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Normalized Leonard pairs and Askey–Wilson relations ☆

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Abstract

Let V denote a vector space with finite positive dimension, and let (A, A^*) denote a Leonard pair on V. As is known, the linear transformations A, A^* satisfy the Askey–Wilson relations

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I,$$
$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I$$

for some scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$. The scalar sequence is unique if the dimension of V is at least 4.

If c, c^*, t, t^* are scalars and t, t^* are not zero, then $(tA + c, t^*A^* + c^*)$ is a Leonard pair on V as well. These affine transformations can be used to bring the Leonard pair or its Askey–Wilson relations into a convenient form. This paper presents convenient normalizations of Leonard pairs by the affine transformations, and exhibits explicit Askey–Wilson relations satisfied by them.

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1. Introduction

Throughout the paper, $\mathbb{K}$ denotes an algebraically closed field. Apart from one remark, we assume the characteristic of $\mathbb{K}$ is not equal to 2.

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Recall that a tridiagonal matrix is a square matrix which has nonzero entries only on the main diagonal, on the superdiagonal and the subdiagonal. A tridiagonal matrix is called irreducible whenever all entries on the superdiagonal and subdiagonal are nonzero.

Definition 1.1. Let V be a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations which satisfy the following two conditions:

(i) There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

Remark 1.2. In this paper we do not use the conventional notation A^* for the conjugate-transpose of A. In a Leonard pair (A, A^*), the linear transformations A and A^* are arbitrary subject to the conditions (i) and (ii) above.

Leonard pairs occur in the theory of orthogonal polynomials, combinatorics, the representation theory of the Lie algebra sl_2 or the quantum group $U_q(sl_2)$. We refer to [8] as a survey on Leonard pairs, and as a source of further references.

We have the following result [9, Theorem 1.5].

Theorem 1.3. *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let (A, A^*) be a Leonard pair on V. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ taken from $\mathbb{K}$ such that*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \tag{1}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \tag{2}$$

The sequence is uniquely determined by the pair (A, A^*) provided the dimension of V is at least 4.

Eqs. (1) and (2) are called the *Askey–Wilson relations*. They first appeared in the work [10] of Zhedanov, where he showed that the Askey–Wilson polynomials give pairs of infinite-dimensional matrices which satisfy the Askey–Wilson relations. We denote this pair of equations by $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$.

It is easy to notice that if (A, A^*) is a Leonard pair, then

$$(tA + c, t^*A^* + c^*), \text{ with } c, c^*, t, t^* \in \mathbb{K} \text{ and } t, t^* \neq 0, \tag{3}$$

is a Leonard pair as well. We say that the two Leonard pairs are related by the *affine transformation* $(A, A^*) \mapsto (tA + c, t^*A^* + c^*)$. Affine transformations act on Askey–Wilson relations as well, as explained in Section 4 below. For example, if $\beta \neq 2$ then the Askey–Wilson relations can be normalized so that $\gamma = 0$ and $\gamma^* = 0$. Affine transformations can be used to normalize Leonard pairs, parameter arrays representing them, or the Askey–Wilson relations conveniently.

This paper presents convenient normalizations of Leonard pairs and their Askey–Wilson relations. We generally assume that the dimension of the underling vector space is at least 4, and use Terwilliger's classification [7] (or [8, Section 35]) of parameter arrays representing Leonard pairs. For parameter arrays of the q-type, we present two normalizations: one that is close to Terwilliger's general expressions in [7], and one where Askey–Wilson coefficients are normalized most attractively. For other parameter arrays, we give one normalization. This work is more of bookkeeping kind than of deep research. Examples of Askey–Wilson relations for normalized Leonard pairs are given in [9,5]. Indirectly, Askey–Wilson relations for Leonard pairs arising from certain distince regular graphs are computed in [1,2].

We note that Terwilliger's classification of parameter arrays by certain families of orthogonal polynomials from the Askey–Wilson scheme can be largely imitated to categorize Leonard pairs and Askey–Wilson relations; see Sections 2 and 8 below. We have the same types of Leonard pairs and of Askey–Wilson relations, except that the quantum q-Krawtchouk and affine q-Krawtchouk types are merged.

The paper is organized as follows. In the next section we discuss the relation between Leonard pairs and parameter arrays. In Section 3 we recall expressions of the Askey–Wilson coefficients in (1) and (2) in terms of parameter arrays. Section 4 deals with possible normalizations of Askey–Wilson relations. Sections 5 and 6 present two normalizations of q-parameter arrays and Askey–Wilson relations for them. Section 7 presents normalizations of other parameter arrays and Askey–Wilson relations for them. In Section 8 we give a classification of Askey–Wilson relations consistent with the classification of Leonard pairs. In Section 9 we discuss uniqueness of normalizations of Leonard pairs and Askey–Wilson relations.

2. Leonard pairs and parameter arrays

Leonard pairs are represented and classified by parameter arrays. More precisely, parameter arrays are in one-to-one correspondence with *Leonard systems* [8, Definition 3.2], and to each Leonard pair one associates 4 Leonard systems or parameter arrays.

From now on, let d be a non-negative integer, and let V be a vector space with dimension $d+1$ over $\mathbb{K}$.

Definition 2.1 [3]. Let (A, A^*) denote a Leonard pair on V. Let W denote a vector space over $\mathbb{K}$ with finite positive dimension, and let (B, B^*) denote a Leonard pair on W. By an *isomorphism of Leonard pairs* we mean an isomorphism of vector spaces $\sigma : V \mapsto W$ such that $\sigma A \sigma^{-1} = B$ and $\sigma A^* \sigma^{-1} = B^*$. We say that (A, A^*) and (B, B^*) are *isomorphic* if there is an isomorphism of Leonard pairs from (A, A^*) to (B, B^*).

Definition 2.2 [7]. By a *parameter array* over $\mathbb{K}$, of diameter d, we mean a sequence

$$(\theta_0, \theta_1, \ldots, \theta_d; \theta_0^*, \theta_1^*, \ldots, \theta_d^*; \varphi_1, \ldots, \varphi_d; \phi_1, \ldots, \phi_d) \tag{4}$$

of scalars taken from $\mathbb{K}$, that satisfy the following conditions:

PA1. $\theta_i \neq \theta_j$ and $\theta_i^* \neq \theta_j^*$ if $i \neq j$, for $0 \leqslant i, j \leqslant d$.
PA2. $\varphi_i \neq 0$ and $\phi_i \neq 0$, for $1 \leqslant i \leqslant d$.
PA3. $\varphi_i = \phi_1 \sum_{j=0}^{i-1} \frac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$, for $1 \leqslant i \leqslant d$.
PA4. $\phi_i = \varphi_1 \sum_{j=0}^{i-1} \frac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$, for $1 \leqslant i \leqslant d$.
PA5. The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i, for $2 \leqslant i \leqslant d-1$.

To get a Leonard pair from parameter array (4), one must choose a basis for V and define the two linear transformations by the following matrices (with respect to that basis):

$$\begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \quad \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{5}$$

Alternatively, the following two matrices define an isomorphic Leonard pair on V:

$$\begin{pmatrix} \theta_d & & & & \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_0 \end{pmatrix}, \quad \begin{pmatrix} \theta_0^* & \phi_1 & & & \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{6}$$

Conversely, if (A, A^*) is a Leonard pair on V, there exists [8, Section 21] a basis for V with respect to which the matrices for A, A^* have the bidiagonal forms in (5), respectively. There exists another basis for V with respect to which the matrices for A, A^* have the bidiagonal forms in (6), respectively, with the same scalars $\theta_0, \theta_1, \dots, \theta_d; \theta_0^*, \theta_1^*, \dots, \theta_d^*$. Then the following 4 sequences are parameter arrays of diameter d:

$$(\theta_0, \theta_1, \dots, \theta_d; \theta_0^*, \theta_1^*, \dots, \theta_d^*; \varphi_1, \dots, \varphi_d; \phi_1, \dots, \phi_d), \tag{7}$$
$$(\theta_0, \theta_1, \dots, \theta_d; \theta_d^*, \dots, \theta_1^*, \theta_0^*; \phi_d, \dots, \phi_1; \varphi_d, \dots, \varphi_1), \tag{8}$$
$$(\theta_d, \dots, \theta_1, \theta_0; \theta_0^*, \theta_1^*, \dots, \theta_d^*; \phi_1, \dots, \phi_d; \varphi_1, \dots, \varphi_d), \tag{9}$$
$$(\theta_d, \dots, \theta_1, \theta_0; \theta_d^*, \dots, \theta_1^*, \theta_0^*; \varphi_d, \dots, \varphi_1; \phi_d, \dots, \phi_1). \tag{10}$$

If we apply to any of these parameter arrays the construction above, we get back a Leonard pair isomorphic to (A, A^*). These are all parameter arrays which correspond to (A, A^*) in this way.

The parameter arrays in (7)–(10) are related by permutations. The permutation group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group action is without fixed points, since the eigenvalues θ_i's (or θ_i^*'s) are distinct. Let $\downarrow$ and $\Downarrow$ denote the permutations which transform (7) into, respectively, (8) and (9). Observe that the composition $\downarrow\Downarrow$ transforms (7) into (10). We refer to the permutations $\downarrow$, $\Downarrow$ and $\downarrow\Downarrow$ as *relation operators*, because in [8, Section 4] the parameter arrays in (7)–(10) corresponding to (A, A^*) and the 4 similar parameter arrays corresponding to the Leonard pair (A^*, A) are called *relatives* of each other.

Parameter arrays are classified by Terwilliger in [7]; alternatively, see [8, Section 35]. For each parameter array, certain orthogonal polynomials naturally occur in entries of the transformation matrix between two bases characterized in Definition 1.1 for the corresponding Leonard pair. Terwilliger's classification largely mimics the terminating branch of orthogonal polynomials in the Askey–Wilson scheme [4]. Specifically, the classification comprises Racah, Hahn, Krawtchouk polynomials and their q-versions, plus Bannai–Ito and orphan polynomials. Classes of parameter arrays can be identified by the type of corresponding orthogonal polynomials; we refer to them as *Askey–Wilson types*. The type of a parameter array is unambiguously defined if $d \geqslant 3$. We recapitulate Terwilliger's classification in Sections 5 through 7 by giving general normalized parameter arrays of each type.

By inspecting Terwilliger's general parameter arrays [8, Section 35], one can observe that the relation operators $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$ do not change the Askey–Wilson type of a parameter array (but only the free parameters such as q, h, h^*, s there), except that the $\Downarrow$ and $\downarrow\Downarrow$ relations mix up the quantum q-Krawtchouk and affine q-Krawtchouk types. Consequently, given a Leonard pair, all 4 associated parameter arrays have the same type, except when parameter arrays of the quantum q-Krawtchouk or affine q-Krawtchouk type occur. Therefore we can use the same classifying terminology for Leonard pairs, except that we have to merge the quantum q-Krawtchouk and affine q-Krawtchouk types.

3. Parameter arrays and AW relations

Let us consider a parameter array as in (7). Suppose that the corresponding Leonard pair satisfies Askey–Wilson relations $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$. Note that the Askey–Wilson relations are invariant under isomorphism of Leonard pairs. Expressions for the 8 Askey–Wilson coefficients in terms of parameter arrays are presented in [9, Theorems 4.5 and 5.3]. Here are the formulas:

$$\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}, \tag{11}$$

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \tag{12}$$

$$\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}, \tag{13}$$

$$\varrho = \theta_i^2 - \beta\theta_i\theta_{i-1} + \theta_{i-1}^2 - \gamma(\theta_i + \theta_{i-1}), \tag{14}$$

$$\varrho^* = \theta_i^{*2} - \beta\theta^*_i\theta^*_{i-1} + \theta_{i-1}^{*2} - \gamma^*(\theta^*_i + \theta^*_{i-1}), \tag{15}$$

$$\omega = a_i(\theta^*_i - \theta^*_{i+1}) + a_{i-1}(\theta^*_{i-1} - \theta^*_{i-2}) - \gamma(\theta^*_i + \theta^*_{i-1}) \tag{16}$$

$$\quad = a^*_i(\theta_i - \theta_{i+1}) + a^*_{i-1}(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1}), \tag{17}$$

$$\eta = a^*_i(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i, \tag{18}$$

$$\eta^* = a_i(\theta^*_i - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1}) - \gamma\theta_i^{*2} - \omega\theta^*_i. \tag{19}$$

The expressions for $\beta + 1$ and ω are valid for $2 \leqslant i \leqslant d - 1$, the expressions for ϱ, ϱ^* are valid for $1 \leqslant i \leqslant d$, and the expressions for $\gamma, \gamma^*, \eta, \eta^*$ are valid for $1 \leqslant i \leqslant d - 1$. The numbers a_i, a^*_i are the diagonal entries in the tridiagonal forms of A, A^* of Definition 1.1; see [8, Section 7]. In terms of parameter arrays, we have [6, Section 10]:

$$a_i = \theta_i + \frac{\varphi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\varphi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \tag{20}$$

$$\quad = \theta_{d-i} + \frac{\phi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\phi_{i+1}}{\theta^*_i - \theta^*_{i+1}}, \tag{21}$$

$$a^*_i = \theta^*_i + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \tag{22}$$

$$\quad = \theta^*_{d-i} + \frac{\phi_{d-i+1}}{\theta_i - \theta_{i-1}} + \frac{\phi_{d-i}}{\theta_i - \theta_{i+1}}. \tag{23}$$

Here for $i \in \{0, d\}$ we should take

$$\varphi_0 = 0, \quad \varphi_{d+1} = 0, \quad \phi_0 = 0, \quad \phi_{d+1} = 0. \tag{24}$$

The numbers $\theta_{-1}, \theta_{d+1}, \theta^*_{-1}, \theta_{d+1}$ can be left undetermined. Surely, the Askey–Wilson coefficients are invariant under the action of $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$ on parameter arrays.

As stated in Theorem 1.3, the coefficient sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ is unique if $d \geqslant 3$. If $d = 2$, we can take β freely and other coefficients get determined uniquely. If $d = 1$, we can take the 3 coefficients β, γ, γ^* freely. If $d = 0$, we can take the 6 coefficients $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega$ freely.

4. Normalized Askey–Wilson relations

Let (A, A^*) denote a Leonard pair on V. Suppose that it satisfies the Askey–Wilson relations $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$. It can be computed that Leonard pair (3) then satisfies

$$\begin{aligned} AW\Big(&\beta, \gamma t + (2-\beta)c, \gamma^* t^* + (2-\beta)c^*, \varrho t^2 - 2\gamma c t + (\beta-2)c^2,\\ &\varrho^* t^{*2} - 2\gamma^* c^* t^* + (\beta-2)c^{*2}, \omega t t^* - 2\gamma c^* t - 2\gamma^* c t^* + 2(\beta-2)cc^*,\\ &\eta t^2 t^* - \varrho c^* t^2 - \omega c t t^* + \gamma^* c^2 t^* + 2\gamma^* c c^* t + (2-\beta)c^2 c^*,\\ &\eta^* t t^{*2} - \varrho^* c t^{*2} - \omega c^* t t^* + \gamma c^{*2} t + 2\gamma c c^* t^* + (2-\beta) c c^{*2}\Big). \end{aligned} \tag{25}$$

Note that β stays invariant. The affine transformations

$$(A, A^*) \mapsto (tA + c, t^*A^* + c^*), \text{ with } c, c^*, t, t^* \in \mathbb{K}, t, t^* \neq 0, \tag{26}$$

can be used to normalize Leonard pairs so that their Askey–Wilson relations would have a simple form. We refer to a transformations of the form $(A, A^*) \mapsto (A + c, A^* + c^*)$ as an *affine translation*, and to a transformation of the form $(A, A^*) \mapsto (tA, t^*A^*)$ as an *affine scaling*. Generally, we can use an affine translation to set some two Askey–Wilson coefficients to zero, and then use an affine scaling to normalize some two nonzero coefficients. Specifically, by affine translations we can achieve the following.

Lemma 4.1. *The Askey–Wilson relations $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$ can be normalized as follows*:

1. *If $\beta \neq 2$, we can set $\gamma = 0, \gamma^* = 0$.*
2. *If $\beta = 2, \gamma \neq 0, \gamma^* \neq 0$, we can set $\varrho = 0, \varrho^* = 0$.*
3. *If $\beta = 2, \gamma = 0, \gamma^* \neq 0$, we can set $\varrho^* = 0, \omega = 0$.*
4. *If $\beta = 2, \gamma^* = 0, \gamma \neq 0$, we can set $\varrho = 0, \omega = 0$.*
5. *If $\beta = 2, \gamma = 0, \gamma^* = 0, \omega^2 \neq \varrho\varrho^*$, we can set $\eta = 0, \eta^* = 0$.*
6. *If $\beta = 2, \gamma = 0, \gamma^* = 0$, $\mathrm{rk}\left(\begin{smallmatrix} \omega & \varrho & \eta \\ \varrho^* & \omega & \eta^* \end{smallmatrix}\right) \leqslant 1$, we can set $\eta = 0, \eta^* = 0$.*
7. *Otherwise, we have*

$$\beta = 2, \quad \gamma = 0, \quad \gamma^* = 0, \quad \omega^2 = \varrho\varrho^*, \ \mathrm{rk}\begin{pmatrix} \omega & \varrho & \eta \\ \varrho^* & \omega & \eta^* \end{pmatrix} = 2.$$

Then we can set either $\eta = 0$ or $\eta^* = 0$, but not both.

In the first 5 cases, there is a unique affine translation to make the normalization. In the last 2 cases, there are infinitely many normalizations by affine translations.

Proof. The first 4 cases are straightforward, including the uniqueness statement. If $\beta = 2, \gamma = 0$, $\gamma^* = 0$, the new Askey–Wilson relations (25) are

$$AW(2, 0, 0, \varrho t^2, \varrho^* t^{*2}, \omega t t^*, (\eta - \omega a - \varrho a^*)t^2 t^*, (\eta^* - \varrho^* a - \omega a^*)t t^{*2}),$$

where $a = c/t$ and $a^* = c^*/t^*$. To set the last two parameters to zero, we have to solve two linear equations in a, a^*. If we have det $\binom{\omega\ \varrho}{\varrho^*\ \omega} \neq 0$, the solution is unique. Otherwise we have either infinitely many or none solutions, which leads us to the last two cases. □

As it turns out, cases 6 and 7 of Lemma 4.1 do not occur for Askey–Wilson relations satisfied by Leonard pairs if $d \geqslant 3$. See part 3 of Theorem 8.1 below.

In Section 5, we normalize the general q-parameter arrays in Terwilliger's classification [8, Section 35] with most handy changes in the explicit expressions. We use the following simplest action of (26) on parameter arrays, consistent with the transformation of Leonard pairs:

$$\theta_i \mapsto t\theta_i + c, \quad \theta_i^* \mapsto t^*\theta_i^* + c^*, \quad \varphi_i \mapsto tt^*\varphi_i, \quad \phi_i \mapsto tt^*\phi_i. \tag{27}$$

It turns out that the corresponding Askey–Wilson relations follow the specification of part 1 of Lemma 4.1 immediately.

Suppose that we normalized a pair of Askey–Wilson relations to satisfy implications of Lemma 4.1, and suppose that cases 6 and 7 do not apply. Then the only affine transformations which preserve two specified zero coefficients are affine scalings. One can use affine scalings to normalize some two nonzero coefficients to convenient values. Sections 6 and 7 present such normalized parameter arrays that in their Askey–Wilson relations two nonzero coefficients are basically constants. (More precisely, in the q-cases they depend on q, or equivalently, on β.) The scaling normalization is explained more thoroughly in Section 8.

5. Normalized q-parameter arrays

Here we present the most straightforward normalizations of the general parameter arrays in [8, Section 35] with the q-parameter. Lemma 5.2 below gives the Askey–Wilson relations for the corresponding Leonard pairs. The Askey–Wilson relations turn out to be normalized according to part 1 of Lemma 4.1.

Lemma 5.1. *The parameter arrays in* [8, Examples 35.2–35.8] *can be normalized by affine transformations* (27) *to the following forms*:

- *The q-Racah case*: $\theta_i = q^{-i} + sq^{i+1}$, $\theta_i^* = q^{-i} + s^*q^{i+1}$.

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i)(r - ss^*q^{d+1+i})/r,$$
$$\phi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1})(r - s^*q^i)(sq^{d+1} - rq^i)/r.$$

- *The q-Hahn case*: $\theta_i = q^{-i}$, $\theta_i^* = q^{-i} + s^*q^{i+1}$,

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = -q^{1-i}(1-q^i)(1-q^{i-d-1})(r - s^*q^i).$$

- *The dual q-Hahn case*: $\theta_i = q^{-i} + sq^{i+1},\ \theta_i^* = q^{-i}$,

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}).$$

- *The q-Krawtchouk case*: $\theta_i = q^{-i},\ \theta_i^* = q^{-i} + s^*q^{i+1}$,

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = s^*q(1-q^i)(1-q^{i-d-1}).$$

- *The dual q-Krawtchouk case*: $\theta_i = q^{-i} + sq^{i+1},\ \theta_i^* = q^{-i}$,

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = sq^{d+2-2i}(1-q^i)(1-q^{i-d-1}).$$

- *The quantum q-Krawtchouk case*: $\theta_i = q^{i+1}$, $\theta_i^* = q^{-i}$,

$$\varphi_i = -rq^{1-i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(1-rq^{i-d-1}).$$

- *The affine q-Krawtchouk case*: $\theta_i = q^{-i}$, $\theta_i^* = q^{-i}$,

$$\varphi_i = q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = -rq^{1-i}(1-q^i)(1-q^{i-d-1}).$$

In each case, q, s, s^*, r *are nonzero scalar parameters such that* $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ *for* $0 \leqslant i < j \leqslant d$, *and* $\varphi_i \neq 0, \phi_i \neq 0$ *for* $1 \leqslant i \leqslant d$.

Proof. By affine translations, we adjust Terwilliger's parameters θ_0, θ_0^* so that we have only summands depending on i in the expanded expressions for θ_i, θ_i^* in [8, Examples 35.2–35.8]. By affine scalings, we set Terwilliger's parameters h, h^* to the value 1. In the quantum q-Krawtchouk case [8, Example 35.5] there is no parameter h, so we set $s = 1$. Other parameters remain unchanged, except that in the q-Racah case we rename r_1 to r and set $r_2 = ss^*q^{d+1}/r$. $\square$

Lemma 5.2. *Let* q, s, s^*, r *denote the same scalar parameters as in the previous lemma. We use the following notations*:

$$S = sq^{d+1} + 1, \quad S^* = s^*q^{d+1} + 1, \quad R = r + \frac{ss^*q^{d+1}}{r}, \tag{28}$$

$$Q = q^{d+1} + 1, \quad K = -\frac{(q^2-1)^2}{q}, \quad K^* = \frac{(q-1)^2}{q^{d+1}}. \tag{29}$$

The Askey–Wilson relations for the parameter arrays of Lemma 5.1 are:

- *For the q-Racah case*:

$$AW(q+q^{-1}, 0, 0, sK, s^*K, -K^*(SS^*+RQ),\\ (q+1)K^*(SR+sS^*Q), (q+1)K^*(S^*R+s^*SQ)). \tag{30}$$

- *For the q-Hahn case*:

$$AW(q+q^{-1}, 0, 0, 0, s^*K, -K^*(S^*+rQ),\\ (q+1)K^*r, (q+1)K^*(S^*r+s^*Q)). \tag{31}$$

- *For the dual q-Hahn case*:

$$AW(q+q^{-1}, 0, 0, sK, 0, -K^*(S+rQ),\\ (q+1)K^*(Sr+sQ), (q+1)K^*r). \tag{32}$$

- *For the q-Krawtchouk case*:

$$AW(q+q^{-1}, 0, 0, 0, s^*K, -K^*S^*, 0, (q+1)K^*s^*Q). \tag{33}$$

- *For the dual q-Krawtchouk case*:

$$AW(q+q^{-1}, 0, 0, sK, 0, -K^*S, (q+1)K^*sQ, 0). \tag{34}$$

- *For the quantum q-Krawtchouk case*:

$$AW(q+q^{-1}, 0, 0, 0, 0, -K^*(q^{d+1}+rQ), (q+1)(q-1)^2r, (q+1)K^*r). \tag{35}$$

- *For the affine q-Krawtchouk case*:

$$AW(q+q^{-1}, 0, 0, 0, 0, -K^*(1+rQ), (q+1)K^*r, (q+1)K^*r). \tag{36}$$

Proof. Direct computations with formulas (11)–(22). □

6. Alternative normalized q-arrays

Here we present alternative normalizations of the general parameter arrays in [8, Section 35] with the general q-parameter. The parameters are rescaled, and the free parameters q, s, s^*, r are different. In particular, the q of the previous section is replaced by q^2. The normalization for the q-Racah case is proposed in [5].

The corresponding Askey–Wilson relations are normalized according to part 1 of Lemma 4.1. Advantages of this normalization are: the two nonzero values normalized by affine scaling are q-constants; expressions for normalized parameter arrays are more symmetric; the set of normalized parameter arrays is preserved by the $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$ operations (see Section 9).

Lemma 6.1. *The parameter arrays in* [8, *Examples* 35.2–35.8] *can be normalized by affine transformations* (27) *to the following forms*:

- *The q-Racah case*: $\theta_i = sq^{d-2i} + \frac{q^{2i-d}}{s},\ \theta^*_i = s^*q^{d-2i} + \frac{q^{2i-d}}{s^*}$.

$$\varphi_i = \frac{q^{2d+2-4i}}{ss^*r}(1-q^{2i})(1-q^{2i-2d-2})(ss^*-rq^{2i-d-1})(ss^*r-q^{2i-d-1}),$$
$$\phi_i = \frac{q^{2d+2-4i}}{ss^*r}(1-q^{2i})(1-q^{2i-2d-2})(s^*r-sq^{2i-d-1})(s^*-srq^{2i-d-1}).$$

- *The q-Hahn case*: $\theta_i = rq^{d-2i},\ \theta^*_i = s^*q^{d-2i} + \frac{q^{2i-d}}{s^*}$,

$$\varphi_i = \frac{q^{2d+2-4i}}{r}(1-q^{2i})(1-q^{2i-2d-2})(s^*r^2-q^{2i-d-1}),$$
$$\phi_i = -\frac{q^{d+1-2i}}{rs^*}(1-q^{2i})(1-q^{2i-2d-2})(s^*-r^2q^{2i-d-1}).$$

- *The dual q-Hahn case*: $\theta_i = sq^{d-2i} + \frac{q^{2i-d}}{s},\ \theta^*_i = rq^{d-2i}$,

$$\varphi_i = \frac{q^{2d+2-4i}}{r}(1-q^{2i})(1-q^{2i-2d-2})(sr^2-q^{2i-d-1}),$$
$$\phi_i = \frac{q^{2d+2-4i}}{rs}(1-q^{2i})(1-q^{2i-2d-2})(r^2-sq^{2i-d-1}).$$

- *The q-Krawtchouk*: $\theta_i = q^{d-2i},\ \theta^*_i = s^*q^{d-2i} + \frac{q^{2i-d}}{s^*}$,

$$\varphi_i = s^*q^{2d+2-4i}(1-q^{2i})(1-q^{2i-2d-2}),$$
$$\phi_i = \frac{1}{s^*}(1-q^{2i})(1-q^{2i-2d-2}).$$

- *The dual q-Krawtchouk*: $\theta_i = sq^{d-2i} + \frac{q^{2i-d}}{s},\ \theta^*_i = q^{d-2i}$,

$$\varphi_i = sq^{2d+2-4i}(1-q^{2i})(1-q^{2i-2d-2}),$$
$$\phi_i = \frac{q^{2d+2-4i}}{s}(1-q^{2i})(1-q^{2i-2d-2}).$$

- *The quantum q-Krawtchouk*: $\theta_i = rq^{2i-d}$, $\theta^*_i = rq^{d-2i}$,

$$\varphi_i = -\frac{q^{d+1-2i}}{r}(1-q^{2i})(1-q^{2i-2d-2}),$$
$$\phi_i = \frac{q^{2d+2-4i}}{r}(1-q^{2i})(1-q^{2i-2d-2})(r^3-q^{2i-d-1}).$$

- *The affine q-Krawtchouk*: $\theta_i = rq^{d-2i}$, $\theta^*_i = rq^{d-2i}$,

$$\varphi_i = \frac{q^{2d+2-4i}}{r}(1-q^{2i})(1-q^{2i-2d-2})(r^3-q^{2i-d-1}),$$
$$\phi_i = -\frac{q^{d+1-2i}}{r}(1-q^{2i})(1-q^{2i-2d-2}).$$

In each case, q, s, s^*, r *are nonzero scalar parameters such that* $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ *for* $0 \leqslant i < j \leqslant d$, *and* $\varphi_i \neq 0, \phi_i \neq 0$ *for* $1 \leqslant i \leqslant d$.

Proof. In every case of Lemma 5.1, we substitute

$$q \mapsto q^2, \quad s \mapsto \frac{1}{s^2 q^{2d+2}}, \quad s^* \mapsto \frac{1}{s^{*2} q^{2d+2}}.$$

Besides, in the q-Racah, q-Hahn, dual q-Hahn, quantum q-Krawtchouk and affine q-Krawtchouk cases we substitute r by, respectively,

$$\frac{r}{ss^* q^{d+1}}, \quad \frac{1}{s^* r^2 q^{d+1}}, \quad \frac{1}{s r^2 q^{d+1}}, \quad \frac{q^{d+1}}{r^3}, \quad \frac{1}{r^3 q^{d+1}}.$$

After that, we apply affine scaling. We use formula (27) with $c = 0$, $c^* = 0$ and (t, t^*) equal to, respectively in the listed order,

$$(sq^d, s^*q^d), \quad (rq^d, s^*q^d), \quad (sq^d, rq^d), \quad (q^d, s^*q^d), \quad (sq^d, q^d),$$
$$(rq^{-d-2}, rq^d), \quad (rq^d, rq^d). \qquad \square$$

Lemma 6.2. *As in the previous lemma, let* q, s, s^*, r *denote nonzero scalar parameters. We use the following notations*:

$$Q_j = q^j + q^{-j}, \quad Q_j^* = q^j - q^{-j}, \text{ for } j = 1, 2, \ldots, \tag{37}$$

$$S = s + \frac{1}{s}, \quad S^* = s^* + \frac{1}{s^*}, \quad R = r + \frac{1}{r}. \tag{38}$$

The Askey–Wilson relations for the parameter arrays of Lemma 6.1 are:

- *For the q-Racah case*:

$$AW(Q_2, 0, 0, -Q_2^{*2}, -Q_2^{*2}, -Q_1^{*2}(SS^* + Q_{d+1}R),$$
$$Q_1 Q_1^{*2}(SR + Q_{d+1}S^*), Q_1 Q_1^{*2}(S^*R + Q_{d+1}S)). \tag{39}$$

- *For the q-Hahn case*:

$$AW(Q_2, 0, 0, 0, -Q_2^{*2}, -Q_1^{*2}(S^*r + Q_{d+1}r^{-1}),$$
$$Q_1 Q_1^{*2}, Q_1 Q_1^{*2}(S^*r^{-1} + Q_{d+1}r)). \tag{40}$$

- *For the dual q-Hahn case*:

$$AW(Q_2, 0, 0, -Q_2^{*2}, 0, -Q_1^{*2}(Sr + Q_{d+1}r^{-1}),$$
$$Q_1 Q_1^{*2}(Sr^{-1} + Q_{d+1}r), Q_1 Q_1^{*2}). \tag{41}$$

- *For the q-Krawtchouk case*:

$$AW(Q_2, 0, 0, 0, -Q_2^{*2}, -Q_1^{*2}S^*, 0, Q_1 Q_1^{*2} Q_{d+1}). \tag{42}$$

- *For the dual q-Krawtchouk case*:

$$AW(Q_2, 0, 0, -Q_2^{*2}, 0, -Q_1^{*2}S, Q_1Q_1^{*2}Q_{d+1}, 0). \tag{43}$$

- *For the quantum q-Krawtchouk and affine q-Krawtchouk cases*:

$$AW(Q_2, 0, 0, 0, 0, -Q_1^{*2}(r^2 + Q_{d+1}r^{-1}), Q_1Q_1^{*2}, Q_1Q_1^{*2}). \tag{44}$$

Proof. Transform the Askey–Wilson relations in Lemma 5.2 with the same substitutions and affine scalings as in the previous proof. In the notation of this lemma, the expressions S, S^*, R, Q, K, K^* of Lemma 5.2 get replaced by, respectively, S/s, S^*/s^*, $R/q^{d+1}ss^*$, $q^{d+1}Q_{d+1}$, $-q^2Q_2^{*2}$ and $q^{-2d}Q_1^{*2}$. □

7. Other parameter arrays

Here we present normalizations of the remaining general parameter arrays in [8, Section 35]. The corresponding Askey–Wilson relations are normalized according to Lemma 4.1, and two nonzero values are constants. Since we generally assume that char $\mathbb{K} \neq 2$, the orphan case is missing in the lemmas below. It is briefly discussed in Remark 7.3.

Lemma 7.1. *The parameter arrays in* [8, *Examples* 35.9–35.13] *can be normalized by affine transformations* (27) *to the following forms*:

- *The Racah case*: $\theta_i = (i+u)(i+u+1)$, $\theta_i^* = (i+u^*)(i+u^*+1)$,

$$\varphi_i = i(i-d-1)(i+u+u^*-v)(i+u+u^*+v+d+1),$$
$$\phi_i = i(i-d-1)(i-u+u^*+v)(i-u+u^*-v-d-1).$$

- *The Hahn case*: $\theta_i = i+v-\frac{d}{2}$, $\theta_i^* = (i+u^*)(i+u^*+1)$,

$$\varphi_i = i(i-d-1)(i+u^*+2v),$$
$$\phi_i = -i(i-d-1)(i+u^*-2v).$$

- *The dual Hahn case*: $\theta_i = (i+u)(i+u+1)$, $\theta_i^* = i+v-\frac{d}{2}$,

$$\varphi_i = i(i-d-1)(i+u+2v),$$
$$\phi_i = i(i-d-1)(i-u+2v-d-1).$$

- *The Krawtchouk case*: $\theta_i = i-\frac{d}{2}$, $\theta_i^* = i-\frac{d}{2}$,

$$\varphi_i = vi(i-d-1),$$
$$\phi_i = (v-1)i(i-d-1).$$

- *The Bannai–Ito case*: $\theta_i = (-1)^i \left(i + u - \frac{d}{2}\right), \theta^*_i = (-1)^i \left(i + u^* - \frac{d}{2}\right),$

$$\varphi_i = \begin{cases} -i\left(i+u+u^*+v-\frac{d+1}{2}\right), & \text{for } i \text{ even, } d \text{ even.} \\ -(i-d-1)\left(i+u+u^*-v-\frac{d+1}{2}\right), & \text{for } i \text{ odd, } d \text{ even.} \\ -i(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\ v^2-\left(i+u+u^*-\frac{d+1}{2}\right)^2, & \text{for } i \text{ odd, } d \text{ odd.} \end{cases}$$

$$\phi_i = \begin{cases} i\left(i-u+u^*-v-\frac{d+1}{2}\right), & \text{for } i \text{ even, } d \text{ even.} \\ (i-d-1)\left(i-u+u^*+v-\frac{d+1}{2}\right), & \text{for } i \text{ odd, } d \text{ even.} \\ -i(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\ v^2-\left(i-u+u^*-\frac{d+1}{2}\right)^2, & \text{for } i \text{ odd, } d \text{ odd.} \end{cases}$$

In each case, u, u^*, v *are scalar parameters such that* $\theta_i \neq \theta_j, \theta^*_i \neq \theta^*_j$ *for* $0 \leqslant i < j \leqslant d$, *and* $\varphi_i \neq 0, \phi_i \neq 0$ *for* $1 \leqslant i \leqslant d$.

Proof. Like in the proof of Lemma 5.1, we adjust Terwilliger's parameters θ_0, θ^*_0 by an affine translation, and then adjust other two parameters by an affine scaling. We also make linear substitutions for the remaining parameters. In the Racah case, we substitute

$$s \mapsto 2u, \quad s^* \mapsto 2u^*, \quad r_1 \mapsto u + u^* - v, \text{ so that } r_2 = u + u^* + v + d + 1.$$

Then we adjust $\theta_0 = u^2 + u$, $\theta^*_0 = u^{*2} + u^*$, $h = 1$, $h^* = 1$. In the Hahn case, we substitute $s^* \mapsto 2u^*, r \mapsto u^* + 2v$ and adjust $\theta_0 = v - \frac{d}{2}, \theta^*_0 = u^{*2} + u^*, h^* = 1, s = 1$. In the dual Hahn case, we substitute $s \mapsto 2u, r \mapsto u + 2v$ and adjust $\theta_0 = u^2 + u$, $\theta^*_0 = v - \frac{d}{2}$, $h = 1, s = 1$. In the Krawtchouk case, we substitute $r \mapsto v$ and adjust $\theta_0 = -\frac{d}{2}, \theta^*_0 = -\frac{d}{2}, s = 1, s^* = 1$. In the Bannai–Ito case, we substitute

$$s \mapsto d + 1 - 2u, \quad s^* \mapsto d + 1 - 2u^*, \quad r_1 \mapsto u + u^* + v - \frac{d+1}{2},$$

so that $r_2 \mapsto u + u^* - v - \frac{d+1}{2}$, and adjust $\theta_0 = u - \frac{d}{2}, \theta^*_0 = u^* - \frac{d}{2}, h = \frac{1}{2}, h^* = \frac{1}{2}$. □

Lemma 7.2. *Let* u, u^*, v *denote the same scalar parameters as in the previous lemma. The Askey–Wilson relations for the parameter arrays of Lemma* 7.1 *are*:

- *For the Racah case*:

$$\begin{aligned} AW(&2, 2, 2, 0, 0, -2u^2 - 2u^{*2} - 2v^2 - 2(d+1)(u+u^*+v) - 2d^2 - 4d, \\ &2u(u+d+1)(v-u^*)(v+u^*+d+1), \\ &2u^*(u^*+d+1)(v-u)(v+u+d+1)). \end{aligned} \tag{45}$$

- *For the Hahn case*:

$$AW\left(2, 0, 2, 1, 0, 0, -(u^*+1)(u^*+d) - 2v^2 - \frac{d^2}{2}, -4u^*(u^*+d+1)v\right). \tag{46}$$

- *For the dual Hahn case*:

$$AW\left(2, 2, 0, 0, 1, 0, -4u(u+d+1)v, -(u+1)(u+d) - 2v^2 - \frac{d^2}{2}\right). \tag{47}$$

- *For the Krawtchouk case*:

$$AW(2, 0, 0, 1, 1, 2v-1, 0, 0). \tag{48}$$

- *For the Bannai–Ito case, if d is even*:

$$AW(-2, 0, 0, 1, 1, 4uu^* - 2(d+1)v, 2uv - (d+1)u^*, 2u^*v - (d+1)u). \tag{49}$$

- *For the Bannai–Ito case, if d is odd*:

$$AW\left(-2, 0, 0, 1, 1, -2u^2 - 2u^{*2} + 2v^2 + \frac{(d+1)^2}{2},\right.$$
$$\left. -u^2 + u^{*2} - v^2 + \frac{(d+1)^2}{4}, u^2 - u^{*2} - v^2 + \frac{(d+1)^2}{4}\right). \tag{50}$$

Proof. Direct computations with formulas (11)–(22). □

Remark 7.3. In the characteristic 2, the normalization of part 1 in Lemma 4.1 is not available, hence our results are incomplete if char $\mathbb{K} = 2$. In particular, we miss the orphan case (with char $\mathbb{K} = 2$ and $d = 3$) completely. A general parameter array of the orphan type can be normalized as follows:

$$(\theta_0, \theta_1, \theta_2, \theta_3) = (0, s+1, 1, s),$$
$$(\theta_0^*, \theta_1^*, \theta_2^*, \theta_3^*) = (0, s^*+1, 1, s^*),$$
$$(\varphi_1, \varphi_2, \varphi_3) = (r, 1, r+s+s^*),$$
$$(\phi_1, \phi_2, \phi_3) = (r+s+ss^*, 1, r+s^*+ss^*).$$

Here adjusted $\theta_0 = 0, \theta_0^* = 0, h = 1, h^* = 1$ in [8, Example 35.14]. The Askey–Wilson relations are

$$AW(0, 1, 1, s^2+s, s^{*2}+s^*, ss^*, rs, rs^*). \tag{51}$$

The relations can be renormalized to $\rho = 0, \rho^* = 0$ by affine translations (in 4 ways, generally). The normalized coefficients η, η^*, ω in (51) are dependent on two free parameters, so there is a relation between them. Here is the relation, in the form invariant under affine rescaling:

$$(\omega^2 - \varrho\varrho^*)^2 = \omega(\gamma\omega - \gamma^*\varrho)(\gamma^*\omega - \gamma\varrho^*). \tag{52}$$

8. Classification of AW relations

Askey–Wilson relations can be consistently classified by families of orthogonal polynomials in the same way as Leonard pairs. The classification is presented in the first two columns of Table 1. In each line, the underlined equalities can be achieved by using affine translations if the preceding conditions are satisfied. If $\beta \neq 2$, by $\widehat{\varrho}, \widehat{\varrho}^*, \widehat{\omega}, \widehat{\eta}, \widehat{\eta}^*$ we denote the Askey–Wilson coefficients in a normalization specified by part 1 of Lemma 4.1.

The first part of the following theorem establishes the consistency of Askey–Wilson types for Leonard pairs and for Askey–Wilson relations.

Table 1
Classification of Askey–Wilson relations

Askey–Wilson type	Askey–Wilson coefficients	with Leonard pairs
q-Racah	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho\varrho}^* \neq 0$	–
q-Hahn	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = 0$, $\widehat{\varrho}^*\widehat{\eta} \neq 0$	–
Dual q-Hahn	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}^* = 0$, $\widehat{\varrho\eta}^* \neq 0$	–
q-Krawtchouk	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = \widehat{\eta} = 0$	$\widehat{\varrho}^*\widehat{\eta}^* \neq 0$
Dual q-Krawtchouk	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}^* = \widehat{\eta}^* = 0$	$\widehat{\varrho\eta} \neq 0$
Quantum/affine q-Krawtchouk	$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = \widehat{\varrho}^* = 0$	$\widehat{\eta\eta}^* \neq 0$
Racah	$\beta = 2$, $\gamma\gamma^* \neq 0$, $\underline{\varrho = \varrho^* = 0}$	–
Hahn	$\beta = 2$, $\gamma = 0$, $\gamma^* \neq 0$, $\underline{\varrho^* = 0}$	$\varrho \neq 0$, $\underline{\omega = 0}$
Dual Hahn	$\beta = 2$, $\gamma^* = 0$, $\gamma \neq 0$, $\underline{\varrho = 0}$	$\varrho^* \neq 0$, $\underline{\omega = 0}$
Krawtchouk	$\beta = 2$, $\gamma = \gamma^* = 0$	$\varrho\varrho^* \neq 0$, $\underline{\eta = \eta^* = 0}$
Bannai–Ito	$\beta = -2$, $\underline{\gamma = \gamma^* = 0}$	$\widehat{\varrho\varrho}^* \neq 0$

Theorem 8.1. *Assume that $d \geqslant 3$. Let (A, A^*) denote a Leonard pair on V, and let AW denote the Askey–Wilson relations satisfied by (A, A^*).*

1. *The Askey–Wilson relations AW have the same Askey–Wilson type as the Leonard pair (A, A^*).*
2. *If there is other Leonard pair on V that satisfies AW, it has the same Askey–Wilson type as (A, A^*).*
3. *There exist unique affine translation which normalizes AW according to the specifications of Lemma 4.1.*
4. *The Askey–Wilson relations AW satisfy all inequalities in the last two columns of Table 1 on the corresponding line. All underlined equalities can be achieved after an affine translation, and such an affine translation is unique. The indicated nonzero coefficients can be normalized to any chosen values by an affine scaling.*

Proof. For the first statement, check the results in Section 5 (or Section 6) and Section 7, and observe that the Askey–Wilson relations associated to any parameter array have the same Askey–Wilson type as the parameter array, with the exception of the ambiguity between the quantum q-Krawtchouk and affine q-Krawtchouk types.

The second statement is an immediate consequence.

For the third statement, we have to prove that cases 6 and 7 of Lemma 4.1 do not apply to AW. Assuming the contrary, AW would have the Krawtchouk type. In the corresponding normalized form of Lemma 7.2 we would have $v \in \{0, 1\}$. But then the Krawtchouk parameter array of Lemma 7.1 degenerates, since $\phi_i = 0$ or $\psi_i = 0$ for all $i = 1, 2, \ldots, d$. The third statement follows.

The inequalities of the last column of Table 1 can be checked by inspecting all Askey–Wilson relations in Lemmas 5.2, 6.2 and 7.2. Normalization by affine translations follows from the Lemma 4.1 and the previous part here. Normalization by affine scalings is clear. □

Note that the normalization specified by Lemma 4.1 follows implications of part 4 of Theorem 8.1. By part 3 of Theorem 8.1, there is a unique affine translation to set two specified Askey–

Wilson coefficients to zero. For each type of Leonard pairs, we get two Askey–Wilson coefficients which are certainly nonzero after the normalizing affine translation. These coefficients can be characterized as follows: they are the first nonzero (after the normalizing translation) coefficients in the two sequences

$$(\gamma, \varrho, \eta, \eta^*) \quad \text{and} \quad (\gamma^*, \varrho^*, \eta^*, \eta). \tag{53}$$

By affine scalings, the two coefficients can be normalized to any convenient values. In the Askey–Wilson relations of Lemmas 6.2 and 7.2, the normalized values depend only on β:

$$\begin{array}{ll} \gamma, \gamma^*: & 2 \text{ (if } \beta = 2); \\ \varrho, \varrho^*: & \begin{cases} 4-\beta^2, & \text{if } \beta \neq \pm 2, \\ 1, & \text{if } \beta = \pm 2; \end{cases} \\ \eta, \eta^*: & \begin{cases} \sqrt{\beta+2}(\beta-2), & \text{if } \eta\eta^* \neq 0 \text{ or } \omega = 0, \\ \sqrt{\beta+2}(\beta-2)Q_{d+1}, & \text{if } \eta\eta^* = 0 \text{ and } \omega \neq 0. \end{cases} \end{array} \tag{54}$$

Q_{d+1} can be independently defined by the linear recurrence $Q_{n+2} = \beta Q_n - Q_{n-2}$ with the initial values $Q_{-1} = Q_1 = \sqrt{\beta+2}$, $Q_0 = 2$, $Q_2 = \beta$. One can take for $\sqrt{\beta+2}$ any of the two values of the square root. In the context of Lemma 6.2, we should identify $\sqrt{\beta+2}$ with $q + q^{-1}$. The effect of changing the sign of $\sqrt{\beta+2}$ is multiplication of A and/or A^* by -1.

9. Uniqueness of normalizations

The results in Sections 5 through 7 can be used to compute the Askey–Wilson relations for any Leonard pair. To do this, one may take a parameter array corresponding to a given Leonard pair; then find an affine transformation (26) which normalizes the parameter array by (27) to one of the forms of Lemmas 5.1, 6.1 or 7.1; then pick up the corresponding normalized Askey–Wilson relations in Lemmas 5.2, 6.2 or 7.2; and then apply the inverse affine transformation to the normalized relations using formula (25). This procedure can be applied for any d, although for $d < 3$ the type of a representing parameter array is ambiguous and the Askey–Wilson relations are not unique.

For the rest of this section, we refer to the results of Sections 6 and 7. We assume $d \leqslant 3$ and adopt the following terminology. A pair of Askey–Wilson relations is called *normalized* if it satisfies the specifications of Lemma 4.1 and the description in the previous section; see (53) and (54). A Leonard pair is *normalized* if it satisfies normalized Askey–Wilson relations. A parameter array is *normalized* if it can be expressed in one of the forms of Lemma 6.1 or Lemma 7.1.

We consider the following questions:

Question 9.1. How unique is normalization of Askey–Wilson relations?

Question 9.2. Given a Leonard pair, how unique is its normalization?

Question 9.3. Is every normalized Leonard pair representable by a normalized parameter array?

Question 9.4. Are normalized parameter arrays represented uniquely by the forms in Lemmas 6.2 and 7.2?

Question 9.5. Do the relation operators $\downarrow, \Downarrow, \downarrow\Downarrow$ preserve the set of normalized parameter arrays?

Regarding the first question, non-uniqueness occurs for two reasons:

- There exist affine scalings by small roots of unity that leave the first nonzero coefficients in both sequences of (53) invariant. The list of these affine scalings is given by the first two columns of Table 2. By ζ_3 we denote a primitive cubic root of unity. In the q-Racah, Krawtchouk and Bannai–Ito cases, two given scalings can be composed. In the q-Hahn, dual q-Hahn and quantum/affine q-Krawtchouk cases, there are non-trivial iterations of the given scalings. The third column of Table 2 gives corresponding conversions of parameter arrays.
- In all q-Hahn and q-Krawtchouk cases, there exists an alternative normalization of the two nonzero Askey–Wilson coefficients from (53), with the other sign of $\sqrt{\beta+2}$. This effectively multiplies A or A^* (or both) by -1. The corresponding action on parameter arrays is given by the second column of Table 3.

Normalization of Askey–Wilson relations is unique in the Racah case. Otherwise, the normalization is unique in the q-Racah, Hahn, dual Hahn, Krawtchouk and Bannai–Ito cases if (and only if) the Askey–Wilson relations remain invariant under the respective affine scalings. This means

Table 2
Reparametrization of different normalizations

Askey–Wilson type	Affine scaling (t, t^*)	Conversion of normalized parameter arrays
q-Racah	$(-1, 1)$	$s \mapsto -s,\ r \mapsto -r$
	$(1, -1)$	$s^* \mapsto -s^*,\ r \mapsto -r$
q-Hahn	$(\sqrt{-1}, -1)$	$s^* \mapsto -s^*,\ r \mapsto \sqrt{-1}r$
Dual q-Hahn	$(-1, \sqrt{-1})$	$s \mapsto -s,\ r \mapsto \sqrt{-1}r$
q-Krawtchouk	$(1, -1)$	$s^* \mapsto -s^*$
Dual q-Krawtchouk	$(-1, 1)$	$s \mapsto -s$
Quantum and affine q-Krawtchouk	(ζ_3, ζ_3)	$r \mapsto \zeta_3 r$
Racah	–	–
Hahn	$(-1, 1)$	$\Downarrow$ and $v \mapsto -v$
Dual Hahn	$(1, -1)$	$\downarrow$ and $v \mapsto -v$
Krawtchouk	$(-1, 1)$	$\Downarrow$
	$(1, -1)$	$\downarrow$
Bannai–Ito	$(-1, 1)$	If d even: $\Downarrow$ and $u \mapsto -u,\ v \mapsto -v$
	$(1, -1)$	If d even: $\downarrow$ and $u^* \mapsto -u^*,\ v \mapsto -v$

Table 3
Alternative normalization and invariant reparametrizations

Askey–Wilson type	Change of sign of $\sqrt{\beta+2}$	Parameter array stays invariant
q-Racah	–	$r \mapsto 1/r$; also (56), (57)
q-Hahn	$q \mapsto -q,\ s^* \mapsto (-1)^{d+1}s^*$	–
Dual q-Hahn	$q \mapsto -q,\ s \mapsto (-1)^{d+1}s$	–
q-Krawtchouk	If d odd: $q \mapsto -q,\ s^* \mapsto -s^*$	If d even: $q \mapsto -q$
Dual q-Krawtchouk	If d odd: $q \mapsto -q,\ s \mapsto -s$	If d even: $q \mapsto -q$
Quantum and affine q-Krawtchouk	$q \mapsto -q,\ r \mapsto (-1)^{d+1}r$	–
Racah	–	$v \mapsto -v-d-1$
Bannai–Ito	–	If d odd: $v \mapsto -v$

that the non-scaled coefficients (such as $\widehat{\omega}$, $\widehat{\eta}$, $\widehat{\eta}^*$ in the q-Racah and Bannai–Ito cases) in the normalized relations are equal to zero.

Question 9.2 is equivalent to Question 9.1. However, existence and uniqueness of representation of a normalized Leonard pair by a normalized parameter array is determined by Questions 9.3 through 9.5.

An important discrepancy between normalization of Askey–Wilson relations and normalization of parameter arrays occurs in the Bannai–Ito case if d is odd. As Table 2 implies, affine scalings that preserve normalization of Askey–Wilson relations cannot be realized by transformations of normalized parameter arrays then. This has implications for Question 9.3.

Lemma 9.6

1. *Any normalized Leonard pair can be represented by a normalized parameter array, except when the Askey–Wilson type is Bannai–Ito, and d is odd.*
2. *Suppose that d is odd. Let (B, B^*) denote the Leonard pair represented by the parameter array of the Bannai–Ito type in Lemma 7.1. Then the following four Leonard pairs satisfy normalized Askey–Wilson relations of the Bannai–Ito type:*

$$(B, B^*), \quad (-B, B^*), \quad (B, -B^*), \quad (-B, -B^*). \tag{55}$$

Of these Leonard pairs, only (B, B^*) can be represented by a normalized parameter array.

Proof. Let (A, A^*) denote a normalized Leonard pair on V. Let Φ denote a parameter array for (A, A^*). Let $\Phi^\#$ denote a normalization of Φ by (27); it can be expressed in one of the forms of Lemmas 6.2 and 7.2. The parameter arrays Φ and $\Phi^\#$ differ by an affine scaling from Table 2, plus (in some q-cases) possibly the change of the sign of $\sqrt{\beta+2}$ in the Askey–Wilson relations. If the sign of $\sqrt{\beta+2}$ is changed, one can apply a corresponding reparametrization in the second column of Table 3. Reparametrizations for relevant affine scalings are indicated in Table 2, except for the Bannai–Ito case with odd d. Hence Φ is normalized as well, except perhaps when it has the Bannai–Ito type and d is odd.

Now we prove the second part. The four Leonard pairs in (55) are normalized according to our discussion of Questions 9.1 and 9.2. The Bannai–Ito parameter array of Lemma 7.1 has the following property: the even-indexed θ_i's and the even indexed θ_i^*'s form increasing sequences. Since d is assumed odd, the relation operators $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$ preserve this property. But affine scalings (27) with $t=-1$ or $t^*=-1$ reverse this property for θ_i's or θ_i^*'s, respectively. Hence, in all parameter arrays representing $(-B, B^*)$, $(B, -B^*)$ or $(-B, -B^*)$ the even-indexed θ_i's and/or the even indexed θ_i^*'s are in the decreasing order. The conclusion is that the four Leonard pairs $(\pm B, \pm B^*)$ cannot be transformed to each other by change of the parameters u, u^*, v or the relation operators. Hence only (B, B^*) can be represented as a specialization of the Bannai–Ito parameter array of Lemma 7.1. □

Questions 9.4 and 9.5 determine how unique are representations of normalized Leonard pairs by normalized parameter arrays. Invariant reparametrization of parameter arrays do occur. They are given in the third column of Table 3. In the q-Racah case, we additionally have the following invariant transformations:

$$q \mapsto 1/q, \quad s \mapsto 1/s, \quad s^* \mapsto 1/s^*; \tag{56}$$

$$q \mapsto -q, \quad s \mapsto (-1)^d s, \quad s^* \mapsto (-1)^d s^*, \quad r \mapsto (-1)^{d+1} r. \tag{57}$$

Table 4
Relative parameter arrays

Askey–Wilson type	Conversion to $\Downarrow$	Conversion to $\downarrow$
q-Racah	$s \mapsto 1/s$	$s^* \mapsto 1/s^*$
q-Hahn	$q \mapsto 1/q,\ s^* \mapsto 1/s^*$	$s^* \mapsto 1/s^*$
Dual q-Hahn	$s \mapsto 1/s$	$q \mapsto 1/q,\ s \mapsto 1/s$
q-Krawtchouk	$q \mapsto 1/q,\ s^* \mapsto 1/s^*$	$s^* \mapsto 1/s^*$
Dual q-Krawtchouk	$s \mapsto 1/s$	$q \mapsto 1/q,\ s \mapsto 1/s$
Quantum/affine q-Krawtchouk	Switch	Switch and $q \mapsto 1/q$
Racah	$u \mapsto -u - d - 1$	$u^* \mapsto -u^* - d - 1$
Hahn	–	$u^* \mapsto -u^* - d - 1$
Dual Hahn	$u \mapsto -u - d - 1$	–
Bannai–Ito, d odd	$u \mapsto -u$	$u^* \mapsto -u^*$

Question 9.5 is thoroughly answered in Table 4. There, "Switch" means interchanging the quantum q-Krawtchouk and affine q-Krawtchouk types of parameter arrays. Of course, $\downarrow\Downarrow$ is the composition of $\downarrow$ and $\Downarrow$. As wee see, the relation operators preserve normalization of parameter arrays in all q-cases, in the Racah case, and in the Bannai–Ito case with odd d.

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