions* (I, II′, III, IV), *we have*

$$\mathscr{B}_{w^{(k)}}(\lambda) \simeq \begin{cases} u_{\lambda_1} \otimes B_a^{(1)} & \text{if } j = 1, \\ u_{\lambda_j} \otimes B_a^{(j,j-1)} \otimes B^{\otimes (j-2)} & \text{if } j \ge 2. \end{cases}$$

We move to the first topic. Let us consider the case of $\mathfrak{g} = C_n^{(1)}$. We set $\lambda = l\Lambda_i$ ($i \neq 0, n$), and try to find a sequence $\{w^{(k)}\}$ for the perfect crystal B in Section 8 satisfying $B_d^{(j)} = B$, i.e., $\kappa = 1$. Take $n = 2$, $l = 1$, $i = 1$. Even in this simplest case, it is impossible to find such a sequence, but we can easily find a sequence satisfying $B_d^{(j+1,j)} = B_d^{(j+1)} \otimes B$, i.e., $\kappa = 2$.

Checking a few more examples leads to a conjecture that this should be true for any n, l, and i ($\neq 0, n$). Now let us choose another perfect crystal B', which is isomorphic to $B(l\overline{\Lambda}_n)$ as $U_q(C_n)$ crystal (see [KMN2]). Then it appears in this case that for $\lambda = l\Lambda_i$ ($0 \leq i \leq n$), we can find a sequence with $\kappa = 1$. This shows that the mixing index may depend on the choice of the perfect crystals.

We explain what we call $\kappa = 2$ conjecture. This originates from a natural question of how the mixing index changes when we change the highest weight λ from $l\Lambda_0$, keeping the sequence $\{w^{(k)}\}$ unchanged. Several simple examples and computations at the character level lead us to the following conjecture.

Conjecture 9.1. Let B be a perfect crystal. If the sequence of Weyl group elements $\{w^{(k)}\}$ satisfies the conditions (II), (III), and (IV) for $\lambda = l\Lambda_0$, then the conditions (II$'$) and (III) are also satisfied for any level l dominant integral weight λ.

This conjecture has been proved for the case of $\mathfrak{g} = A_n^{(1)}$, $B = B(l\overline{\Lambda}_1)$ (as $U_q(A_n)$ crystal) in [KMOU]. It is easy to prove for the cases treated in the remarks of each section. (Note that in the remarks $\{w^{(k)}\}$ is changed according to the change in the highest weights, whereas in the conjecture $\{w^{(k)}\}$ is the same as in the case of $l\Lambda_0$.)

In [KMOTU1], we have shown the classical invariance of the Demazure module $V_{w^{(Ld)}}(l\Lambda_0)$ for the case of $\mathfrak{g} = A_n^{(1)}$. In a similar manner, we can also show the classical invariance for the cases treated in this paper. Let V be the finite-dimensional $U_q'(\mathfrak{g})$-module corresponding to the perfect crystal B. Then we have the following isomorphism:

$$V_{w^{(Ld)}}(l\Lambda_0) \simeq V^{\otimes L} \qquad \text{as } U_q(\mathfrak{g}_{I\setminus\{i_L\}})\text{-modules.}$$

Here $\mathfrak{g}_{I\setminus\{i\}}$ is the finite-dimensional simple Lie algebra corresponding to the Dynkin diagram obtained by removing the vertex i, and i_L is determined from $\sigma^L(\Lambda_0) = \Lambda_{i_L}$. Note that $U_q(\mathfrak{g}_{I\setminus\{i_L\}})$ can be viewed as a subalgebra of $U_q(\mathfrak{g})$ in the canonical way.

We have seen in this paper that a perfect crystal B is intimately related to a sequence of Weyl group elements $\{w^{(k)}\}$, and the Demazure crystal $\mathscr{B}_{w^{(k)}}(l\Lambda_0)$ is well described in terms of paths. Denote by $B^{m,l}$, if it exists, the level l perfect crystal, which is isomorphic to $B(l\gamma_m\overline{\Lambda}_m) \oplus \cdots$ as $U_q(\mathfrak{g}_{I\setminus\{0\}})$ crystal. (γ_m is some positive integer. In all known cases, $\gamma_m = \max(1, a_m/a_m^\vee)$, where a_i (resp., $a_i^\vee$) is defined by $\delta = \sum_{i\in I} a_i\alpha_i$ (resp., $c = \sum_{i\in I} a_i^\vee \alpha_i^\vee$).) Then it appears that the sequence $\{w^{(k)}\}$ does not depend on l. However, as seen in the $A_n^{(1)}$ case, it is considered that the sequence does depend on m. This motivates the search for new perfect crystals. We also think it important to investigate Demazure characters and clarify their relation to Kostka-type polynomials and q-analogues of products of the classical characters. In [KMOTU1], we have picked the $A_n^{(1)}$ case, and discussed the relation between the Demazure character and the product of Schur functions or the Kostka–Foulkes polynomial. We hope the classical invariance discussed here sheds new light on the investigation of these polynomials. We give a unified treatment of level one Demazure characters in terms of one-dimensional configuration sums in another publication [KMOTU2].

ACKNOWLEDGMENTS

A. K. thanks T. Nakanishi and A. Varchenko for their warm hospitality at the University of North Carolina. K. C. M. and M. O. thank O. Foda for his hospitality at the University of Melbourne, and for collaboration in their earlier work [FMO]. K. C. M. is partially supported by NSA/MSP grant 96-1-0013.

REFERENCES

[FMO] O. Foda, K. C. Misra, and M. Okado, Demazure modules and vertex models: the $sl(2)$ case, *J. Math. Phys.* **39**, No. 3 (1998), 1601–1622.

[K1] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, *Duke Math. J.* **63** (1991), 465–516.

[K2] M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula, *Duke Math. J.* **71** (1993), 839–858.

[K3] M. Kashiwara, Global crystal bases of quantum groups, *Duke Math. J.* **69** (1993), 455–485.

[KKM] S-J. Kang, M. Kashiwara, K. C. Misra, Crystal bases of Verma modules for quantum affine Lie algebras, *Compositio Math.* **92** (1994), 299–325.

[KMN1] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, Affine crystals and vertex models, *Internat. J. Modern Phys. A* **7** (Suppl. 1A) (1992), 449–484.

[KMN2] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, *Duke Math. J.* **68** (1992), 499–607.

[KMOU] A. Kuniba, K. C. Misra, M. Okado, and J. Uchiyama, Demazure modules and perfect crystals, *Comm. Math. Phys.* **192** (1998), 555–567.

[KMOTU1] A. Kuniba, K. C. Misra, M. Okado, T. Takagi, and J. Uchiyama, Paths, Demazure crystals and symmetric functions, *in* "Proceedings of the Nankai-CRM meeting, Extended and Quantum Algebras and Their Applications to Physics, Tianjin, August 19–24," 1996, Springer-Verlag, Berlin/New York.

[KMOTU2] A. Kuniba, K. C. Misra, M. Okado, T. Takagi, and J. Uchiyama, Characters of Demazure modules and solvable lattice models, *Nucl. Phys.* B510[PM] (1998), 555–576.

[KN] M. Kashiwara and T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, *J. Algebra* **165** (1994), 295–345.