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Some results for uniformly L-Lipschitzian mappings in Banach spaces

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ABSTRACT

The purpose of this work is to prove a strong convergence theorem for a pair of uniformly L-Lipschitzian mappings in Banach spaces. The results presented in the work improve and extend some recent results of Chang [S.S. Chang, Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129 (2001) 845–853], Cho et al [Y.J. Cho, J.I. Kang, H.Y. Zhou, Approximating common fixed points of asymptotically nonexpansive mappings, Bull. Korean Math. Soc. 42 (2005) 661–670], Ofoedu [E.U. Ofoedu, Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in a real Banach space, J. Math. Anal. Appl. 321 (2006) 722–728], Schu [J. Schu, Iterative construction of fixed point of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407–413] and Zeng [L.C. Zeng, On the iterative approximation for asymptotically pseudo-contractive mappings in uniformly smooth Banach spaces, Chinese Math. Ann. 26 (2005) 283–290 (in Chinese); L.C. Zeng, On the approximation of fixed points for asymptotically nonexpansive mappings in Banach spaces, Acta Math. Sci. 23 (2003) 31–37 (in Chinese)].

1. Introduction and preliminaries

Throughout this work, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. Let $T : K \to K$ be a mapping.

(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that, for any $x, y \in K$,

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1;$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for any given $x, y \in K$,

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall n \ge 1;$$

(3) T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that, for any $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le k_n\|x - y\|^2, \quad \forall n \ge 1.$$

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Remark 1. (1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\geq 1} k_n$. And every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the inverse is not true, in general.

(2) The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4], while the concept of asymptotically pseudo-contractive mappings was introduced by Schu [7] who proved the following theorem:

Theorem 1 (*Schu [7]*). *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ satisfying the following conditions:*

(i) $k_n \to 1$ *as* $n \to \infty$;
(ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ such that $\varepsilon < \alpha_n < \beta_n \leq b$, $\forall n \geq 1$, where $\varepsilon > 0$ and $b \in (0,\ L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$ are some positive numbers. For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T in K.

In [1], the first author extended Theorem 1 to a real uniformly smooth Banach space and proved the following theorem:

Theorem 2 (*Chang [1]*). *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \leq k_n\|x_n - x^*\|^2 - \phi(\|x_n - x^*\|), \quad \forall n \geq 0,$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$.

Very recently, in [6] Ofoedu proved the following theorem:

Theorem 3 (*Ofoedu [6]*). *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$ be a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $x^* \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall x \in K,$$

then $\{x_n\}$ converges strongly to x^*.

Remark 2. It should be pointed out that although Theorem 3 extends Theorem 2 from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K, the proof of [6, Theorem 3.1] has some problems.

The purpose of this work is, by using a simple and quite different method, to prove some strong convergence theorems for a pair of L-Lipschitzian mappings instead of making the assumption that T is a uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping in a Banach space. Our results extend and improve some recent results of [1,3,6–9].

For the main results, we first give the following lemmas:

Lemma 1.1 (*Chang [2]*). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 1.2 (*Moore and Nnoli [5]*). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \le \lambda_n \le 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ such that

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = \circ(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

Lemma 1.3. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \le (1 + \lambda_n) a_n + b_n, \quad \forall n \ge n_0,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2. Main results

In this section, we shall prove our main theorems in this work:

Theorem 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K, i = 1, 2$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \bigcap F(T_2) \ne \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K and x^* be a point in $F(T_1) \bigcap F(T_2)$. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_1^n y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T_2^n x_n. \end{cases} \tag{2.1}$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x - x^*) \rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, $i = 1, 2$, then $\{x_n\}$ converges strongly to x^*.

Proof. The proof is divided into two steps.

(I) Define $L = \max\{L_1, L_2\}$. First, we prove that the sequence $\{x_n\}$ defined by (2.1) is bounded.

In fact, it follows from (2.1) and Lemma 1.1 that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n (T_1^n y_n - x^*)\|^2 \\ &\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T_1^n y_n - x^*, j(x_{n+1} - x^*) \rangle \\ &\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T_1^n x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle + 2\alpha_n \langle T_1^n y_n - T_1^n x_{n+1},\ j(x_{n+1} - x^*) \rangle \\ &\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\} + 2\alpha_n L \|y_n - x_{n+1}\| \|x_{n+1} - x^*\|. \end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|(1 - \alpha_n)(x_n - y_n) + \alpha_n (T_1^n y_n - y_n)\| \\ &\le (1 - \alpha_n) \|x_n - y_n\| + \alpha_n \|T_1^n y_n - x^* + x^* - y_n\| \\ &\le (1 - \alpha_n) \|x_n - y_n\| + \alpha_n (1 + L) \|y_n - x^*\| \\ &\le (1 - \alpha_n) \|x_n - y_n\| + \alpha_n (1 + L) \{\|y_n - x_n\| + \|x_n - x^*\|\} \\ &= (1 + L\alpha_n) \|x_n - y_n\| + \alpha_n (1 + L) \|x_n - x^*\| \\ &= (1 + L\alpha_n) \{\beta_n \|x_n - T_2^n x_n\|\} + \alpha_n (1 + L) \|x_n - x^*\| \\ &\le (1 + L\alpha_n) \beta_n (1 + L) \|x_n - x^*\| + \alpha_n (1 + L) \|x_n - x^*\| \\ &= d_n \|x_n - x^*\|, \end{aligned} \tag{2.3}$$

where

$$d_n = (1 + L)\{(1 + L\alpha_n)\beta_n + \alpha_n\}.$$

By the conditions (i)–(iii), we know that

$$\sum_{n=0}^{\infty} \alpha_n d_n < \infty. \tag{2.4}$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\} + 2\alpha_n L d_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\} + \alpha_n L d_n \{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \end{aligned} \tag{2.5}$$

and hence

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \frac{A_n}{B_n} \|x_n - x^*\|^2 - \frac{2\alpha_n \phi(\|x_{n+1} - x^*\|)}{B_n} \\ &= \left\{1 + \frac{2\alpha_n (k_n - 1) + 2L d_n \alpha_n + \alpha_n^2}{B_n}\right\} \|x_n - x^*\|^2 - \frac{2\alpha_n \phi(\|x_{n+1} - x^*\|)}{B_n}, \end{aligned} \tag{2.6}$$

where

$$A_n = 1 - 2\alpha_n + \alpha_n^2 + \alpha_n L d_n, \qquad B_n = 1 - (2\alpha_n k_n + \alpha_n L d_n).$$

Since $\alpha_n \to 0$ as $n \to \infty$, there exists a positive integer n_0 such that $\frac{1}{2} < B_n \le 1$ for all $n \ge n_0$. Therefore, it follows from (2.6) that

$$\|x_{n+1} - x^*\|^2 \le \{1 + 2[2\alpha_n (k_n - 1) + 2L d_n \alpha_n + \alpha_n^2]\} \|x_n - x^*\|^2 - 2\alpha_n \phi(\|x_{n+1} - x^*\|), \quad \forall n \ge n_0 \tag{2.7}$$

and so

$$\|x_{n+1} - x^*\|^2 \le \{1 + 2[2\alpha_n (k_n - 1) + 2L d_n \alpha_n + \alpha_n^2]\} \|x_n - x^*\|^2, \quad \forall n \ge n_0. \tag{2.8}$$

By the condition (ii), (iii) and (2.4), we have

$$2 \sum_{n=0}^{\infty} [2\alpha_n (k_n - 1) + 2L d_n \alpha_n + \alpha_n^2] < \infty.$$

It follows from Lemma 1.3 that the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Therefore, the sequence $\{\|x_n - x^*\|\}$ is bounded. Without loss of generality, we can assume that $\|x_n - x^*\|^2 \le M$, where M is a positive constant.

(II) Now, we consider (2.7) and prove that $x_n \to x^*$.

Taking $\theta_n = \|x_n - x^*\|$, $\lambda_n = 2\alpha_n$ and $\sigma_n = 2[2\alpha_n (k_n - 1) + 2L d_n \alpha_n + \alpha_n^2] M$, (2.7) can be written as

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \quad \forall n \ge n_0.$$

By the conditions (i)–(iii), we know that all the conditions in Lemma 1.2 are satisfied. Therefore, it follows that

$$\|x_n - x^*\| \to 0,$$

that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3. (1) Theorem 2.1 extends and improves the corresponding results of Chang [1], Cho et al. [3], Ofoedu [6], Schu [7] and Zeng [8,9].

(2) The method given in the proof of Theorem 2.1 is quite different from the method given in Ofoedu [6].

(3) Theorem 2.1 also corrects some mistakes that appeared in the proof of Theorem 3.1 of Ofoedu [6].

(4) Theorem 2.1 can be generalized from a pair of uniformly L-Lipschitzian mappings to a finite family of uniformly L-Lipschitzian mappings in Banach spaces.

(5) Under suitable conditions, the sequence $\{x_n\}$ defined by (2.1) in Theorem 2.1 can also be generalized to the iterative sequences with errors. Because the proof is straightforward, we omit it here.

The following theorem can be obtained from Theorem 2.1 immediately:

Theorem 2.2. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$ be a uniformly L-Lipschitzian mapping with $F(T) \ne \emptyset$, where $F(T)$ is the set of fixed points of T in K, and x^* be a point in $F(T)$. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0. \tag{2.9}$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x-x^*) \in J(x-x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Remark 4. Theorem 2.2 is also a generalization and improvement of Ofoedu [6, Theorem 3.2].

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